Party Algebra of Type B and Construction of its Irreducible Representations

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Abstract

Suppose that there exist two parties each of which consists of n members. The parties hold meetings splitting into several small groups. Every group consists of even number of members. Some groups may consist of members of just one of the parties. The set of seat-plans of such meetings makes an algebra called *the party algebra of type B*. We show that the party algebra of type B is semisimple by constructing a complete set of irreducible representations.

Supposez que là existent deux parties chacune dont se compose de n membres. Les parties tiennent des réunions coupant en plusieurs petits groupes. Chaque groupe se compose d'un chiffre pair des membres. Quelques groupes peuvent se composer des membres juste d'un des parties. L'ensemble de siège-plans de telles réunions fait une algèbre appelée *l'algèbre de partie du type B*. Nous prouvons que l'algèbre de partie du type B est semisimple en construisant un ensemble complet avec des représentations irréductibles.

1 Introduction

In [3], the author talked about the party algebra $P_{n,\infty} = \mathcal{A}_n$ of type $\tilde{A}$, which was generated by the symmetric group $\mathfrak{S}_n$ together with one special element f. In the talk, he showed that $P_{n,\infty}$ is semisimple and all the irreducible components are indexed by the n-tuple of Young diagrams whose weight some is equal to n. The standard basis of the party algebra $P_{n,\infty}$ was geometrically understood by *seat-plans* of the meetings held by two parties each of which consists of n members. The algebra $P_{n,\infty}$ naturally becomes a subalgebra of the partition algebra $P_{n,1}(Q) = P_n(Q)$ defined by P. Martin in his papers [5, 6]. While the party algebra is isomorphic to the centralizer $\mathrm{End}_{G(1,1,k)}(V^{\otimes n}) = \mathrm{End}_{\mathfrak{S}_k}(V^{\otimes n})$ ($G(1,1,k)$ acts diagonally on $V^{\otimes n}$), the party algebra $P_{n,\infty}$ is isomorphic to the centralizer $\mathrm{End}_{G(r,1,k)}(V^{\otimes n})$ under the condition that $k \geq n$ and $r > n$.

In this talk, we define $P_{n,2}(Q)$ the party algebra of type B slightly changing the definition of the party algebra of type $\tilde{A}$. The standard words of $P_{n,2}(Q)$ will have one to one correspondences with *the seat-plans of type B* of size n for the meetings held by two parties each of which consists of n members under the new conditions (see Section 2). If Q is equal to a positive integer k, then we have a surjective homomorphism from the algebra $P_{n,2}(k)$ onto $\mathrm{End}_{G(2,1,k)}(V^{\otimes n})$. Moreover, if $k \geq n$, then the above homomorphism becomes injective. In particular, in this case we find that the algebra $P_{n,2}(k)$ is semisimple. We show that $P_{n,2}(Q)$ is also semisimple for any generic parameter Q by explicitly constructing a complete set of irreducible representations of the the algebra $P_{n,2}(k)$ and replacing k with Q.

Party algebra $P_{n,r}(Q)$ is also defined in terms of seat-plans, which is defined from the centralizer algebra of the unitary reflection group $G(r,1,k)$. This generalization is presented in Section 4.

Finally we consider the structure of $\mathrm{End}_{G(2,1,3)}V^{\otimes n}$ in Section 5. This algebra is a surjective image of $P_{n,2}(3)$ and its Bratteli diagram grows periodically in accordance with the growth of n. Our representation is also defined on this diagram. This indicates that $\mathrm{End}_{G(2,1,3)}V^{\otimes \infty}$ may give an example of subfactors.

2 Definition of the party algebra of type B

We consider the following situation. Let $D = \{d_1, d_2, \ldots, d_n\}$ and $R = \{r_1, r_2, \ldots, r_n\}$ be two sets each of which consists of n distinct elements such that $D \cap R = \emptyset$. We decompose $D \sqcup R$ into subsets $M_1, M_2, \ldots, M_n$ (some of M_js might be empty) so that they satisfy

$$D \sqcup R = \bigcup_{j=1,2,\ldots,n} M_j, \quad M_i \cap M_j = \emptyset \text{ if } i \neq j,$$
$$|M_i| \in \{0, 2, \ldots, 2n\} \text{ for } 1 \leq i \leq n.$$

We call such a partition into subsets a *type B seat-plan* of size n. Let $P(n)$ be a set of partitions of n. If we sort M_is so that they satisfy $|M_1| \geq |M_2| \geq \cdots \geq |M_n|$, then there exists a partition $\lambda \in P(n)$ such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (|M_1|/2, |M_2|/2, \ldots, |M_n|/2)$. The number of type B seat-plans of size n is

$$\sum_{\lambda \in P(n)} \left(\frac{(2n)!}{(2\lambda_1)!(2\lambda_2)! \cdots (2\lambda_n)!} \right)^2 \cdot \frac{1}{\alpha_1! \alpha_2! \cdots \alpha_n!}, \tag{1}$$

where $\alpha_i = |\{\lambda_k; \lambda_k = i\}|$.

