Partition Algebra, Its Characterization and Representations

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Abstract

In this note we give representations for the partition algebra $A_3(Q)$ in Young's seminormal form. For this purpose, we also give the defining relations of $A_n(Q)$ and $A_{n-\frac{1}{2}}(Q)$.

1 Introduction

1.1 Definition of the partition algebra

Let $M = \{1, 2, \dots, n\}$ be a set of n symbols and $F = \{1', \dots, n'\}$ another set of n symbols. We assume that the elements of M and F are ordered by $1 < 2 < \cdots < n$ and $1' < 2' < \cdots < n'$ respectively. Consider the following set of set partitions:

$$\begin{aligned}\Sigma_n^1 \quad = \quad & \{\{T_1, \dots, T_s\} \mid s = 1, 2, \dots\ , \\ & T_j(\neq \emptyset) \subset M \cup F \ (j = 1, 2, \dots, s), \\ & \cup T_j = M \cup F, \quad T_i \cap T_j = \emptyset \text{ if } i \neq j\}.\end{aligned} \tag{1}$$

We call an element w of Σ_n^1 *a seat-plan* and each element of w a *part* of w. It is easy to see that the number of seat-plans is equal to B_{2n}, the Bell number.

For $w \in \Sigma_n^1$ consider a rectangle with n marked points on the bottom and the same n on the top as in Figure 1. The n marked points on the top are labeled by $1, 2, \dots n$ from left to right. Similarly, the n marked points on the bottom are labeled by $1', 2', \dots, n'$. If w consists of s parts, then put s shaded circles in the middle of the rectangle so that they have no intersections. Then we join the $2n$ marked points and the s circles with $2n$ shaded bands so that each shaded circle represent a part of w.

Using these diagrams, for $w_1, w_2 \in \Sigma_n^1$, an arbitrary pair of seat-plans, we can define a product $w_1 w_2$. The product is obtained by placing w_1 on w_2, gluing the corresponding boundaries and shrinking half along the vertical axis. We then have a new diagram possibly containing some shaded regions which are not connected to the boundaries. If the resulting diagram has p such regions, then

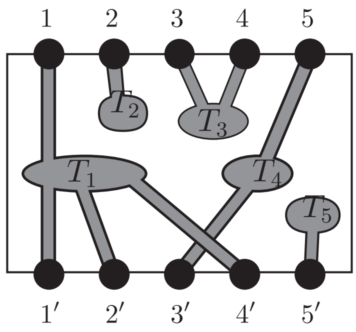

Figure 1: A seat-plan of Σ_5

the product is defined by the diagram with such region removed and multiplied by Q^p. Here Q is an indeterminate. (It is easily checked that the product defined above is closed in the linear span of the set of seat-plans Σ_n^1 over $\mathbb{Z}[Q]$.) For example, if

$$w_1 = \{\{1, 1', 4'\}, \{2, 5\}, \{3, 4\}, \{2'\}, \{3', 5'\}\} \in \Sigma_5^1$$

and

$$w_2 = \{\{1, 1', 3', 4'\}, \{2\}, \{3, 5\}, \{4\}, \{2', 5'\}\} \in \Sigma_5^1,$$

then we have

$$w_1 w_2 = Q^2\{\{1, 1', 3', 4'\}, \{2, 5\}, \{3, 4\}, \{2', 5'\}\} \in \mathbb{Z}[Q]\Sigma_5^1$$

as in Figure 2. By this product, the set of linear combinations of the elements of Σ_n^1 over $\mathbb{Z}[Q]$ makes an algebra $A_n(Q)$ called the *partition algebra*. The identity of $A_n(Q)$ is a diagram which corresponds to the partition

$$1 = \{\{1, 1'\}, \{2, 2'\}, \ldots, \{n, n'\}\}.$$

We put $A_0(Q) = A_1(Q) = \mathbb{Z}[Q]$. We can define $A_n(Q)$ more rigorously in terms of the set partitions (See P. P. Maritin's paper [13]).

Next we define special elements s_i, f_i $(1 \leq i \leq n-1)$ and e_i $(1 \leq i \leq n)$ of Σ_n^1 by

$$\begin{aligned}
s_i &= \{\{1, 1'\}, \ldots, \{i-1, (i-1)'\}, \{i+2, (i+2)'\}, \ldots, \{n, n'\}, \\
&\qquad \{i, (i+1)'\}, \{i+1, i'\}\} \\
f_i &= \{\{1, 1'\}, \ldots, \{i-1, (i-1)'\}, \{i+2, (i+2)'\}, \ldots, \{n, n'\}, \\
&\qquad \{i, i+1, i', (i+1)'\}\} \\
e_i &= \{\{1, 1'\}, \ldots, \{i-1, (i-1)'\}, \{i\}, \{i'\}\{i+1, (i+1)'\}, \ldots, \{n, n'\}\}.
\end{aligned}$$

The diagrams of these special elements are illustrated by the figures in Figure 3. Note that in the picture of e_i, there exist "a male" only part and "a female" only part. We call such a part "defective" (see Section 3.1).

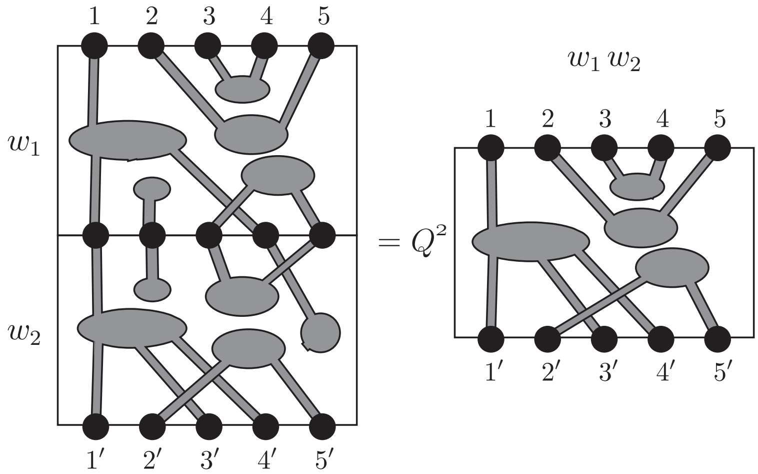

Figure 2: The product of seat-plans

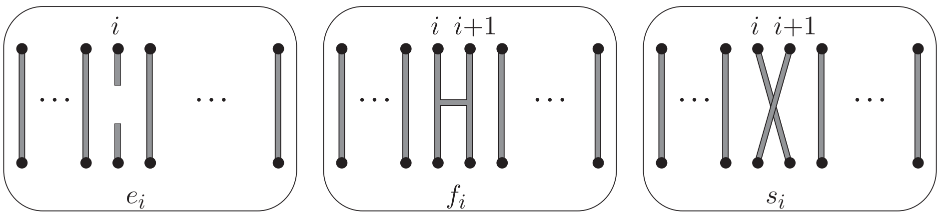

Figure 3: Special elements

We easily find that they satisfy the following basic relations.

$$\begin{aligned} f_{i+1} &= s_i s_{i+1} f_i s_{i+1} s_i \quad (i = 1, 2, \ldots, n-2), \\ e_{i+1} &= s_i e_i s_i \quad (i = 1, 2, \cdots, n-1) \end{aligned} \tag{R0}$$

$$\begin{aligned} s_i^2 &= 1 \quad (i = 1, 2, \ldots, n-1), \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad (i = 1, 2, \ldots, n-2), \\ s_i s_j &= s_j s_i \quad (|i-j| \geq 2), \end{aligned} \tag{R1}$$

$$f_i^2 = f_i, \ f_i f_j = f_j f_i, \tag{R2}$$

$$f_i s_i = s_i f_i = f_i, \tag{R3}$$

$$f_i s_j = s_j f_i \quad (|i-j| \geq 2), \tag{R4}$$

$$e_i^2 = Q e_i, \tag{E1}$$

$$s_i e_i e_{i+1} = e_i e_{i+1} s_i = e_i e_{i+1} \quad (i = 1, 2, \ldots, n-1), \tag{E2}$$

$$e_i s_j = s_j e_i \quad (j - i \geq 1, \ i - j \geq 2), \quad e_i e_j = e_j e_i, \tag{E3}$$

$$\begin{aligned} e_i f_i e_i = e_i \quad & e_{i+1} f_i e_{i+1} = e_{i+1} \quad (i = 1, 2, \ldots, n-1), \\ f_i e_i f_i = f_i, \quad & f_i e_{i+1} f_i = f_i \quad (i = 1, 2, \ldots, n-1), \end{aligned} \tag{E4}$$

$$e_i f_j = f_j e_i \quad (j - i \geq 1, \ i - j \geq 2). \tag{E5}$$

Here we make a remark on the special elements above.

REMARK 1.1. The relation $(R0)$ implies that the special elements $\{f_i\}$ and $\{e_i\}$ are generated by $f = f_1$, $e = e_1$ and $s_1, \ldots, s_{n-1}$.

In this note, firstly we show that the special elements and the basic relations $(R0)$-$(R4)$ and $(E1)$-$(E5)$ above characterize the partition algebra $A_n(Q)$, *i.e.* the special elements generate $A_n(Q)$, and all the possible relations in $A_n(Q)$ are obtained from the basic relations. By Remark 1.1, the basic relations will be translated into the relations among the symbols f, e and s_is. Characterizations will be stated by these symbols.

1.2 Characterization for $A_n(Q)$

Since generators $\{s_i \mid 1 \leq i \leq n-1\}$ of the partition algebra $A_n(Q)$ satisfy the relations of the symmetric group $\mathfrak{S}_n$, we can understand that f_i and e_i are "conjugate" to f and e respectively.

Hence the basic relations $(R2)$-$(R4)$ and $(E1)$-$(E5)$ among the special elements are translated into the relations $(R2')$-$(R4')$ and $(E1')$-$(E5')$ among the generators as follows.

Theorem 1.2. *The partition algebra $A_n(Q)$ is characterized by the generators*

$$f, e, s_1, s_2, \ldots, s_{n-1},$$

and the relations

$$\begin{array}{rcl} s_i^2 &=& 1 \quad (i=1,2,\ldots,n-1), \\ s_is_{i+1}s_i &=& s_{i+1}s_is_{i+1} \quad (i=1,2,\ldots,n-2), \\ s_is_j &=& s_js_i \quad (|i-j|\geq 2,\ i,j=1,2,\ldots,n-1), \end{array} \tag{$R1$}$$

$$f^2=f,\ fs_2fs_2=s_2fs_2f,\ fs_2s_1s_3s_2fs_2s_1s_3s_2=s_2s_1s_3s_2fs_2s_1s_3s_2f, \tag{$R2'$}$$

$$fs_1=s_1f=f, \tag{$R3'$}$$

$$fs_i=s_if \quad (i=3,4,\ldots,n-1), \tag{$R4'$}$$

$$e^2=Qe, \tag{$E1'$}$$

$$es_1es_1=s_1es_1e=es_1e, \tag{$E2'$}$$

$$es_i=s_ie \quad (i=2,3,\ldots,n-1), \tag{$E3'$}$$

$$efe=e,\ fef=f, \tag{$E4'$}$$

$$fs_2s_1es_1s_2=s_2s_1es_1s_2f. \tag{$E5'$}$$

In Sections 2-4 we prove this theorem not using the generators and the relations in the theorem but using the special elements and the basic relations ($R0$)-($R4$) and ($E1$)-($E5$).

The partition algebras $A_n(Q)$ were introduced in early 1990s by Martin [12, 13] and Jones [6] independently and have been studied, for example, in the papers [14, 2, 5]. The theorem above has already shown in the paper [5]. Here we give another poof defining a "standard" expression of a word of the special elements of $A_n(Q)$ according to the papers [8, 10, 11]. From this standard expression, we will find that the partition algebra $A_n(Q)$ is cellular in the sense of Graham and Lehrer [4]. Thus, applying the general representation of cellular algebras to the partition algebras, we will get a description of the irreducible modules of $A_n(Q)$ for any field of arbitrary characteristic. (For the cell representations, we also refer the paper [7].)

Further, we can make the character table of $A_n(Q)$ using the standard expressions. These topics will be studied in near future. For the present we refer the notes [10, 17] and the results about the partition algebras [2, 22].

2 Local moves deduced from the basic relations

Let

$$\mathcal{L}_n^1=\{s_1,s_2,\ldots,s_{n-1},f_1,f_2,\ldots,f_{n-1},e_1,e_2,\ldots,e_n\}$$

be the set of symbols whose words satisfy the basic relations ($R0$)-($R4$) and ($E1$)-($E5$). There are many relations among these symbols which are deduced from the basic relations. These relations are pictorially expressed as local moves. Among them, we frequently use relations $f_{i+1}s_is_{i+1}=s_is_{i+1}f_i$ ($R0$), $f_is_{i+1}f_i=f_if_{i+1}$ ($R2''$) and $e_is_i=e_if_ie_{i+1}=s_ie_{i+1}$ ($E4''$) as in Figure 4,5 and 6 respectively. The latter two relations are deduced from the relations ($R0$)-($R3$) and ($R0$), ($R3$), ($E4$) respectively.

Figure 4: $f_{i+1}s_is_{i+1} = s_is_{i+1}f_i$ ($R0$)

Figure 5: $f_is_{i+1}f_i = f_if_{i+1}$ ($R2''$)

As we noted in the previous paper [11], these basic relations are invariant under the transpositions of indices $i \leftrightarrow n-i+1$ as well as the $\mathbb{Z}[Q]$-linear involution $*$ defined by $(xy)^* = y^*x^*$ $(x, y \in A_n(Q))$. This implies that if one local move is allowed then other three moves —obtained by reflections with respect to the vertical and the horizontal lines and their composition— are also allowed.

Further, we note that if we put

$$e^{[r]} = f_1f_2\cdots f_{r-1}e_1e_2\cdots e_rf_1f_2\cdots f_{r-1}$$

then we can check that $e^{[r]}$, f and s_i $(1 \le i \le n-1)$ satisfy the defining relations of $P_{n,r}(Q)$, the r-modular party algebra, defined in the paper [11]. This means that the local moves shown in the paper [11] also hold in $A_n(Q)$ (in fact, these local moves are more easily verified in $A_n(Q)$). Some of them are pictorially expressed in Figures 7,8,9 and 10.

3 Standard expressions of seat-plans

In this section, for a seat-plan w of Σ_n^1, we define a *basic expression*, as a word in the alphabet Γ_n^1. Then we define more general forms called *crank form expressions*. As a special type of the crank form expression, we define the *standard expression*. In the next section, we show that any two crank form expressions of a seat-plan will be moved to each other by using the basic relations ($R0$)-($R4$) and ($E1$)-($E5$) finite times. Consequently, we find that any seat-plan can be moved to its standard expression. To define these expressions, we introduce some terminologies.

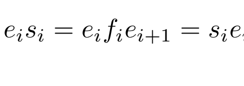

Figure 6: $e_is_i = e_if_ie_{i+1} = s_ie_{i+1}$ ($E4''$)

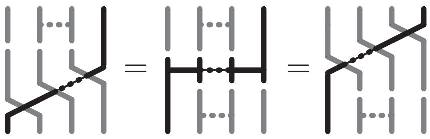

Figure 7: Defective part jump rope ($R13'$)

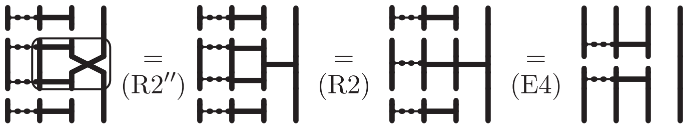

Figure 8: Removal (addition) of excrescences ($R14'$)

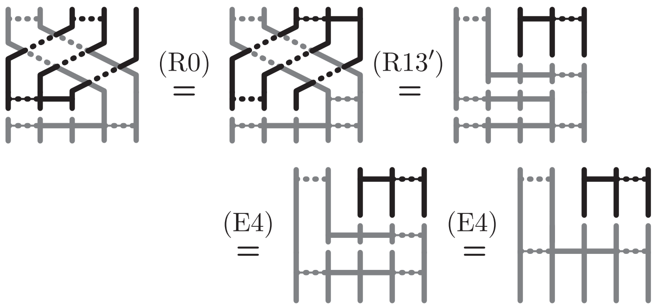

Figure 9: Defective part shift ($R16'$)

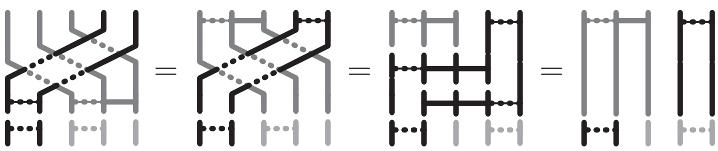

Figure 10: Defective part exchange ($R17'$)

3.1 Propagating number

Let $w = \{T_1, T_2, \ldots, T_s\}$ be a seat-plan of $A_n(Q)$. For a part $T_i \in w$, the intersection with M, or $T_i^M = M \cap T_i$, is called the *upper part* of T_i. Similarly, $T_i^F = F \cap T_i$ is called the *lower part* of T_i. If $T_i^M \neq \emptyset$ and $T_i^F \neq \emptyset$ hold simultaneously, T_i is called *propagating*, otherwise, it is called *non-propagating*, or *defective*. Let $\pi(w) := \{T \in w \mid T : \text{propagating}\}$ be the set of propagating parts. The number of propagating parts $|\pi(w)|$ is called the *propagating number* (of w). If $T_i \in \pi(w)$ then the upper [resp. lower] part T_i^M [resp. T_i^F] of T_i is also called *propagating*. If $T_i \in w \setminus \pi(w)$ and $T_i^M = T_i$ [resp. $T_i^F = T_i$], then T_i^M [resp. T_i^F] is called *defective*.

For example, in Figure 1, $\pi(w) = \{T_1, T_4\}$. Hence $|\pi(w)| = 2$. On the other hand T_2, T_3 and T_5 are defective. The upper and the lower propagating parts are $\{1\}$, $\{5\}$ and $\{1', 2', 4'\}$, $\{3'\}$ respectively. The upper defective parts are T_2 and T_3. The lower defective part is T_5.

3.2 A basic expression of a seat-plan

For a part $T_i \in w$, define $t(T_i)$ by

$$t(T_i) = \begin{cases} 1 & \text{if } T_i \text{ is propagating,} \\ 0 & \text{if } T_i \text{ is defective.} \end{cases}$$

Similarly we define $t(T_i^M)$ [resp. $t(T_i^F)$] to be 1 or 0 in accordance with that T_i^M [resp. T_i^F] is propagating or not.

Using the terminologies above, first we define a *basic expression* of an seat-plan. Let $w \in \Sigma_n^1$ be a seat-plan and $\rho_w = (T_1, \ldots, T_s)$ be an arbitrary sequence of all parts of w. For the sequence ρ_w, we define the sequence of the upper [resp. lower] parts $\mathbb{M} = \mathbb{M}(\rho_w) = (T_{i_1}^M, \ldots, T_{i_u}^M)$ $(i_1 < \cdots < i_u,\ u \leq s)$ [resp. $\mathbb{F} = \mathbb{F}(\rho_w) = (T_{j_1}^F, \ldots, T_{j_v}^F)$ $(j_1 < \cdots < j_v,\ v \leq s)$] omitting empty parts.

Using these data, we define *cranks* $C_{\mathbb{M}}[i]$, $C_{\mathbb{F}}^*[i]$ and $C_{\mathbb{F}}^{\mathbb{M}}[\sigma]$) as products of the generators as in Figure 11, 12 and 13 respectively. Here σ is a word in the alphabet $\{s_1, \ldots, s_{|\pi(w)|-1}\}$.

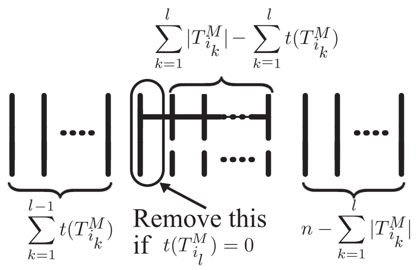

Figure 11: $C_{\mathbb{M}}[l]$

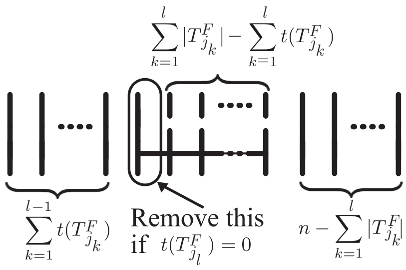

Figure 12: $C^*_{\mathbb{F}}[l]$

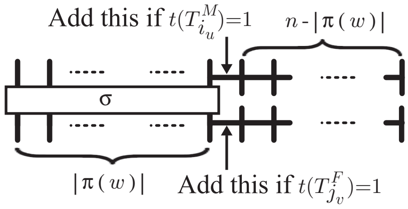

Figure 13: $C^{\mathbb{M}}_{\mathbb{F}}[\sigma]$

Further we define the "product of cranks" $C[\mathbb{M}]$ and $C[\mathbb{F}]$ by

$$C[\mathbb{M}] = C_{\mathbb{M}}[1]C_{\mathbb{M}}[2]\cdots C_{\mathbb{M}}[u-1]$$

and

$$C^*[\mathbb{F}] = C^*_{\mathbb{F}}[v-1]\cdots C^*_{\mathbb{F}}[2]C^*_{\mathbb{F}}[1]$$

respectively. We note that $C_{\mathbb{M}}[l]$ [resp. $C^*_{\mathbb{F}}[l]$] is defined by a composition $\mathbb{E} = (E_1, \ldots, E_s)$ of n whose components have labels either "propagating" or "defective". For example if $\mathbb{M} = (2,1,2,2,3)$, $(t(M_i))_{1\le i\le 5} = (0,1,0,1,1)$, $\mathbb{F} = (3,4,3)$, $(t(F_i))_{i=1,2,3} = (1,1,1)$ and $\sigma = (1,2)(2,3) \in \mathfrak{S}_3$, then the product of cranks $C[\mathbb{M}]C^{\mathbb{M}}_{\mathbb{F}}[\sigma]C^*[\mathbb{F}]$ is presented as in Figure 14.

Let $\overline{\mathbb{M}}$ be the sequence of n symbols obtained from $\mathbb{M} = \mathbb{M}(\rho_w)$ by arranging all elements of $T^M_{i_k}$s in accordance with the sequence $\mathbb{M}$ so that all elements of each $T^M_{i_k}$ are increasingly lined up from left to right. For example, if $\mathbb{M} = (\{3,1,7\},\{6,4\},\{5,2\})$, then $\overline{\mathbb{M}} = (1,3,7,4,6,2,5)$. Similarly $\overline{\mathbb{F}}$ is defined from $\mathbb{F} = \mathbb{F}(\rho_w)$.

Then the following product becomes an expression of a seat-plan w.

$$\mathcal{C}(\mathbb{M}, id, \mathbb{F}) = x_{\overline{\mathbb{M}}}C[\mathbb{M}]C^{\mathbb{M}}_{\mathbb{F}}[id]C^*[\mathbb{F}]x^*_{\overline{\mathbb{F}}}.$$

Here $x_{\overline{\mathbb{M}}}$ [resp. $x^*_{\overline{\mathbb{F}}}$] is a permutation which maps j to the number in the j-th coordinate of $\overline{\mathbb{M}}$. [resp. the number written in the j-th coordinate of $\overline{\mathbb{F}}$ to j'].

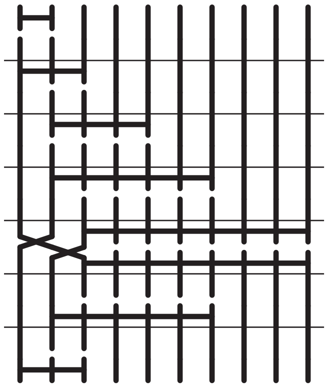

Figure 14: Product of cranks

We call this expression a *basic expression* of w. We note that for a seat-plan w there are several ways to choose ρ_w, a sequence of the parts of w. *i.e.* Several basic expressions can be defined for one seat-plan.