A type B seat-plan of size n is figured as follows. Consider a rectangle with n marked points on the bottom and the same n on the top as in Figure 1. The n marked points on the bottom are labeled by $d_1, d_2, \ldots d_n$ from left to right. Similarly, the n marked points on the top is labeled by $r_1, r_2, \ldots, r_n$ from left to right. If $D \sqcup R$ is divided into non-empty m subsets, then put m shaded circles in the middle of the rectangle so that they have no intersections. Each of the circles corresponds to one of the non-empty M_js. Then we join the $2n$ marked points and the m circles with $2n$ shaded bands so that the marked points

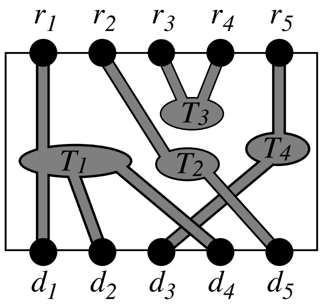

Figure 1: A seat-plan of type B

labeled by the elements of M_j are connected to the corresponding circle with $|M_j|$ bands.

Now we define the product w_1w_2 between two of rectangles w_1, w_2 (each of which corresponds to a seat-plan) by placing w_1 on w_2, gluing the corresponding boundaries and shrinking half along the vertical axis as in Figure 2. We then

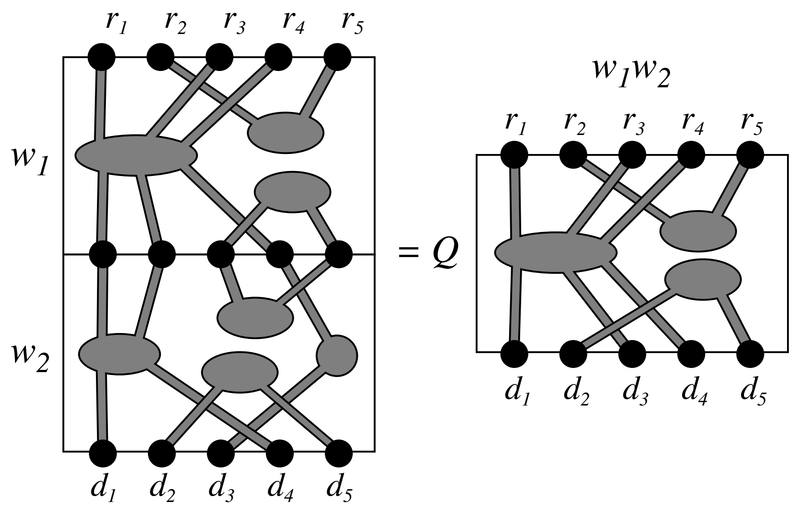

Figure 2: The product of seat-plans

have a new diagram possibly containing some shaded islands. If there p shaded islands occur in the product, first remove holes in the islands (if they exist) and then multiply the resulting diagram by Q^p removing the p islands. It is easy to define this product in terms of seat-plans. By this definition, a set of linear combinations of seat-plans of size n over $\mathbb{C}$ makes an algebra $P_{n,2}(Q)$. We call it the *party algebra of type B*. We put $P_{0,2}(Q) = P_{1,2}(Q) = \mathbb{C}$.

According to the paper [11], the generators of $P_{n,2}(Q)$ is afforded by the seat-plans illustrated in Figure 3. We further have the following proposition.

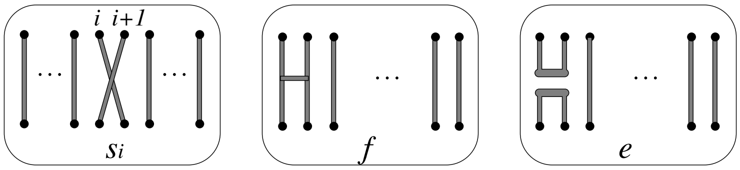

Figure 3: Generators

Proposition 1. *For an integer $n > 1$, the party algebra $P_{n,2}(Q)$ is characterized by the following generators and relations:*