3.3 The standard expressions of a seat-plan

Our claim is that any basic expression of a seat-plan w can be moved to a special expression called the *standard expression* by using the relations $(R0)$-$(R4)$ and $(E1)$-$(E5)$ finite times. In order to show this claim, next we introduce the notion of a *crank form expression* of w.

Consider the propagating parts $\pi(w) = \{T_{i_1}, \dots, T_{i_p}\}$ $(p = |\pi(w)|)$ of w. Let $(M_1, \dots, M_p)$ be a sequence of the upper parts of $\pi(w)$ and $(F_1, \dots, F_p)$ the one of the lower parts. Then there exists a permutation $\sigma \in \mathfrak{S}_p$ such that $\{M_{\sigma(k)} \sqcup F_k \mid k = 1, \dots, p\} = \pi(w)$. As is well known, a permutation σ of degree p is presented by p-strings braid which connects the lower k-th point to the upper $\sigma(k)$-th point.

Now we define a *crank form expression* of w. Let $\mathbb{M} = (M_1, \dots, M_u)$ [resp. $\mathbb{F} = (F_1, \dots, F_v)$] be any fixed sequence of the upper [resp. lower] parts of w (whose empty parts are omitted and propagating parts are specified). From the sequences $\mathbb{M}$ and $\mathbb{F}$, we obtain products of cranks $C[\mathbb{M}]$ and $C^*[\mathbb{F}]$. Further, from $\pi(\mathbb{M})$ and $\pi(\mathbb{F})$, we obtain a permutation $\sigma \in \mathfrak{S}_p$ such that $\{M_{i_{\sigma(k)}} \sqcup F_{j_k}\ ;\ k = 1, \dots, p\} = \pi(w)$. Then the product

$$\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F}) = x_{\overline{\mathbb{M}}} C[\mathbb{M}] C^{\mathbb{M}}_{\mathbb{F}}[\sigma] C^*[\mathbb{F}] x^*_{\overline{F}}$$

becomes a presentation of w. We call this presentation a *crank form expression of w defined by* $\mathbb{M}$ *and* $\mathbb{F}$. If a crank form expression is made from sequences $(M_1, \dots, M_u)$ and $(F_1, \dots, F_v)$ such that

1. $M_1, \dots, M_p$ and $F_1, \dots, F_p$ are propagating,

2. $M_{p+1}, \dots, M_u$ and $F_{p+1}, \dots, F_v$ are defective.

then we call it *in normal form*.

Finally, we define the *standard expression* of w, as a special expression of crank form expressions in normal form by properly choosing the sequences $(M_1, \dots, M_u)$ and $(F_1, \dots, F_v)$. For this purpose first we sort the parts $T_1, \dots, T_s$ of w so that they satisfy:

1. $\pi(w) = \{T_1, T_2, \dots, T_p\}$,
2. $\{T_i \mid i = p+1, p+2, \dots, u\}$ is the set of all upper defective parts,
3. $\{T_i \mid i = u+1, u+2, \dots, u+(v-p)\}$ is the set of all lower defective parts.

For an ordered set E, let $\min E$ be the minimum element in E. Let $T_1, T_2, \dots, T_p$ be the parts of $\pi(w)$. Define $(M_1, M_2, \dots, M_p)$ so that they satisfy

$$\{M_1, M_2, \dots, M_p\} = \{T_1^M, T_2^M, \dots, T_p^M\}$$

and

$$\min M_1 < \min M_2 < \cdots < \min M_p.$$

Similarly $(F_1, F_2, \dots, F_p)$ are defined using the lower parts of $\pi(w)$. In such a method, the sequences of the upper parts $(M_1, \dots, M_p)$ and the lower parts $(F_1, \dots, F_p)$ are uniquely defined from a seat-plan w.

Now we define $(M_{p+1}, \dots, M_u)$ so that they satisfy

$$\{M_{p+1}, M_{p+2}, \dots, M_u\} = \{T_{p+1}, T_{p+2}, \dots, T_u\}$$

and

$$\min M_{p+1} < \min M_{p+2} < \cdots < \min M_u.$$

Similarly we define $(F_{p+1}, \dots, F_v)$ so that they satisfy

$$\{F_{p+1}, F_{p+2}, \dots F_v\} = \{T_{u+1}, T_{u+2}, \dots, T_{u+(v-p)}\}$$

and

$$\min F_{p+1} < \min F_{p+2} < \cdots < \min F_v.$$

Using these upper and lower sequences defined above, we can obtain a crank from expression in normal form called the *standard expression* of w.

4 Proof of Theorem 1.2

In the previous section, we have defined the standard expression of a word in the alphabet $\mathcal{L}_n^1$ as a special expression of the crank form expressions in normal form. In this section, first we show that any two crank form expressions of a seat-plan w are transformed to each other by finitely using the local moves shown in Section 2. Then we show that any word in the alphabet $\mathcal{L}_n^1$ is moved to a scalar multiple of one of the crank form expressions. Thus we can find that any word in the alphabet $\mathcal{L}_n^1$ is reduced to a scalar multiple of a the standard expression. Since the set of seat-plans makes a basis of $A_n(Q)$ and since every seat-plan has its standard expression, this proves that the partition algebra $A_n(Q)$ is characterized by the generators and relations in Theorem 1.2.

First we show that any two crank form expressions are transformed to each other. For $w \in \Sigma_n^1$, let $\mathbb{M} = (M_1, \dots, M_u)$ and $\mathbb{F} = (F_1, \dots, F_v)$ be sequences of the upper and the lower parts of w respectively. Assume that the subsequence $\pi(\mathbb{M}) = (M_{i_1}, \dots, M_{i_p})$ ($i_1 < \cdots < i_p$) of $\mathbb{M}$ is the sequence of the upper propagating parts and $\pi(\mathbb{F}) = (F_{j_1}, \dots, F_{j_p})$ ($j_1 < \cdots < j_p$) is that of the lower propagating parts. Then there exists a permutation σ of degree $p = |\pi(w)|$ which specifies how the propagating parts of w are recovered from $\pi(\mathbb{M})$ and $\pi(\mathbb{F})$. Let $\mathbb{E} = (E_1, \dots, E_s)$ be a sequence of the upper or lower parts. Suppose that $\tau \in \mathfrak{S}_s$ acts on $\mathbb{E}$ by $\tau\mathbb{E} = (E_{\tau^{-1}(1)}, \dots, E_{\tau^{-1}(s)})$. Then the following lemma holds.

Lemma 4.1. *Let $\mathbb{M} = (M_1, \dots, M_u)$ and $\mathbb{F} = (F_1, \dots, F_v)$ be sequences of the upper and the lower (non-empty) parts of a seat-plan respectively. If M_i [resp. F_i] is defective and $\sigma_i = (i, i+1)$, the i-th adjacent transposition, then the crank form expression $\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is moved to another crank form expression $\mathcal{C}(\sigma_i\mathbb{M}, \sigma, \mathbb{F})$ [resp. $\mathcal{C}(\mathbb{M}, \sigma, \sigma_i\mathbb{F})$].*

Proof. We consider the case M_i is defective. In case F_i is defective, the similar proof will hold. Let $P_{\mathbb{M},i} \in \mathfrak{S}_n$ be a permutation defined by

$$P_{\mathbb{M},i}(x) := \begin{cases} x + |M_{i+1}| & \text{if } \sum_{j=1}^{i-1} |M_j| < x \le \sum_{j=1}^{i} |M_j|, \\ x - |M_i| & \text{if } \sum_{j=1}^{i} |M_j| < x \le \sum_{j=1}^{i+1} |M_j|, \\ x & \text{otherwise.} \end{cases}$$

Then we find that $x_{\overline{\mathbb{M}}} P_{\mathbb{M},i}^{-1}$ maps j to the j-th coordinate of $\overline{\sigma_i\mathbb{M}}$. Hence we have $x_{\overline{\mathbb{M}}} P_{\mathbb{M},i}^{-1} = x_{\overline{\sigma_i\mathbb{M}}}$. (For the definition of $\overline{\mathbb{M}}$, see Section 3.2.)

On the other hand, since M_i is defective, we have $P_{\mathbb{M},i} C[\mathbb{M}] = C[\sigma_i\mathbb{M}]$ by removing an excrescence of M_i and iteratively using "defective part exchange" ($R17'$) in Figure 10 (if M_{i+1} is defective) or iteratively using "defective part shift" ($R16'$) in Figure 9 (if M_{i+1} is propagating), and then adding an excrescence to M_i just moved. Thus we obtain

$$\begin{aligned} \mathcal{C}(\mathbb{M}, \sigma, \mathbb{F}) &= x_{\overline{\mathbb{M}}} C[\mathbb{M}] C_{\mathbb{F}}^{\mathbb{M}}[\sigma] C^*[\mathbb{F}] x_{\overline{\mathbb{F}}}^* \\ &= (x_{\overline{\mathbb{M}}} P_{\mathbb{M},i}^{-1})(P_{\mathbb{M},i} C[\mathbb{M}]) C_{\mathbb{F}}^{\mathbb{M}}[\sigma] C^*[\mathbb{F}] x_{\overline{\mathbb{F}}}^* \\ &= x_{\overline{\sigma_i\mathbb{M}}} C[\sigma_i\mathbb{M}] C_{\mathbb{F}}^{\mathbb{M}}[\sigma] C^*[\mathbb{F}] x_{\overline{\mathbb{F}}}^* \\ &= \mathcal{C}(\sigma_i\mathbb{M}, \sigma, \mathbb{F}). \end{aligned}$$

□

Remark 4.2. Lemma 4.1 also holds if M_{i+1} [resp. F_{i+1}] is defective.

By Lemma 4.1 and Remark 4.2 we may assume that any crank form expression is given in normal form.

Lemma 4.3. *Let $\mathcal{C}(\mathbb{M},\sigma,\mathbb{F})$ be a crank form expression of w in normal form. If M_i and M_{i+1} are propagating then $\mathcal{C}(\mathbb{M},\sigma,\mathbb{F})$ is moved to another crank form expression $\mathcal{C}(\sigma_i\mathbb{M},\sigma_i\sigma,\mathbb{F})$ in normal form. Similarly if F_i and F_{i+1} are propagating, then $\mathcal{C}(\mathbb{M},\sigma,\mathbb{F})$ is moved to $\mathcal{C}(\mathbb{M},\sigma\sigma_i,\sigma_i\mathbb{F})$.*

Proof. Let $C_{\mathbb{M}}[i]$ and $C_{\mathbb{M}}[i+1]$ be i-th and $(i+1)$-st cranks of $C[\mathbb{M}]$. By Figure 15, we have

$$P_{\mathbb{M},i}C_{\mathbb{M}}[i]C_{\mathbb{M}}[i+1] = C_{\sigma_i\mathbb{M}}[i]C_{\sigma_i\mathbb{M}}[i+1]\sigma_i.$$

Thus we obtain

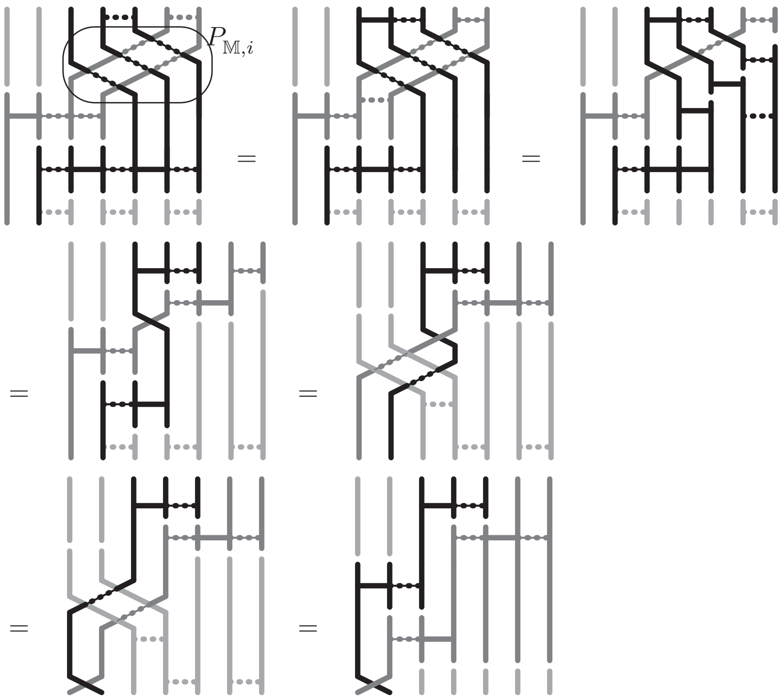

Figure 15: Crank form exchange

$$\begin{aligned}
\mathcal{C}(\mathbb{M},\sigma,\mathbb{F}) &= x_{\overline{\mathbb{M}}}C[\mathbb{M}]C^{\mathbb{M}}_{\mathbb{F}}[\sigma]C^*[\mathbb{F}]x^*_{\overline{\mathbb{F}}} \\
&= (x_{\overline{\mathbb{M}}}P^{-1}_{\mathbb{M},i})(P_{\mathbb{M},i}C[\mathbb{M}])C^{\mathbb{M}}_{\mathbb{F}}[\sigma]C^*[\mathbb{F}]y^*_{\overline{\mathbb{F}}} \\
&= x_{\overline{\sigma_i\mathbb{M}}}C[\sigma_i\mathbb{M}]C^{\mathbb{M}}_{\mathbb{F}}[\sigma_i\sigma]C^*[\mathbb{F}]x^*_{\overline{\mathbb{F}}} \\
&= \mathcal{C}(\sigma_i\mathbb{M},\sigma_i\sigma,\mathbb{F}).
\end{aligned}$$

□

By Lemma 4.1, Remark 4.2 and Lemma 4.3 we obtain the following.

Proposition 4.4. *A crank form expression of a seat-plan is moved to its standard expression.*

Now we prove that any word in the alphabet $\mathcal{L}_n^1$ is moved to a crank form expression. By the above proposition, we will find that any word can be moved to its standard expression.

Proposition 4.5. *If $\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is the standard expression of a seat-plan w, then $s_i\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is moved to a crank form expression of $s_i w$.*

Proof. If i and $i+1$ are both included one of the (upper) parts of w, say M_k, then we have

$$\sum_{j=1}^{k-1} |M_j| < x_{\overline{\mathbb{M}}}^{-1}(i) < x_{\overline{\mathbb{M}}}^{-1}(i+1) = x_{\overline{\mathbb{M}}}^{-1}(i) + 1 \leq \sum_{j=1}^{k} |M_j|$$

and

$$(x_{\overline{\mathbb{M}}}^{-1}(i), x_{\overline{\mathbb{M}}}^{-1}(i+1))C_{\mathbb{M}}[k] = (x_{\overline{\mathbb{M}}}^{-1}(i), x_{\overline{\mathbb{M}}}^{-1}(i)+1)C_{\mathbb{M}}[k] = \mathcal{C}_{\mathbb{M}}[k].$$

Since

$$s_i x_{\overline{\mathbb{M}}} = (i, i+1)x_{\overline{\mathbb{M}}} = x_{\overline{\mathbb{M}}}(x_{\overline{\mathbb{M}}}^{-1}(i), x_{\overline{\mathbb{M}}}^{-1}(i+1)),$$

we find that $s_i\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F}) = \mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is a crank form expression.

If i is included in M_j and $i+1$ is included in M_k ($j \neq k$), then we have $s_i x_{\overline{\mathbb{M}}} = x_{\overline{\mathbb{M}'}}$. Here $\mathbb{M}'$ is the sequence of the upper parts obtained from $\mathbb{M} = (M_1, \dots, M_u)$ by replacing M_j with $M_j' = (M_j \setminus \{i\}) \cup \{i+1\}$ and M_k with $M_k' = (M_k \setminus \{i+1\}) \cup \{i\}$.

Hence we find that $s_i\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is moved to $\mathcal{C}(\mathbb{M}', \sigma, \mathbb{F})$, a crank form expression. In particular this expression again becomes the standard expression, unless $k = j+1$, $t(M_j) = t(M_{j+1})$, and $i = \min M_j$, $i+1 = \min M_{j+1}$. □

Proposition 4.6. *If $\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is the standard expression of a seat-plan w, then $f\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is moved to a crank form expression of fw.*

Proof. First consider the case $\{1, 2\} \subset M_k$ for some k. In this case, there exists an integer i such that $i = x_{\overline{\mathbb{M}}}^{-1}(1)$ and $i+1 = x_{\overline{\mathbb{M}}}^{-1}(2)$. Hence in this case we have $f x_{\overline{\mathbb{M}}} = x_{\overline{\mathbb{M}}} f_i$ and $f_i \mathcal{C}_{\mathbb{M}}[k] = \mathcal{C}_{\mathbb{M}}[k]$. Thus we obtain $f\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F}) = \mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$.

Next consider the case $1 \in M_j$ and $2 \in M_k$ ($j \neq k$). In the following we assume that M_j and M_k are both propagating. Even if either M_j or M_k or both of them are defective, the similar proof will hold. Proposition 4.4 implies that the standard expression $\mathcal{C}(\mathbb{M}, \sigma, \mathbb{F})$ is moved to a crank form expression $\mathcal{C}(\mathbb{M}', id, \mathbb{F}')$ so that the first and the second components of $\mathbb{M}'$ are M_j and M_k respectively and the first and the second components of $\mathbb{F}'$ are jointed to M_j and M_k respectively. Using the relations $(R2'')$, $(R2)$ and $(R12'')$, we find that the first and the second components of $\mathbb{M}'$ and those of $\mathbb{F}'$ are merged by the action of f. For example, if $|M_j| = 5$ and $|M_k| = 4$ then we have Figure 16. The merged propagating parts will be moved to a crank form expression $\mathcal{C}(\mathbb{M}'', id, \mathbb{F}'')$ by

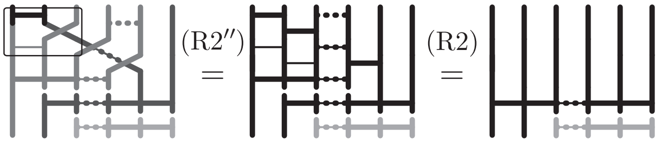

Figure 16: Action of f on w

Figure 17: Bumping

"bumping" as in Figure 17. Here $\mathbb{M}''$ [resp. $\mathbb{F}''$] is a sequence of upper [resp. lower] parts obtained from $\mathbb{M}$ [resp. $\mathbb{F}$] by merging the first two components. □

Proposition 4.7. *If $\mathcal{C}(\mathbb{M},\sigma,\mathbb{F})$ is the standard expression of a seat-plan w, then $e\mathcal{C}(\mathbb{M},\sigma,\mathbb{F})$ is moved to a crank form expression of ew.*

Proof. By the same argument in the previous proposition, we may assume that $\mathcal{C}(\mathbb{M},\sigma,\mathbb{F})$ is moved to a crank form expression

$$\mathcal{C}(\mathbb{M}'', id, \mathbb{F}'')$$

such that the first component M_1'' of $\mathbb{M}''$ contains $\{1\}$.

First consider the case $|M_1''| > 1$. In this case, it is easy to check that $e\mathcal{C}(\mathbb{M}'', id, \mathbb{F}'')$ is again a crank form expression of ew as it is.

Next consider the case $|M_1''| = 1$. If M_1'' is defective, then we have a scalar multiple of a crank form expression $e\mathcal{C}(\mathbb{M}'', id, \mathbb{F}'') = Q\mathcal{C}(\mathbb{M}'', id, \mathbb{F}'')$. If M_1'' is propagating, then applying "addition of excrescences $(R14')$" and "bumping" in Figures 8 and 17 we can move $e\mathcal{C}(\mathbb{M}'', id, \mathbb{F}'')$ to a crank form expression. □

Proof of Theorem 1.2. Let $\widetilde{A_n(Q)}$ be the associative algebra over $\mathbb{Z}[Q]$ abstractly defined by the generators and the relations in Theorem 1.2. So there exists a surjective morphism ψ from $\widetilde{A_n(Q)}$ to $A_n(Q)$. As we noted in Section 1, we may assume that $\widetilde{A_n(Q)}$ is generated by the alphabets $\mathcal{L}_n^1$ which satisfy the relations $(R0)$-$(R4)$ and $(E1)$-$(E5)$. Here we note that the "geometrical moves" we have shown previously can be applied to any algebra which satisfies the relations $(R0)$-$(R4)$ and $(E1)$-$(E5)$. Hence if we associate the alphabets in $\mathcal{L}_n^1$ with the diagrams in Figure 3, then we can apply the notion of *basic expressions*, *crank form expressions* and *standard expressions* to the words in the alphabets $\mathcal{L}_n^1$ of $\widetilde{A_n(Q)}$.

Let w be a word in the alphabet $\mathcal{L}_n^1$ of $\widetilde{A_n(Q)}$. Suppose that w is presented in a standard expression. Then by Proposition 4.5-4.7, $s_i w$, fw and ew are all moved to (possibly scalar multiples of) crank form expressions. By Proposition 4.4, they are moved to the standard expressions. Since s_i $(1 \leq i \leq n-1)$, f, and e are crank form expressions as they are, by induction on the lengths of the words in the alphabets $\mathcal{L}_n$, any word turn out to be equal to (a scalar multiple) of the standard expression of a seat-plan w of Σ_n^1. Hence we have

$$\operatorname{rank} \widetilde{A_n(Q)} \leq |\Sigma_n^1|.$$

As Tanabe showed in [19], Σ_n^1 makes a basis of $\mathbb{C} \otimes A_n(k) = \mathbb{C} \otimes \psi(\widetilde{A_n(k)})$ if $k \geq n$. Hence rank $\mathbb{C} \otimes A_n(z) = |\Sigma_n^1|$ holds as far as z takes any integer value $k \geq n$. This implies that ψ is an isomorphism and we find that the generators and the relations in Theorem 1.2 characterize the partition algebra $A_n(Q)$. $\square$

5 Definition of $A_{n-\frac{1}{2}}(Q)$, a subalgebra of $A_n(Q)$

In this section, we consider a subalgebra $A_{n-\frac{1}{2}}(Q)$ of $A_n(Q)$ generated by the special elements $s_1, \ldots, s_{n-2}$, $f_1, \ldots, f_{n-1}$ and $e_1, \ldots, e_{n-1}$. As we have noted in Remark 1.1, $\{f_i\}$ $(1 \leq n-2)$ and $\{e_i\}$ $(1 \leq n-1)$ are written as products of $f = f_1$, $e = e_1$ and $s_1, \ldots, s_{n-2}$. The special element f_{n-1}, however, can not be expressed as a product of other special elements in $A_{n-\frac{1}{2}}(Q)$, since we deleted s_{n-1} from the generators of $A_n(Q)$. Hence $A_{n-\frac{1}{2}}(Q)$ can be defined as a subalgebra of $A_n(Q)$ generated by the following elements: $s_1, \ldots, s_{n-2}$, $f = f_1$, $f_* = f_{n-1}$ and $e = e_1$. We can obtain the defining relations among these generators just as in the case of $A_n(Q)$.