$$
\begin{array}{rl}
\textit{generators}; & s_1, s_2, \ldots, s_{n-1}, f, e, \\
\textit{relations}; & s_i^2 = 1 \quad (1 \le i \le n-1), \\
& s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \le i \le n-2), \\
& s_i s_j = s_j s_i \quad (|i-j| \ge 2), \\
& e^2 = Qe, \quad f^2 = f, \\
& ef = fe = e, \quad es_1 = s_1 e = e, \quad fs_1 = s_1 f = f, \\
& es_i = s_i e, \quad fs_i = s_i f \quad (i \ge 3), \\
& es_2 e = e, \quad fs_2 fs_2 = s_2 fs_2 f, \quad fs_2 es_2 f = fs_2 f, \\
& xs_2 s_1 s_3 s_2 y s_2 s_1 s_3 s_2 = s_2 s_1 s_3 s_2 y s_2 s_1 s_3 s_2 x \quad (x, y \in \{e, f\}).
\end{array}
$$

Since we have one to one correspondences between the set of *standard words* of the generators above and the set of type B seat-plans, the equation (1) expresses the upper bound of the dimension of the algebra $P_{n,2}(Q)$.

Tanabe also showed the following proposition [11].

Proposition 2. *(Tanabe [11, Theorem 3.1]) Let $G(2,1,k)$ be the group of all the monomial matrices of size n whose non-zero entries are plus or minus one. Let V be a vector space of dimension k with the basis elements $e_1, e_2, \ldots, e_k$ on which $G(2,1,k)$ acts naturally. Let ϕ be the representation of the symmetric group $\mathfrak{S}_n$ on $V^{\otimes n}$ obtained by permuting the tensor product factors, i.e., for $v_1, v_2, \ldots, v_n \in V$ and for $w \in \mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$,*

$$\phi(w)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) := v_{w^{-1}(1)} \otimes v_{w^{-1}(2)} \otimes \cdots \otimes v_{w^{-1}(n)}.$$

Define further $\phi(e)$ and $\phi(f)$ as follows:

$$
\begin{aligned}
\phi(e)(e_{p_1} \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_n}) &:= \delta_{p_1,p_2} \sum_{j=1}^{k} e_j \otimes e_j \otimes e_{p_3} \otimes \cdots \otimes e_{p_n} \\
\phi(f)(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}) &:= \delta_{p_1,p_2} e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}
\end{aligned}
$$

Then $\mathrm{End}_{G(2,1,k)}(V^{\otimes n})$ *is generated by* $\phi(\mathfrak{S}_n)$*,* $\phi(e)$ *and* $\phi(f)$*, and* ϕ *defines a homomorphism from* $P_{n,2}(k)$ *to* $\mathrm{End}_{G(2,1,k)}(V^{\otimes n})$*.*

If $k \geq n$, then we can show that the above ϕ is injective. This implies that if Q is a positive integer k such that $k > n$ then we find that $P_{n,2}(Q)$ is semisimple.

In the following section, we construct a complete set of irreducible representations of $P_{n,2}(Q)$ for any generic value $Q \in \mathbb{C}$ extending the orthogonal representations of the symmetric group $\mathfrak{S}_n$. In particular, $P_{n,2}(Q)$ becomes semisimple if the parameter $Q \in \mathbb{C}$ is generic.

3 Construction

Fix an positive integer $k \geq n$. We define representations of $P_{n,2}(k)$ which turn out to become a complete set of irreducible representations. The representations are constructed on the Bratteli diagram for the sequence $P_{0,2}(k) \subset P_{1,2}(k) \subset \cdots \subset P_{n,2}(k)$ as in Figure 4. (See for example the papers [1, 7, 12, 13].)

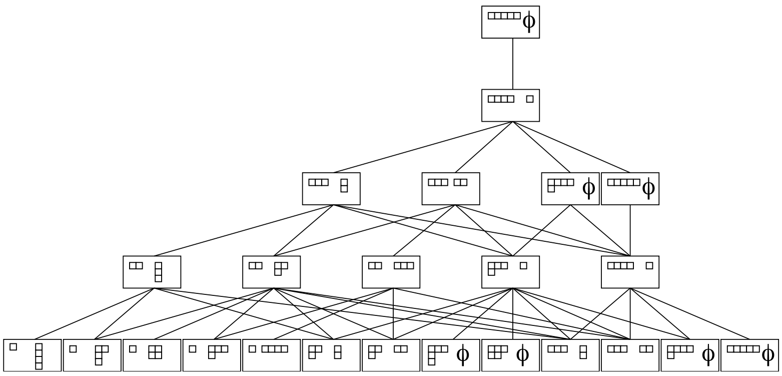

Figure 4: The Bratteli diagram for $P_{4,2}(5)$

Let $\boldsymbol{\beta} = [\alpha, \beta] = [(\alpha_1, \alpha_2, \ldots), (\beta_1, \beta_2, \ldots)]$ be an 2-tuple of Young diagrams. The 1-st [resp. 2-nd] coordinate of the tuple is referred to *the left [resp. right] board*. We consider the following sets:

$$\Lambda_k^B(2i) = \bigcup_{j=0}^{i} \{[\alpha, \beta] \ ; \ |\alpha| = k - 2j, \ \alpha_1 \geq k - i - j, \ |\beta| = 2j\}, \quad (2)$$

$$\Lambda_k^B(2i+1) = \bigcup_{j=0}^{i} \{[\alpha, \beta] \ ; \ |\alpha| = k - 2j - 1, \ \alpha_1 \geq k - i - j - 1, \ |\beta| = 2j + 1\}. \quad (3)$$

Note that $\Lambda_k^B(0) = \{[(k), \emptyset]\}$. Let $\boldsymbol{\beta} \underset{1}{\prec} \tilde{\boldsymbol{\beta}}$ or $\tilde{\boldsymbol{\beta}} \underset{1}{\succ} \boldsymbol{\beta}$ denote that $\tilde{\boldsymbol{\beta}}$ is obtained from $\boldsymbol{\beta}$ by removing one box from the Young diagram on the left board and adding the box to the Young diagram on the right board, or removing one box from the Young diagram on the right board and adding the box to the Young diagram on the left board. We also note that if $\boldsymbol{\beta} \in \Lambda_k^B(m)$, then $\tilde{\boldsymbol{\beta}} \in \Lambda_k^B(m+1)$.

For $\boldsymbol{\beta} \in \Lambda(n)$, a *tableaux* $\mathbb{T}(\boldsymbol{\beta})$ *of shape* $\boldsymbol{\beta}$ is defined by

$$\begin{aligned}\mathbb{T}(\boldsymbol{\beta}) \quad = \quad & \{P = (\boldsymbol{\beta}^{(0)}, \boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(n)}); \boldsymbol{\beta}^{(0)} = [(k), \emptyset] \in \Lambda_k^B(0), \boldsymbol{\beta} \in \Lambda_k^B(n) \\ & \boldsymbol{\beta}^{(i)} \underset{1}{\prec} \boldsymbol{\beta}^{(i+1)} \text{ for } 0 \leq i \leq n-1\}.\end{aligned}$$

Let $V(\boldsymbol{\beta}) = \oplus_{P \in \mathbb{T}(\boldsymbol{\beta})} \mathbb{C} v_P$ be a vector space over $\mathbb{C}$ with the standard basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\beta})\}$.

For a generator s_i of $P_{n,2}(Q)$, we define a linear map on $V(\boldsymbol{\beta})$ giving a matrix $\mathcal{B}_i$ with respect to the basis $\{v_P | P \in \mathbb{T}(\boldsymbol{\beta})\}$. Namely, for a pair of tableaux $P = (\boldsymbol{\beta}^{(0)}, \boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(n)})$ and $Q = (\boldsymbol{\beta}'^{(0)}, \boldsymbol{\beta}'^{(1)}, \ldots, \boldsymbol{\beta}'^{(n)})$ of $\mathbb{T}(\boldsymbol{\beta})$ define $s_i v_P = \sum_{Q \in \mathbb{T}(\boldsymbol{\beta})} (\mathcal{B}_i)_{QP} v_Q$. If there is an $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$ such that $\boldsymbol{\beta}^{(i_0)} \neq \boldsymbol{\beta}'^{(i_0)}$, then we put

$$(\mathcal{B}_i)_{QP} = 0.$$

In the following, we consider the case that $\boldsymbol{\beta}^{(i_0)} = \boldsymbol{\beta}'^{(i_0)}$ for $i_0 \in \{1, 2, \ldots, n-1\} \setminus \{i\}$.