Theorem 5.1. *Let $\mathbb{Z}$ be the ring of rational integers and Q the indeterminate. We put $A_{\frac{1}{2}}(Q) = \mathbb{Z}[Q] \cdot 1$. For an integer $n \geq 2$, $A_{n-\frac{1}{2}}(Q)$ is characterized by the generators*

$$e, f, s_1, s_2, \ldots, s_{n-2}, f_* \text{ (if } n > 2\text{)}$$

and the relations (R0), (R1′)-(R4′) and (E1′)-(E5′) omitting the ones which involve s_{n-1} and adding the following relations:

$$\begin{aligned} & f_* s_{n-2} s_{n-3} \cdots s_3 s_2 s_1 s_2 s_3 \cdots s_{n-3} s_{n-2} f_* \\ &= f_* s_{n-2} s_{n-3} \cdots s_3 s_2 f s_2 s_3 \cdots s_{n-3} s_{n-2} \\ &= s_{n-2} s_{n-3} \cdots s_3 s_2 f s_2 s_3 \cdots s_{n-3} s_{n-2} f_*, \end{aligned} \tag{R2*}$$

$$f f_* = f_* f, \quad e f_* = f_* e, \quad f_* s_i = s_i f_* \ (1 \leq i \leq n-3), \tag{R4*}$$

$$\begin{aligned} f_* s_{n-2} s_{n-3} \cdots s_1 e s_1 \cdots s_{n-3} s_{n-2} f_* &= f_*, \\ e s_1 s_2 \cdots s_{n-2} f_* s_{n-2} \cdots s_2 s_1 e &= e. \end{aligned} \tag{E4*}$$

We understand $A_{1+\frac{1}{2}}(Q) = A_{2-\frac{1}{2}}(Q)$ is defined by the generators 1, e and f with the relations $e^2 = Qe$, $f^2 = f$, $efe = e$, $fef = f$. (Hence, $A_{2-\frac{1}{2}}(Q)$ is a rank 5 module with a basis $\{1, e, f, ef, fe\}$.)

The relations ($R2^*$) correspond to the relations $f_{n-1}s_{n-2}f_{n-1} = f_{n-1}f_{n-2} = f_{n-2}f_{n-1}$. We deduce $f_*s_{n-2}f_* = f_*s_{n-2}f_*s_{n-2} = f_*s_{n-2}f_*s_{n-2}$ from ($R2^*$).

Proof. First we note that all the generators of $A_{n-\frac{1}{2}}(Q)$ have the part which contains n and n' simultaneously.

We consider the transpositions of indices $i \leftrightarrow n-i+1$. These transpositions make $A_{n-\frac{1}{2}}(Q)$ a subalgebra of $A_n(Q)$ generated by

$$\mathcal{L}^1_{n-\frac{1}{2}} = \{f_1, \dots, f_{n-1}, s_2, \dots, s_{n-1}, e_2, \dots, e_n\}.$$

By the relation ($R0$), $A_{n-\frac{1}{2}}(Q)$ is actually generated by letters $\{f_1,\ f_2,\ e_2$ and $s_2, \dots, s_{n-1}\}$. Each of these generators has a part which includes $\{1, 1'\}$. In the following in this section, we suppose that $A_{n-\frac{1}{2}}(Q)$ is generated by the letters in $\mathcal{L}^1_{n-\frac{1}{2}}$. The $\mathbb{Z}[Q]$ bases of $A_{n-\frac{1}{2}}(Q)$ consist of $\Sigma^1_{n-\frac{1}{2}}$ a subset of seat-plans in Σ^1_n which have at least one propagating part which contains 1 and $1'$ simultaneously. In the diagram of the standard expression of a seat-plan of $\Sigma^1_{n-\frac{1}{2}}$, the vertices 1 and $1'$ are joined by a vertical line. Shrinking this vertical line to one vertex, we have one to one correspondences between $\Sigma^1_{n-\frac{1}{2}}$ and the set of the set-partitions of order $2n-1$. (Hence we find $|\Sigma^1_{n-\frac{1}{2}}| = B_{2n-1}$, the Bell number.)

Under this preparation, we prove the theorem. Since the relations in the theorem allow us to use all the required local moves, we can show just in the course of the arguments of Section 4 that any word in the alphabet $\mathcal{L}^1_{n-\frac{1}{2}}$ is equal to (possibly a scalar multiple of) a standard expression in the abstract algebra $\widetilde{A_{n-\frac{1}{2}}}(Q)$.

Hence we have

$$\operatorname{rank} \widetilde{A_{n-\frac{1}{2}}}(Q) \le |\Sigma^1_{n-\frac{1}{2}}|.$$

As Murtin and Rollet showed in [15], $\Sigma^1_{n-\frac{1}{2}}$ makes a basis of $\mathbb{C}\otimes A_{n-\frac{1}{2}}(k) = \mathbb{C}\otimes\psi(\widetilde{A_{n-\frac{1}{2}}}(k))$ if $k > n$. Hence rank $\mathbb{C}\otimes A_n(z) = |\Sigma^1_{n-\frac{1}{2}}|$ holds as far as z takes any integer value $k > n$. This implies that ψ is an isomorphism and we find that the generators and the relations in the theorem characterize the subalgebra $A_{n-\frac{1}{2}}(Q)$. $\square$

6 Bratteli diagram of the partition algebras

In this section, we get back to the original definition of $A_{n-\frac{1}{2}}(Q)$. (*i. e.* $A_{n-\frac{1}{2}}(Q)$ is generated by $s_1, \dots, s_{n-2},\ f_1, \dots, f_{n-1}$ and $e_1, \dots, e_{n-1}$.) Since, $A_{n-\frac{1}{2}}(Q)$ contains all the generators of $A_{n-1}(Q)$, it becomes a subalgebra of $A_{n-\frac{1}{2}}(Q)$. Hence we obtain the sequence of inclusions $A_0(Q) \subset A_{\frac{1}{2}}(Q) \subset \cdots \subset A_{i-\frac{1}{2}}(Q) \subset A_i(Q) \subset A_{i+\frac{1}{2}}(Q) \subset \cdots$.

First we define a graph Γ_n [resp. $\Gamma_{n+\frac{1}{2}}$] for a non-negative integer $n \in \mathbb{Z}_{\ge 0}$. Then we define the sets of *tableaux* as sets of paths on this graph. Figure 18 will help the reader to understand the recipe.

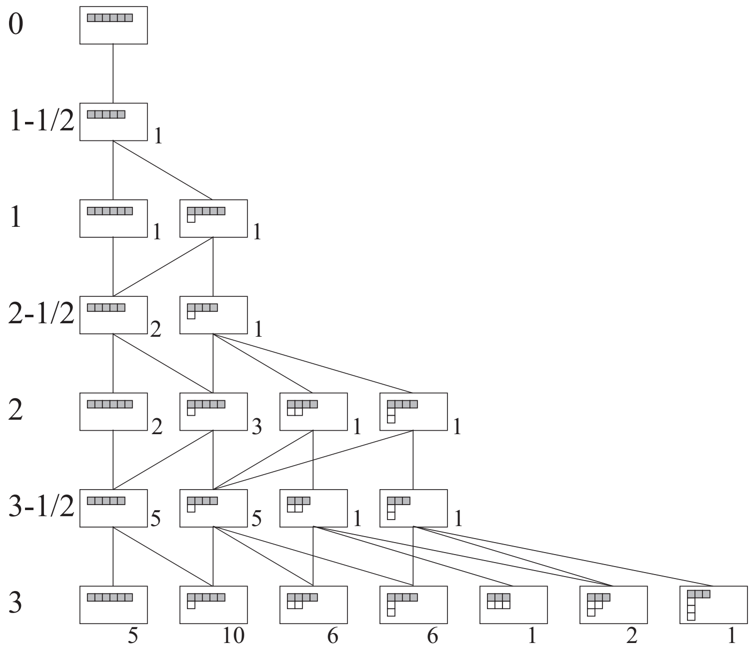

Figure 18: Γ_4

For the moment, we assume that Q is a sufficiently large integer. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition. For this λ, define

$$\begin{aligned} \widetilde{\lambda} &= (Q - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_l) \\ \big[\text{resp. } \widehat{\lambda} &= (Q - 1 - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_l)\big] \end{aligned}$$

to be a partition of size Q [resp. $Q-1$]. Pictorially, $\widetilde{\lambda}$ [resp. $\widehat{\lambda}$] is obtained by adding $Q - |\lambda|$ [resp. $Q - 1 - |\lambda|$] boxes on the top of λ.

Let $P_{\leq i} = \bigcup_{j=0}^{i} \{\lambda \mid \lambda \vdash j\}$ be a set of Young diagrams of size less than or equal to i. We define $\mathbf{\Lambda}_i$ and $\mathbf{\Lambda}_{i+\frac{1}{2}}$ to be

$$\mathbf{\Lambda}_i = \{\widetilde{\lambda} \mid \lambda \in P_{\leq i}\} \text{ and } \mathbf{\Lambda}_{i+\frac{1}{2}} = \{\widehat{\lambda} \mid \lambda \in P_{\leq i}\},$$

which are set of Young diagrams of size Q and $Q-1$ respectively.

Under these preparations we define a graph Γ_n [resp. $\Gamma_{n+\frac{1}{2}}$] which consists of the vertices labeled by:

$$\left(\bigsqcup_{i=0,1,\ldots,n-1} (\mathbf{\Lambda}_i \sqcup \mathbf{\Lambda}_{i+\frac{1}{2}}) \right) \bigsqcup \mathbf{\Lambda}_n \quad \left[\text{resp. } \left(\bigsqcup_{i=0,1,\ldots,n} (\mathbf{\Lambda}_i \sqcup \mathbf{\Lambda}_{i+\frac{1}{2}}) \right) \right]$$

and the edges joined by either of the following rule:

- join $\widetilde{\lambda} \in \mathbf{\Lambda}_i$ and $\widehat{\mu} \in \mathbf{\Lambda}_{i+\frac{1}{2}}$ if $\widehat{\mu}$ is obtained from $\widetilde{\lambda}$ by removing a box $(i = 0, 1, 2, \ldots n-1)$ [resp. $(i = 0, 1, 2, \ldots, n)$],
- join $\widehat{\mu} \in \mathbf{\Lambda}_{i-\frac{1}{2}}$ and $\widetilde{\lambda} \in \mathbf{\Lambda}_i$ if $\widetilde{\lambda}$ is obtained from $\widehat{\mu}$ by adding a box $(i = 1, 2, \ldots n)$.

For a pair of Young diagrams $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, if $\boldsymbol{\beta}$ is obtained from $\boldsymbol{\alpha}$ by one of the method above, we write this as $\boldsymbol{\alpha} \smile \boldsymbol{\beta}$.

Finally, we define the sets of the tableaux. For a half integer $n \in \frac{1}{2}\mathbb{Z}$ and $\boldsymbol{\alpha} \in \mathbf{\Lambda}_n$, we define $\mathbb{T}(\boldsymbol{\alpha})$, *tableaux of shape* $\boldsymbol{\alpha}$, to be

$$\begin{aligned} \mathbb{T}(\boldsymbol{\alpha}) &= \{P = (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1/2)}, \ldots, \boldsymbol{\alpha}^{(n)}) \mid \boldsymbol{\alpha}^{(j)} \in \mathbf{\Lambda}_j \ (j = 0, 1/2, \ldots, n), \\ &\qquad \boldsymbol{\alpha}^{(n)} = \boldsymbol{\alpha}, \boldsymbol{\alpha}^{(j)} \smile \boldsymbol{\alpha}^{(j+1/2)} \ (j = 0, 1/2, \ldots, n-1/2)\}. \end{aligned}$$

7 Construction of representation

Now we have defined the sets of tableaux, we define linear transformations among the tableaux.

Let $\mathbb{Q}$ be the field of rational numbers and $K_0 = \mathbb{Q}(Q)$ its extension. In the following, the linear transformations are defined over K_0. If they preserve the relations defined in the previous sections, they define representations of $A_n = A_n(Q) \otimes K_0$. Similar methods are used for example in the references [1, 3, 16, 20, 21, 9].

Let $\mathbb{V}(\boldsymbol{\alpha}) = \oplus_{P\in\mathbb{T}(\boldsymbol{\alpha})} K_0 v_P$ be a vector space over K_0 with the standard basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\alpha})\}$.

For generators e_i, f_i and s_i of A_n, we define linear maps $\rho_{\boldsymbol{\alpha}}(e_i)$, $\rho_{\boldsymbol{\alpha}}(f_i)$ and $\rho_{\boldsymbol{\alpha}}(s_i)$ on $\mathbb{V}(\boldsymbol{\alpha})$ giving the matrices E_i F_i and M_i respectively with respect to the basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\alpha})\}$.

7.1 Definition of $\rho_{\boldsymbol{\alpha}}(e_i)$

Firstly, we define a linear map for e_i.

For a tableaux $P = (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1/2)}, \ldots, \boldsymbol{\alpha}^{(n)})$ of $\mathbb{T}(\boldsymbol{\alpha})$, we define $\rho_{\boldsymbol{\alpha}}(e_i)(v_P) = \sum_{Q\in\mathbb{T}(\boldsymbol{\alpha})} (E_i)_{QP} v_Q$. Let $Q = (\boldsymbol{\alpha}'^{(0)}, \boldsymbol{\alpha}'^{(1/2)}, \ldots, \boldsymbol{\alpha}'^{(n)})$.

If there is an $i_0 \in \{1/2, 1, \ldots, n-1/2\} \setminus \{i-1/2\}$ such that $\boldsymbol{\alpha}^{(i_0)} \neq \boldsymbol{\alpha}'^{(i_0)}$, then we put

$$(E_i)_{QP} = 0.$$

In the following, we consider the case that $\boldsymbol{\alpha}^{(i_0)} = \boldsymbol{\alpha}'^{(i_0)}$ for $i_0 \in \{0, 1/2, 1, \ldots, n-1/2\} \setminus \{i-1/2\}$.

If $\boldsymbol{\alpha}^{(i-1)}$ and $\boldsymbol{\alpha}^{(i)}$ are not labeled by the same Young diagram, then we put

$$(E_i)_{QP} = 0.$$

We consider the case $\boldsymbol{\alpha}^{(i-1)}$ and $\boldsymbol{\alpha}^{(i)}$ have the same label $\widetilde{\lambda}$. In this case, the possible vertices as $\boldsymbol{\alpha}^{(i-1/2)}$ have labels $\{\widetilde{\lambda}^-_{(s)}\}$, which are obtained by removing one box from $\widetilde{\lambda}$. Let $\{Q_s\}$ be the set of tableaux obtained from P by replacing $\boldsymbol{\alpha}^{(i-1/2)}$ with $\widetilde{\lambda}^-_{(s)}$.

Then we define $(E_i)_{QP}$ to be

$$(E_i)_{Q_sP} = \frac{h(\widetilde{\lambda})}{h(\widetilde{\lambda}^-_{(s)})}.$$

Here $h(\lambda)$ is the product of hook lengths defined by

$$h(\lambda) = \prod_{x\in\lambda} h_\lambda(x)$$

and $h_\lambda(x)$ is the *hook-length* at $x \in \lambda$.

Note that the matrix E_i is determined by the label $\widetilde{\lambda}$ itself not by the vertex at which the tableau P goes through. In other words, if another vertex in different level, say i', has the same label $\widetilde{\lambda}$, then $E_{i'}$ becomes the same matrix.

Let $v(\lambda^-_{(s)}, \lambda)$ be the standard vector which corresponds to a tableau whose $(i-1)$-st, $(i-1/2)$-th and i-th coordinate $(\boldsymbol{\alpha}^{(i-1)}, \boldsymbol{\alpha}^{(i-1/2)}, \boldsymbol{\alpha}^{(i)})$ are labeled by $(\lambda, \lambda^-_{(s)}, \lambda)$. Then for a tableau P which goes through λ at the $(i-1)$-st and the i-th coordinate of P, we have

$$\rho(e_i)(v_P) = \sum_{s'} \frac{h(\lambda)}{h(\lambda^-_{(s')})} v(\lambda^-_{(s')}, \lambda).$$

Here $\lambda^-_{(s')}$ runs through Young diagrams obtained from λ by removing one box.

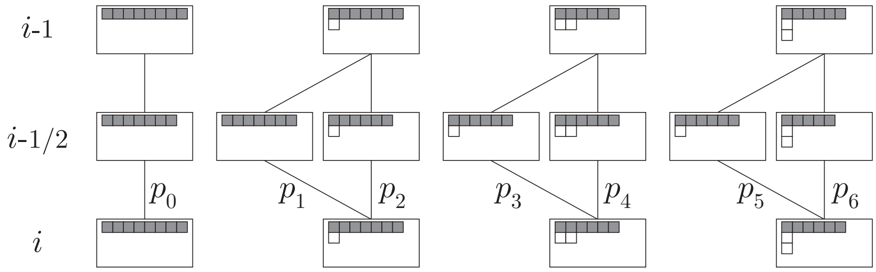

Figure 19: Representation spaces for $\rho(e_i)$

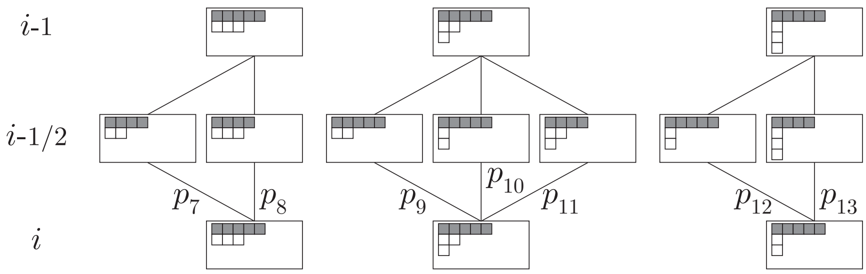

Figure 20: Representation spaces for $\rho(e_i)$

Example 7.1. Suppose that tableaux $\{p_r\}$ goes through paths in pictures illustrated in Figure 19 or 20. Then we have

$$
\begin{aligned}
\rho(e_i)(v_0) &= \frac{h(\widetilde{\emptyset})}{h(\widehat{\emptyset})} v_0 = Q v_0, \\
\rho(e_i)(v_1\ v_2) &= (v_1\ v_2) \begin{pmatrix} h(\widetilde{\square})/h(\widehat{\emptyset}) & h(\widetilde{\square})/h(\widehat{\emptyset}) \\ h(\widetilde{\square})/h(\widehat{\square}) & h(\widetilde{\square})/h(\widehat{\square}) \end{pmatrix} \\
&= (v_1\ v_2) \begin{pmatrix} \frac{Q}{Q-1} & \frac{Q}{Q-1} \\ \frac{Q(Q-2)}{Q-1} & \frac{Q(Q-2)}{Q-1} \end{pmatrix} \\
\rho(e_i)(v_3\ v_4) &= (v_3\ v_4) \begin{pmatrix} h(\widetilde{\square\square})/h(\widehat{\square}) & h(\widetilde{\square\square})/h(\widehat{\square}) \\ h(\widetilde{\square\square})/h(\widehat{\square\square}) & h(\widetilde{\square\square})/h(\widehat{\square\square}) \end{pmatrix}, \\
&= (v_3\ v_4) \begin{pmatrix} \frac{2(Q-2)}{Q-3} & \frac{2(Q-2)}{Q-3} \\ \frac{(Q-1)(Q-4)}{Q-3} & \frac{(Q-1)(Q-4)}{Q-3} \end{pmatrix} \\
\rho(e_i)(v_5\ v_6) &= (v_5\ v_6) \begin{pmatrix} h(\widetilde{\begin{smallmatrix}\square\\\square\end{smallmatrix}})/h(\widehat{\square}) & h(\widetilde{\begin{smallmatrix}\square\\\square\end{smallmatrix}})/h(\widehat{\square}) \\ h(\widetilde{\begin{smallmatrix}\square\\\square\end{smallmatrix}})/h(\widehat{\begin{smallmatrix}\square\\\square\end{smallmatrix}}) & h(\widetilde{\begin{smallmatrix}\square\\\square\end{smallmatrix}})/h(\widehat{\begin{smallmatrix}\square\\\square\end{smallmatrix}}) \end{pmatrix}, \\
&= (v_5\ v_6) \begin{pmatrix} \frac{2Q}{Q-1} & \frac{2Q}{Q-1} \\ \frac{Q(Q-3)}{Q-1} & \frac{Q(Q-3)}{Q-1} \end{pmatrix}.
\end{aligned}
$$

Here v_i is the standard vector which corresponds to p_i. Similarly for the bases $\langle v_7, v_8\rangle$, $\langle v_9, v_{10}, v_{11}\rangle$ and $\langle v_{12}, v_{13}\rangle$, we have the following matrices respectively:

$$
\begin{pmatrix} \frac{3(Q-4)}{Q-5} & \frac{3(Q-4)}{Q-5} \\ \frac{(Q-2)(Q-6)}{Q-5} & \frac{(Q-2)(Q-6)}{Q-5} \end{pmatrix},
$$

$$
\begin{pmatrix} \frac{3(Q-1)}{2(Q-2)} & \frac{3(Q-1)}{2(Q-2)} & \frac{3(Q-1)}{2(Q-2)} \\ \frac{3(Q-3)}{2(Q-4)} & \frac{3(Q-3)}{2(Q-4)} & \frac{3(Q-3)}{2(Q-4)} \\ \frac{(Q-1)(Q-3)(Q-5)}{(Q-2)(Q-4)} & \frac{(Q-1)(Q-3)(Q-5)}{(Q-2)(Q-4)} & \frac{(Q-1)(Q-3)(Q-5)}{(Q-2)(Q-4)} \end{pmatrix},
$$

$$
\begin{pmatrix} \frac{3Q}{Q-1} & \frac{3Q}{Q-1} \\ \frac{Q(Q-4)}{Q-1} & \frac{Q(Q-4)}{Q-1} \end{pmatrix}.
$$

7.2 Definition of $\rho_{\boldsymbol{\alpha}}(f_i)$

Next, we define a linear map for f_i.

For a tableaux $P = (\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1/2)}, \ldots, \boldsymbol{\alpha}^{(n)})$ of $\mathbb{T}(\boldsymbol{\alpha})$, we define $\rho_{\boldsymbol{\alpha}}(f_i)(v_P) = \sum_{Q\in\mathbb{T}(\boldsymbol{\alpha})} (F_i)_{QP} v_Q$. Let $Q = (\boldsymbol{\alpha}'^{(0)}, \boldsymbol{\alpha}'^{(1/2)}, \ldots, \boldsymbol{\alpha}'^{(n)})$.

If there is an $i_0 \in \{1/2, 1, \ldots, n-1/2\} \setminus \{i\}$ such that $\boldsymbol{\alpha}^{(i_0)} \neq \boldsymbol{\alpha}'^{(i_0)}$, then we put

$$(F_i)_{QP} = 0.$$

In the following, we consider the case that $\boldsymbol{\alpha}^{(i_0)} = \boldsymbol{\alpha}'^{(i_0)}$ for $i_0 \in \{0, 1/2, 1, \ldots, n-1/2\} \setminus \{i\}$.

If $\boldsymbol{\alpha}^{(i-1/2)}$ and $\boldsymbol{\alpha}^{(i+1/2)}$ are not labeled by the same Young diagram, then we put

$$(F_i)_{QP} = 0.$$

We consider the case $\boldsymbol{\alpha}^{(i-1/2)}$ and $\boldsymbol{\alpha}^{(i+1/2)}$ have the same label $\widehat{\mu}$. In this case, the possible vertices as $\boldsymbol{\alpha}^{(i)}$ have labels $\{\widehat{\mu}^{+}_{(r)}\}$, which are obtained by adding one box to $\widetilde{\mu}$. Suppose that $\boldsymbol{\alpha}^{(i)}$, the i-th coordinate of P, has its label $\widetilde{\mu}^{+}_{(r_0)}$. Let Q be a tableau obtained from P by replacing $\boldsymbol{\alpha}^{(i)}$ with one of $\{\widehat{\mu}^{+}_{(r)}\}$. Then we define $(F_i)_{QP}$ to be

$$(F_i)_{Q_r P} = \frac{h(\widehat{\mu})}{h(\widehat{\mu}^{+}_{(r_0)})}.$$

Let $v(\mu^{+}_{(r)}, \mu)$ be the standard vector which corresponds to a tableau whose $(i-1/2)$-th, i-th and $(i+1/2)$-th coordinate $(\boldsymbol{\alpha}^{(i-1/2)}, \boldsymbol{\alpha}^{(i)}, \boldsymbol{\alpha}^{(i+1/2)})$ are labeled by $(\mu, \mu^{+}_{(r)}, \mu)$. Then for a tableau P which goes through μ at the $(i-1/2)$-th and the $(i+1/2)$-th coordinate of P, we have

$$\rho(f_i)(v_P) = \sum_r \frac{h(\mu)}{h(\mu^{+}_{(r_0)})} v(\mu^{+}_{(r)}, \mu).$$

Here $\mu^{+}_{(r)}$ runs through Young diagrams obtained from μ by adding one box.