First, we consider the case $\boldsymbol{\beta}^{(i)}$ is obtained from $\boldsymbol{\beta}^{(i-1)}$ by moving a box in the Young diagram on the left [resp. right] board to the Young diagram on the other board and $\boldsymbol{\beta}^{(i+1)}$ is obtained from $\boldsymbol{\beta}^{(i)}$ by moving another box in the Young diagram again on the left [resp. right] board to the Young diagram on the other board. Denote the Young diagram on the left board of $\boldsymbol{\beta}^{(i-1)}$ [resp. $\boldsymbol{\beta}^{(i)}$, $\boldsymbol{\beta}^{(i+1)}$] by $\lambda^{(i-1)}$ [resp. $\lambda^{(i)}$, $\lambda^{(i+1)}$] and denote the Young diagram on the right board of $\boldsymbol{\beta}^{(i-1)}$ [resp. $\boldsymbol{\beta}^{(i)}$, $\boldsymbol{\beta}^{(i+1)}$] by $\mu^{(i-1)}$ [resp. $\mu^{(i)}$, $\mu^{(i+1)}$]. Let $\lambda' \underset{1}{\subset} \lambda$ or $\lambda \underset{1}{\supset} \lambda'$ denote that λ' is obtained from λ by removing one box. Recall that if $\nu \underset{1}{\subset} \mu \underset{1}{\subset} \lambda$, then we can define the *axial distance* $d = d(\nu, \mu, \lambda)$. Namely if μ differs from ν in its r_0-th row and c_0-th column only, and if λ differs from μ in its r_1-th row and c_1-th column only, then $d = d(\nu, \mu, \lambda)$ is defined by

$$d = d(\nu, \mu, \lambda) = (c_1 - r_1) - (c_0 - r_0) = \begin{cases} h_\lambda(r_1, c_0) - 1 & \text{if } r_0 \leq r_1, \\ 1 - h_\lambda(r_0, c_1) & \text{if } r_0 > r_1. \end{cases}$$

Here $h_\lambda(i, j)$ is the *hook-length* at (i, j) in λ and for $\lambda = (\lambda_1, \lambda_2, \ldots)$ the hook-length $h_\lambda(i, j)$ is defined by

$$h_\lambda(i, j) = \lambda_i - j + |\{\lambda_l; \lambda_l \geq j\}| - i + 1.$$

If $\lambda^{(i-1)} \underset{1}{\supset} \lambda^{(i)} \underset{1}{\supset} \lambda^{(i+1)}$, then $\mu^{(i-1)} \underset{1}{\subset} \mu^{(i)} \underset{1}{\subset} \mu^{(i+1)}$. Hence we can define the axial distance $d_1 = d(\lambda^{(i+1)}, \lambda^{(i)}, \lambda^{(i-1)})$ and $d_2 = d(\mu^{(i-1)}, \mu^{(i)}, \mu^{(i+1)})$. If $|d_1| \geq 2$ [resp. $|d_2| \geq 2$], then there is a unique Young diagram $\lambda' \neq \lambda$ [resp. $\mu' \neq \mu$] which satisfies $\lambda^{(i-1)} \underset{1}{\supset} \lambda' \underset{1}{\supset} \lambda^{(i+1)}$ [resp. $\mu^{(i-1)} \underset{1}{\subset} \mu' \underset{1}{\subset} \mu^{(i+1)}$]. Similarly, if $\lambda^{(i-1)} \underset{1}{\subset} \lambda^{(i)} \underset{1}{\subset} \lambda^{(i+1)}$, then $\mu^{(i-1)} \underset{1}{\supset} \mu^{(i)} \underset{1}{\supset} \mu^{(i+1)}$, and we can define the axial distance $d_1 = d(\lambda^{(i-1)}, \lambda^{(i)}, \lambda^{(i)})$ and $d_2 = d(\mu^{(i+1)}, \mu^{(i)}, \mu^{(i-1)})$. If $|d_1| \geq 2$ [resp. $|d_2| \geq 2$], then λ' [resp. $\mu^{(i-1)}$] is defined as before. Let Q_1, Q_2, Q_3 be tableaux of shape $\boldsymbol{\beta}$ which are obtained from P by replacing $\boldsymbol{\beta}^{(i)} = [\lambda^{(i)}, \mu^{(i)}]$ on the j-th and the $(j+1)$-st board of $\boldsymbol{\beta}^{(i)}$ with $[\lambda^{(i)}, \mu']$, $[\lambda', \mu^{(i)}]$, $[\lambda', \mu']$ respectively. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(v_P, v_{Q_1}, v_{Q_2}, v_{Q_3}) \longmapsto (v_P, v_{Q_1}, v_{Q_2}, v_{Q_3})\mathcal{B}_i,$$

where

$$\mathcal{B}_i = \begin{pmatrix} \frac{1}{d_1 d_2} & \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\frac{1}{d_1 d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} \\ \sqrt{\frac{d_1^2-1}{d_1^2}}\sqrt{\frac{d_2^2-1}{d_2^2}} & -\sqrt{\frac{d_1^2-1}{d_1^2}}\frac{1}{d_2} & -\frac{1}{d_1}\sqrt{\frac{d_2^2-1}{d_2^2}} & \frac{1}{d_1 d_2} \end{pmatrix}.$$

Second, we consider the case that the only left boards of $\boldsymbol{\beta}^{(i-1)}$ and $\boldsymbol{\beta}^{(i+1)}$ coincide. Suppose that $\boldsymbol{\beta}^{(i-1)} = [\lambda, \mu]$. Then we can write $\boldsymbol{\beta}^{(i+1)} = [\lambda, \mu']$ $(\mu \neq \mu')$. Let $\{\lambda^+_{(r)} | r = 1, 2, \ldots, b(\lambda)\}$ [resp. $\{\lambda^-_{(r')} | r' = 1, 2, \ldots, b(\lambda)'\}$] be the set of all the Young diagrams which satisfy $\lambda^+_{(r)} \underset{1}{\supset} \lambda$ [resp. $\lambda^-_{(r')} \underset{1}{\subset} \lambda$] and let $P_1, P_2, \ldots, P_{b(\lambda)}$ [resp. $Q_1, Q_2, \ldots, Q_{b(\lambda)'}$] be all the tableaux which are obtained from P by replacing $\boldsymbol{\beta}^{(i)}$ with $[\lambda^+_{(r)}, \mu \cap \mu']$ [resp. $[\lambda^-_{(r')}, \mu \cup \mu']$]. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$\begin{aligned} (\mathcal{B}_i)_{P_r, P_{r'}} &= \sqrt{\frac{h(\lambda)^2}{h(\lambda^+_{(r)})h(\lambda^+_{(r')})}}, \\ (\mathcal{B}_i)_{P_r, Q_{r'}} &= (\mathcal{B}_i)_{Q_{r'}, P_r} = \frac{1}{d(\lambda^-_{(r')}, \lambda, \lambda^+_{(r)})}\sqrt{\frac{h(\lambda)^2}{h(\lambda^-_{(r')})h(\lambda^+_{(r)})}}, \\ (\mathcal{B}_i)_{Q_r, Q_{r'}} &= 0. \end{aligned}$$

Here $h(\nu)$ is the product of all the hook-lengths in ν:

$$h(\nu) = \prod_{(i,j)\in\nu} h_\nu(i,j).$$

If $\boldsymbol{\beta}^{(i-1)} = [\lambda, \mu]$ and $\boldsymbol{\beta}^{(i+1)} = [\lambda', \mu]$, then the matrix $(\mathcal{B}_i)$ is similarly defined by replacing λ with μ in the argument above. For example, let

$$\begin{aligned} P_1 &= ([k,0], [k-1,1], [k-2,1^2], [1(k-2),1]), \\ P_2 &= ([k,0], [k-1,1], [k-2,2], [1(k-2),1]), \\ Q_1 &= ([k,0], [k-1,1], [1(k-1),0], [1(k-2),1]) \end{aligned}$$

be the tableaux of shape $[1(k-2),1]$. Then the matrix $\mathcal{B}_2$ with respect to this basis is

$$\begin{pmatrix} 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}.$$

Next, we consider the case $\boldsymbol{\beta}^{(i-1)} = \boldsymbol{\beta}^{(i+1)}$. We put $\boldsymbol{\beta}^{(i-1)} = \boldsymbol{\beta}^{(i+1)} = [\lambda, \mu]$. Let $\{\lambda^+_{(r)}\}$, $\{\lambda^-_{(r')}\}$, $\{\mu^+_{(s)}\}$ and $\{\mu^-_{(s')}\}$ be the sets of Young diagrams previously defined and let $\{Q_{r',s}\}$ and $\{P_{r,s'}\}$ be the sets of tableaux obtained from P by replacing $\boldsymbol{\beta}^{(i)}$ with $[\lambda^-_{(r')}, \mu^+_{(s)}]$ and $[\lambda^+_{(r)}, \mu^-_{(s')}]$ respectively. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(\mathcal{B}_i)_{P,P'} = \begin{cases} \frac{1}{d(\lambda^-_{(r')},\lambda,\lambda^+_{(r)})d(\mu^-_{(s')},\mu,\mu^+_{(s)})}\sqrt{\frac{h(\lambda)^2 h(\mu)^2}{h(\lambda^-_{(r')})h(\lambda^+_{(r)})h(\mu^-_{(s')})h(\mu^+_{(s)})}} & \\ \qquad \text{if } (P,P') = (P_{r,s'}, Q_{r',s}) \text{ or } (Q_{r',s}, P_{r,s'}), & \\ \sqrt{\frac{h(\lambda)^2}{h(\lambda^+_{(r)})h(\lambda^+_{(r')})}} & \text{if } (P,P') = (P_{r,s}, P_{r',s}), \\ \sqrt{\frac{h(\mu)^2}{h(\mu^+_{(s)})h(\mu^+_{(s')})}} & \text{if } (P,P') = (Q_{r,s}, Q_{r,s'}), \\ 0 & \text{otherwise.} \end{cases}$$