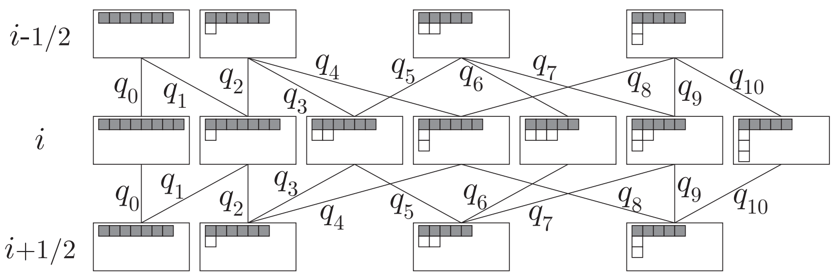

Figure 21: Representation spaces for $\rho(f_i)$

Example 7.2. Suppose that tableau $\{q_r\}$ go through paths in the picture illustrated in Figure 21. Then we have

$$\rho(f_i)(v_0\ v_1) = (v_0\ v_1)\begin{pmatrix} h(\widehat{\emptyset})/h(\widetilde{\emptyset}) & h(\widehat{\emptyset})/h(\widetilde{\square}) \\ h(\widehat{\emptyset})/h(\widetilde{\emptyset}) & h(\widehat{\emptyset})/h(\widetilde{\square}) \end{pmatrix} = (v_0\ v_1)\begin{pmatrix} \frac{1}{Q} & \frac{Q-1}{Q} \\ \frac{1}{Q} & \frac{Q-1}{Q} \end{pmatrix}$$

and

$$
\begin{aligned}
\rho(f_i)(v_2\ v_3\ v_4) &= (v_2\ v_3\ v_4) \begin{pmatrix} h(\widehat{\square})/h(\widetilde{\square}) & h(\widehat{\square})/h(\widetilde{\square\square}) & h(\widehat{\square})/h(\widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}) \\ h(\widehat{\square})/h(\widetilde{\square}) & h(\widehat{\square})/h(\widetilde{\square\square}) & h(\widehat{\square})/h(\widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}) \\ h(\widehat{\square})/h(\widetilde{\square}) & h(\widehat{\square})/h(\widetilde{\square\square}) & h(\widehat{\square})/h(\widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}) \end{pmatrix} \\
&= (v_2\ v_3\ v_4) \begin{pmatrix} \frac{Q-1}{Q(Q-2)} & \frac{Q-3}{2(Q-2)} & \frac{Q-1}{2Q} \\ \frac{Q-1}{Q(Q-2)} & \frac{Q-3}{2(Q-2)} & \frac{Q-1}{2Q} \\ \frac{Q-1}{Q(Q-2)} & \frac{Q-3}{2(Q-2)} & \frac{Q-1}{2Q} \end{pmatrix}.
\end{aligned}
$$

Here v_i is the standard vector which corresponds to q_i. Similarly, for the bases $\langle v_5, v_6, v_7\rangle$ and $\langle v_8, v_9, v_{10}\rangle$ we have the following matrices respectively:

$$
\begin{pmatrix} \frac{Q-3}{(Q-1)(Q-4)} & \frac{Q-5}{3(Q-4)} & \frac{2(Q-2)}{3(Q-1)} \\ \frac{Q-3}{(Q-1)(Q-4)} & \frac{Q-5}{3(Q-4)} & \frac{2(Q-2)}{3(Q-1)} \\ \frac{Q-3}{(Q-1)(Q-4)} & \frac{Q-5}{3(Q-4)} & \frac{2(Q-2)}{3(Q-1)} \end{pmatrix}, \quad \begin{pmatrix} \frac{Q-1}{Q(Q-3)} & \frac{2(Q-4)}{3(Q-3)} & \frac{Q-1}{3Q} \\ \frac{Q-1}{Q(Q-3)} & \frac{2(Q-4)}{3(Q-3)} & \frac{Q-1}{3Q} \\ \frac{Q-1}{Q(Q-3)} & \frac{2(Q-4)}{3(Q-3)} & \frac{Q-1}{3Q} \end{pmatrix}.
$$

7.3 Definition of $\rho_{\boldsymbol{\alpha}}(s_i)$

Finally, we define linear maps for s_i. Unfortunately, we do not have uniform description for $\rho_{\boldsymbol{\alpha}}(s_i)$, except for "non-reductive" paths. So first we define $\rho_{\boldsymbol{\alpha}}(s_i)$ for the non-reductive paths. Then we define $\rho_{\boldsymbol{\alpha}}(s_1)$ and $\rho_{\boldsymbol{\alpha}}(s_2)$ for "reductive" paths one by one.

Non-Reductive Case

In the following, we use notation $\mu \lhd \lambda$ if a Young diagram λ is obtained from a Young diagram μ by adding one box.

For $1 \leq j \leq i$, let ν, μ, λ be Young diagrams of size $j-1$, j and $j+1$ respectively such that $\nu \lhd \mu \lhd \lambda$. If a tableau P of $\mathbb{T}(\boldsymbol{\alpha})$ goes through $\widetilde{\nu}$, $\widetilde{\mu}$ and $\widetilde{\lambda}$ at the $(i-2)$-nd, the $(i-1)$-st and the i-th coordinate, then P goes through $\widehat{\nu}$ and $\widehat{\mu}$ at the $(i-3/2)$-th and the $(i-1/2)$-th coordinate. We call such a tableau *non-reductive* at i. If a tableau P does not satisfy the property above, then we call P *reductive* at i.

Recall that if $\nu \lhd \mu \lhd \lambda$, then we can define the *axial distance* $d = d(\nu, \mu, \lambda)$. Namely, if μ differs from ν in the r_0-th row and the c_0-th column only, and λ differs from μ in the r_1-th row and the c_1-th column only, then $d = d(\nu, \mu, \lambda)$ is defined by

$$
d = d(\nu, \mu, \lambda) = (c_1 - r_1) - (c_0 - r_0) = \begin{cases} h_\lambda(r_1, c_0) - 1 & \text{if } r_0 \geq r_1, \\ 1 - h_\lambda(r_0, c_1) & \text{if } r_0 < r_1. \end{cases}
$$

Here $h_\lambda(i,j)$ is the *hook-length* at (i,j) in λ.

If $|d| \geq 2$, then there is a unique Young diagram $\mu' \neq \mu$ which satisfies $\nu \lhd \mu' \lhd \lambda$. Let P' be a tableau of shape $\boldsymbol{\alpha}$ which are obtained from P by replacing

$(i-1)$-st and $(i-1/2)$-th coordinates of P with $\widetilde{\mu'}$ and $\widehat{\mu'}$ respectively. For the standard vectors v_P and $v_{P'}$ which correspond to P and P', we define the linear map $\rho_{\boldsymbol{\alpha}}(s_i)$ as follows:

$$\rho_{\boldsymbol{\alpha}}(s_i)\ :\ (v_P, v_{P'}) \longmapsto (v_P, v_{P'}) \begin{pmatrix} 1/d & \left(1-1/d^2\right)/c \\ c & -1/d \end{pmatrix}, \tag{2}$$

where we can arbitrarily chose $c \in K_0 \setminus \{0\}$. If we put

$$a_d = 1/d \quad \text{and} \quad b_d = 1 - a_d^2, \tag{3}$$

then the matrix in the expression (2) is written as follows:

$$\begin{pmatrix} a_d & b_d/c \\ c & -a_d \end{pmatrix}.$$

If $|d_1| = 1$, then there does not exist a distinct Young diagram μ' which satisfies $\nu \rhd \mu' \rhd \lambda$ other than μ. In this case, we define $\rho_{\boldsymbol{\alpha}}(s_i)$ to be

$$\rho_{\boldsymbol{\alpha}}(s_i)\ :\ v_P \longmapsto a_d v_P.$$

Here a_d is the one defined by (3).

EXAMPLE 7.3. Suppose that a tableau p_1 of $\mathbb{T}(\boldsymbol{\alpha})$ goes through $\widetilde{\emptyset}$, $\widetilde{\square}$ and $\widetilde{\square\square}$ at the 0-th, the 1-st and the 2-nd coordinates respectively, then for the standard vector u_1 which corresponds to p_1 we have

$$\rho_{\boldsymbol{\alpha}}(s_1)u_1 = u_1.$$

For the standard vector v_2 which corresponds to p_2, a tableau of $\mathbb{T}(\boldsymbol{\alpha})$ which goes through $\widetilde{\emptyset}$, $\widetilde{\square}$ and $\widetilde{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ at the 0-th, the 1-st and the 2-nd coordinates respectively, we have

$$\rho_{\boldsymbol{\alpha}}(s_1)u_2 = -u_2.$$

EXAMPLE 7.4. Let $\lambda^{(1)} = (3)$, $\lambda^{(2)} = (2,1)$ and $\lambda^{(3)} = (1,1,1)$ be partitions of 3. Suppose that tableaux q_1 and q_2 of $\mathbb{T}(\boldsymbol{\alpha})$ both go through $\widetilde{\square}$ and $\widetilde{\square\square}$ at the 1-st and the 2-nd coordinates respectively, and tableaux q_3 and q_4 of $\mathbb{T}(\boldsymbol{\alpha})$ both go through $\widetilde{\square}$ and $\widetilde{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ at the 1-st and the 2-nd coordinates respectively. Further, the tableaux q_1, q_2, q_3 and q_4 go through $\widetilde{\lambda^{(1)}}$, $\widetilde{\lambda^{(2)}}$, $\widetilde{\lambda^{(2)}}$ and $\widetilde{\lambda^{(3)}}$ at the 3-rd coordinates respectively. Then we have

$$\rho_{\boldsymbol{\alpha}}(s_2)(v_1\ v_2\ v_3\ v_4) = (v_1\ v_2\ v_3\ v_4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/2 & 3/(4c) & 0 \\ 0 & c & 1/2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Here v_i is the standard vector which corresponds to q_i.

Reductive Case

Consider the case a tableau P is reductive at i. So far, we do not have uniform description for $\rho_{\boldsymbol{\alpha}}(s_i)$. So we define $\rho_{\boldsymbol{\alpha}}(s_1)$ and $\rho_{\boldsymbol{\alpha}}(s_2)$ one by one.

First we define $\rho_{\boldsymbol{\alpha}}(s_1)$. For tableaux p_1 and p_2 of $\mathbb{T}(\boldsymbol{\alpha})$ which go through $(\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\emptyset})$ and $(\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\emptyset})$ at the 0-th, the $1-\frac{1}{2}$-th, the 1-st, the $2-\frac{1}{2}$-th and the 2-nd coordinate respectively, let u_1 and u_2 be the corresponding standard vectors. Then we define $\rho_{\boldsymbol{\alpha}}(s_1)(u_1\ u_2)$ by

$$\rho_{\boldsymbol{\alpha}}(s_1)(u_1\ u_2) = (u_1\ u_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For tableaux p_3, p_4 and p_5 of $\mathbb{T}(\boldsymbol{\alpha})$ which go through $(\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square})$, $(\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\square})$ and $(\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\square}, \widetilde{\square})$ at the 0-th, the $1-\frac{1}{2}$-th, the 1-st, the $2-\frac{1}{2}$-th and the 2-nd coordinate respectively, let u_3, u_4 and u_5 be the corresponding standard vectors. Then we define $\rho_{\boldsymbol{\alpha}}(s_1)(u_1\ u_2\ u_3)$ by

$$\rho_{\boldsymbol{\alpha}}(s_1)(u_1\ u_2\ u_3) = (u_1\ u_2\ u_3)\begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{Q-1} & \frac{Q-2}{Q-1} & \frac{-1}{Q-1} \\ \frac{Q-2}{Q-1} & -\frac{Q-2}{Q-1} & \frac{1}{Q-1} \end{pmatrix}.$$

Next we define $\rho_{\boldsymbol{\alpha}}(s_2)$. In the following, we write

$$p = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}, \lambda^{(5)})$$

to mean the tableau p goes through $\lambda^{(1)}$, $\lambda^{(2)}$, $\lambda^{(3)}$, $\lambda^{(4)}$, $\lambda^{(5)}$ at the 1-st, the $(2-\frac{1}{2})$-th, the 2-nd, the $(3-\frac{1}{2})$-th and the 3-rd coordinates respectively.