For example, let

$$\begin{aligned} Q_1 &= ([k,0],[k-1,1],[k-2,1^2],[k-1,1]), \\ Q_2 &= ([k,0],[k-1,1],[k-2,2],[k-1,1]), \\ P_1 &= ([k,0],[k-1,1],[1(k-1),0],[k-1,1]), \\ P_2 &= ([k,0],[k-1,1],[k,0],[k-1,1]) \end{aligned}$$

be the tableaux of shape $[k-1,1]$. Then the matrix $\mathcal{B}_2$ with respect to this basis is

$$\begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2k} & -\sqrt{k-1}/\sqrt{2k} \\ 1/2 & 1/2 & -1/\sqrt{2k} & \sqrt{k-1}/\sqrt{2k} \\ 1/\sqrt{2k} & -1/\sqrt{2k} & (k-1)/k & \sqrt{k-1}/k \\ -\sqrt{k-1}/\sqrt{2k} & \sqrt{k-1}/\sqrt{2k} & \sqrt{k-1}/k & 1/k \end{pmatrix}.$$

Finally, we consider the remaining cases. In these cases, we can put $\boldsymbol{\beta}^{(i-1)} = [\lambda, \mu]$ and $\boldsymbol{\beta}^{(i+1)} = [\lambda', \mu']$ ($\lambda \neq \lambda', \mu \neq \mu'$ and $|\lambda| = |\lambda'|, |\mu| = |\mu'|$). Then $\boldsymbol{\beta}^{(i)}$ must be of the form $[\lambda \cup \lambda', \mu \cap \mu']$ or $[\lambda \cap \lambda', \mu \cup \mu']$. If $\boldsymbol{\beta}^{(i)}$ if the former [resp. latter] one, then the tableau P' is obtained from P by replacing $\boldsymbol{\beta}^{(i)}$ with the latter [resp. former] one. For the basis elements given by the above tableaux, we define the linear map by the following matrix:

$$(v_P, v_{P'}) \longmapsto (v_P, v_Q)\mathcal{B}_i = (v_P, v_Q)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we have completed the preparation, we state the following main result.

Theorem 3. *Let $\boldsymbol{\beta} = [\alpha, \beta]$ be an ordered pair of Young diagrams. If $k \geq n$, then the following statements hold:*

(1) Define $\rho_{\boldsymbol{\beta}}$ as follows:

$$\begin{aligned}
\rho_{\boldsymbol{\beta}}(s_i)v_P &= \sum_{P' \in \mathbb{T}(\boldsymbol{\beta})} (\mathcal{B}_i)_{P'P} v_{P'}, \\
\rho_{\boldsymbol{\beta}}(f)v_P &= \begin{cases} v_P & \textit{if } \boldsymbol{\beta}^{(2)} = [(k), \emptyset] \textit{ or } [(k-1,1), \emptyset] \\ 0 & \textit{otherwise.} \end{cases} \\
\rho_{\boldsymbol{\beta}}(e)v_P &= \begin{cases} kv_P & \textit{if } \boldsymbol{\beta}^{(2)} = [(k), \emptyset] \\ 0 & \textit{otherwise.} \end{cases}
\end{aligned}$$

Then $(\rho_{\boldsymbol{\beta}}, V(\boldsymbol{\beta}))$ defines an irreducible representation of $P_{n,2}(k)$.

(2) For $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \Lambda_k^B(n)$, the irreducible representations $\rho_{\boldsymbol{\beta}}$ and $\rho_{\boldsymbol{\beta}'}$ of $P_{n,2}(k)$ are equivalent if and only if $\boldsymbol{\beta} = \boldsymbol{\beta}'$.

(3) Conversely, for any irreducible representation ρ of $P_{n,2}(k)$, there exists an $\boldsymbol{\beta} \in \Lambda_k^B(n)$ such that ρ and $\rho_{\boldsymbol{\beta}}$ are equivalent.