Suppose that

$$\begin{array}{rcl} q_1 & = & (\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\emptyset}), \\ q_2 & = & (\widetilde{\square}, \widehat{\emptyset}, \widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\emptyset}), \\ q_3 & = & (\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\emptyset}), \end{array} \qquad \begin{array}{rcl} q_4 & = & (\widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\emptyset}), \\ q_5 & = & (\widetilde{\square}, \widehat{\square}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\emptyset}). \end{array}$$

Then for the standard vectors $(v_j)_{j=1}^5$ which correspond to $(q_j)_{j=1}^5$ we define $\rho_{\boldsymbol{\alpha}}(s_2)(v_1\ v_2\ v_3\ v_4\ v_5)$ by

$$\rho_{\boldsymbol{\alpha}}(s_2)(v_1\ v_2\ v_3\ v_4\ v_5) = (v_1\ v_2\ v_3\ v_4\ v_5)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{Q-1} & 0 & \frac{Q-2}{Q-1} & \frac{-1}{Q-1} \\ 0 & \frac{Q-2}{Q-1} & 0 & -\frac{Q-2}{Q-1} & \frac{1}{Q-1} \end{pmatrix}.$$

Assume that

$$\begin{array}{rcl} q_6 & = & (\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}), \\ q_7 & = & (\widetilde{\square}, \widehat{\emptyset}, \widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}), \\ q_8 & = & (\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}), \\ q_9 & = & (\widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}), \\ q_{10} & = & (\widetilde{\square}, \widehat{\square}, \widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}), \end{array} \qquad \begin{array}{rcl} q_{11} & = & (\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\square}, \widetilde{\square}), \\ q_{12} & = & (\widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\square}, \widetilde{\square}), \\ q_{13} & = & (\widetilde{\square}, \widehat{\square}, \widetilde{\square}, \widehat{\square}, \widetilde{\square}), \\ q_{14} & = & (\widetilde{\square}, \widehat{\square}, \widetilde{\square\square}, \widehat{\square}, \widetilde{\square}), \\ q_{15} & = & (\widetilde{\square}, \widehat{\square}, \widetilde{\substack{\square \\ \square}}, \widehat{\square}, \widetilde{\square}). \end{array}$$

Then for the standard vectors $(v_j)_{j=6}^{15}$ which correspond to $(q_j)_{j=6}^{15}$ we define $\rho_{\boldsymbol{\alpha}}(s_2)(v_6\ v_8\ v_{11})$ and $\rho_{\boldsymbol{\alpha}}(s_2)(v_7\ v_9\ v_{10}\ v_{12}\ v_{13}\ v_{14}\ v_{15})$ by

$$\rho_{\boldsymbol{\alpha}}(s_2)(v_6\ v_8\ v_{11}) = (v_6\ v_8\ v_{11}) \begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{(Q-1)} & \frac{Q-2}{Q-1} & \frac{-1}{(Q-1)} \\ \frac{Q-2}{Q-1} & -\frac{Q-2}{Q-1} & \frac{1}{Q-1} \end{pmatrix}$$

and

$$\rho_{\boldsymbol{\alpha}}(s_2)(v_7\ v_9\ v_{10}\ v_{12}\ v_{13}\ v_{14}\ v_{15}) = (v_7\ v_9\ v_{10}\ v_{12}\ v_{13}\ v_{14}\ v_{15})M_i$$

Here the matrix M_i is

$$\begin{pmatrix} \frac{1}{Q-1} & \frac{Q-2}{Q-1} & \frac{-1}{Q-1} & \frac{-1}{Q-1} & \frac{1}{(Q-1)(Q-2)} & \frac{(Q-1)(Q-2)-2}{2(Q-1)(Q-2)} & -1/2 \\ \frac{Q-2}{(Q-1)^2} & \frac{Q^2-3Q+3}{(Q-1)^2} & \frac{1}{(Q-1)^2} & \frac{1}{(Q-1)^2} & \frac{-1}{(Q-1)^2(Q-2)} & \frac{-Q(Q-3)}{2(Q-1)^2(Q-2)} & \frac{1}{2(Q-1)} \\ \frac{-(Q-2)}{(Q-1)^2} & \frac{Q-2}{(Q-1)^2} & \frac{-1}{(Q-1)^2} & \frac{Q(Q-2)}{(Q-1)^2} & \frac{1}{(Q-1)^2(Q-2)} & \frac{Q(Q-3)}{2(Q-1)^2(Q-2)} & \frac{-1}{2(Q-1)} \\ \frac{-(Q-2)}{(Q-1)^2} & \frac{Q-2}{(Q-1)^2} & \frac{Q(Q-2)}{(Q-1)^2} & \frac{-1}{(Q-1)^2} & \frac{1}{(Q-1)^2(Q-2)} & \frac{Q(Q-3)}{2(Q-1)^2(Q-2)} & \frac{-1}{2(Q-1)} \\ \frac{Q-2}{(Q-1)^2} & \frac{-(Q-2)}{(Q-1)^2} & \frac{1}{(Q-1)^2} & \frac{1}{(Q-1)^2} & \frac{(Q-1)^2(Q-2)-1}{(Q-1)^2(Q-2)} & \frac{-Q(Q-3)}{2(Q-1)^2(Q-2)} & \frac{1}{2(Q-1)} \\ \frac{Q-2}{Q-1} & \frac{-(Q-2)}{Q-1} & \frac{1}{Q-1} & \frac{1}{Q-1} & \frac{-1}{(Q-1)(Q-2)} & \frac{Q^2-3Q+4}{2(Q-1)(Q-2)} & 1/2 \\ \frac{-(Q-2)}{Q-1} & \frac{Q-2}{Q-1} & \frac{-1}{Q-1} & \frac{-1}{Q-1} & \frac{1}{(Q-1)(Q-2)} & \frac{Q(Q-3)}{2(Q-1)(Q-2)} & 1/2 \end{pmatrix}.$$

Next assume that

$$\begin{aligned} q_{16} &= (\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\square}, \widetilde{\square\!\square}), & q_{19} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\square\!\square}, \widehat{\square}, \widetilde{\square\!\square}), \\ q_{17} &= (\widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\square}, \widetilde{\square\!\square}), & q_{20} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, \widehat{\square}, \widetilde{\square\!\square}), \\ q_{18} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\square}, \widehat{\square}, \widetilde{\square\!\square}), & q_{21} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\square\!\square}, \widehat{\square\!\square}, \widetilde{\square\!\square}). \end{aligned}$$

Then for the standard vectors $(v_j)_{j=16}^{21}$ which correspond to $(q_j)_{j=16}^{21}$ we define $\rho_{\boldsymbol{\alpha}}(s_2)(v_{16}\ v_{17}\ v_{18}\ v_{19}\ v_{20}\ v_{21})$ by

$$\rho_{\boldsymbol{\alpha}}(s_2)(v_{16}\ v_{17}\ v_{18}\ v_{19}\ v_{20}\ v_{21}) = (v_{16}\ v_{17}\ v_{18}\ v_{19}\ v_{20}\ v_{21})M_i.$$

Here the matrix M_i is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{(Q-1)} & \frac{1}{(Q-1)(Q-2)} & \frac{Q(Q-3)}{2(Q-1)(Q-2)} & -1/2 & 0 \\ 0 & \frac{1}{(Q-1)} & \frac{-1}{(Q-1)(Q-2)} & \frac{Q^2-3Q+4}{2(Q-1)(Q-2)} & 1/2 & 1 \\ 0 & 1 & \frac{Q^2-3Q+4}{Q(Q-3)(Q-2)} & \frac{(Q-1)(Q-4)}{2(Q-2)(Q-3)} & \frac{(Q-1)(Q-4)}{2Q(Q-3)} & \frac{-1}{(Q-3)} \\ 0 & -1 & \frac{1}{(Q-2)} & \frac{Q-4}{2(Q-2)} & 1/2 & \frac{-1}{(Q-1)} \\ 0 & 0 & \frac{(Q-1)^2(Q-4)}{Q(Q-3)(Q-2)} & \frac{-(Q-1)(Q-4)}{2(Q-2)(Q-3)} & \frac{-(Q-1)(Q-4)}{2Q(Q-3)} & \frac{1}{(Q-3)} \end{pmatrix}.$$

Finally assume that

$$
\begin{aligned}
q_{22} &= (\widetilde{\emptyset}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\square}, \widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}), & q_{25} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\square\square}, \widehat{\square}, \widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}),\\
q_{23} &= (\widetilde{\square}, \widehat{\emptyset}, \widetilde{\square}, \widehat{\square}, \widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}), & q_{26} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}, \widehat{\square}, \widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}),\\
q_{24} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\square}, \widehat{\square}, \widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}), & q_{27} &= (\widetilde{\square}, \widehat{\square}, \widetilde{\square\square}, \widehat{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}, \widetilde{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}).
\end{aligned}
$$

Then for the standard vectors $(v_j)_{j=22}^{27}$ which correspond to $(q_j)_{j=22}^{27}$ we define $\rho_{\boldsymbol{\alpha}}(s_2)(v_{22}\ v_{23}\ v_{24}\ v_{25}\ v_{26}\ v_{27})$ by

$$\rho_{\boldsymbol{\alpha}}(s_2)(v_{22}\ v_{23}\ v_{24}\ v_{25}\ v_{26}\ v_{27}) = (v_{22}\ v_{23}\ v_{24}\ v_{25}\ v_{26}\ v_{27})M_i.$$

Here the matrix M_i is

$$
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{(Q-1)} & \frac{-1}{(Q-1)(Q-2)} & \frac{-Q(Q-3)}{2(Q-1)(Q-2)} & 1/2 & 0\\
0 & \frac{-1}{(Q-1)} & \frac{1}{(Q-1)(Q-2)} & \frac{Q(Q-3)}{2(Q-1)(Q-2)} & 1/2 & 1\\
0 & -1 & \frac{1}{(Q-2)} & \frac{Q-4}{2(Q-2)} & 1/2 & \frac{-1}{(Q-3)}\\
0 & 1 & \frac{1}{(Q-2)} & \frac{Q(Q-3)}{2(Q-1)(Q-2)} & \frac{Q-3}{2(Q-1)} & \frac{-1}{(Q-1)}\\
0 & 0 & \frac{Q-3}{Q-2} & \frac{-Q(Q-3)}{2(Q-1)(Q-2)} & \frac{-(Q-3)}{2(Q-1)} & \frac{1}{(Q-1)}
\end{pmatrix}.
$$

8 Discussion

In the previous section, we gave linear maps $\rho_{\boldsymbol{\alpha}}(e_i)$ and $\rho_{\boldsymbol{\alpha}}(f_i)$ for all the tableaux on Γ_n. and defined $\rho_{\boldsymbol{\alpha}}(s_i)$ for non-reductive tableaux on Γ_n. We also defined $\rho_{\boldsymbol{\alpha}}(s_1)$ and $\rho_{\boldsymbol{\alpha}}(s_2)$ for the reductive tableaux on Γ_n. (So far, we have further obtained $\rho_{\boldsymbol{\alpha}}(s_3)$ for almost all reductive tableaux on Γ_4.) These linear maps preserve the relations in Theorem 1.2 and Theorem 5.3. Hence they give representations of $A_n(Q)$ for all $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_n$ ($n = 2 - \frac{1}{2}, 2, 3 - \frac{1}{2}, 3, 4 - \frac{1}{2}$) and for almost all $\boldsymbol{\alpha} \in \boldsymbol{\Lambda}_4$.

These representations also coincide with the ones calculated through the Murphy's operators which are introduced in the paper [5] and programmed by Naruse. Moreover, the traces of the representation matrices above coincide with the "characters" which is defined by Naruse in the paper [17]. This means that the representations we have presented in this note will be irreducible and define Young's seminormal form representations of the partition algebras $A_n(Q)$.

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