In the process of the construction of $\rho_{\boldsymbol{\beta}}$, even if we replace the positive integer k with an indeterminate Q, the matrix elements of $(\mathcal{B}_i)_{P,P'}$ are similarly defined. This means the theorem above is valid for any generic parameter Q. More over if $Q = k$ and $k \geq n$, then by the Schur-Weyl reciprocity, we find that the dimension of $P_{n,2}(k)$ is equal to the square sum of the degree of $\rho_{\boldsymbol{\beta}}$ and it is also equal to the number of the seat-plans of type B, which is presented by the expression (1). Since the degree of $\rho_{\boldsymbol{\beta}}$ does not vary even if we replace the positive integer k with the indeterminate Q, we obtain the following.

Theorem 4. *Let $\Lambda_k^B(n)$ be the set defined by (2) and (3). If $Q \notin \{0, 1, \ldots, n-1\}$, then the party algebra $P_{n,2}(Q)$ is semisimple and $\{\rho_{\boldsymbol{\beta}}; \boldsymbol{\beta} \in \Lambda_B(n)\}$ gives a complete representatives of irreducible representations of $P_{n,2}(Q)$.*

4 Party algebra $P_{n,r}(Q)$

The party algebra $P_{n,r}(Q)$ is defined from the centralizer algebra of the unitary reflection group $G(r, 1, k)$. In this section we explain how the party algebra $P_{n,r}(Q)$ is introduced from the unitary reflection group $G(r, 1, k)$. Although in the paper [11] Tanabe studied the centralizer of the unitary reflection group even for the type $G(r,p,k)$, in the following we consider only the case $p=1$.

The unitary reflection group $G(r,1,k)$ is the subgroup of $GL(k,\mathbb{C})$ generated by the set of all permutation matrices of size k and $\mathrm{diag}(\zeta,1,1,\ldots,1)$ where ζ is a primitive r-th root of unity. Let V be the vector space of dimension k and suppose that it has the standard basis $\{e_1,\ldots,e_k\}$. The unitary reflection group $G(r,1,k)$ acts on V naturally and it also acts on $V^{\otimes n}$ diagonally. For $X\in\mathrm{End}V^{\otimes n}$, we denote by $X^{f_1,\ldots,f_n}_{m_1,\ldots,m_n}$ the matrix coefficients of X with respect to the basis $\{e_{m_1}\otimes\cdots\otimes e_{m_n}\mid m_1,\ldots,m_n\in[k]\}$. Since we can write $G(r,1,k)=(\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_k$, in order to check whether X commutes with the action of $G(r,1,k)$ or not we first examine the following action in the tensor space. For $\sigma\in\mathfrak{S}_k$, we have

$$\sigma^{-1}X\sigma(e_{m_1}\otimes\cdots\otimes e_{m_n})=\sum_{f_1,\ldots,f_n\in[k]}X^{\sigma(f_1),\ldots,\sigma(f_n)}_{\sigma(m_1),\ldots,\sigma(m_n)}e_{f_1}\otimes\cdots\otimes e_{f_n}$$

Hence we have the basis of $\mathrm{End}_{\mathfrak{S}_k}V^{\otimes n}$

$$\left\{T_{\sim}\Big|\begin{array}{l}\sim\text{ is an equivalence relation on }\{1,\ldots 2n\}\\ \text{whose number of classes is less than or equal to }n\end{array}\right\},$$

where

$$(T_{\sim})^{m_{n+1},\ldots,m_{2n}}_{m_1,\ldots,m_n}:=\begin{cases}1 & \text{if } (m_i=m_j \text{ if and only if } i\sim j),\\ 0 & \text{otherwise.}\end{cases}$$

Here we set $m_{n+i}:=f_i$ $(1\le i\le n)$. Note that $\sim$ is zero if the number of classes for $\sim$ is more than k.

In addition to the argument above, considering the action of $\xi\in\mathbb{Z}/r\mathbb{Z}$ we find that the following equivalence relation becomes a basis of the centralizer.

Lemma 5. *Let Π_{2n} be the set of all the partitions of $[2n]$ into subsets. For $B=\{B_1,\ldots,B_k\}\in\Pi_{2n}$ (some of the parts may be empty), let $bot(B_i):=B_i\cap[n]$ and $top(B_i):=B_i\cap([2n]\setminus[n])$ $(1\le i\le k)$. Let*

$$\Pi_{2n}(r,1,k):=\{B=\{B_1,\ldots,B_k\}\ ;\ |top(B_i)|\equiv|bot(B_i)|(mod\ r)(1\le i\le k)\}.$$

Then $\{T_{\sim B}\ ;\ B\in\Pi_{2n}(r)\}$ is a basis of $\mathrm{End}_{G(r,1,k)}V^{\otimes n}$.

The set Σ_n of seat-plans of type $\tilde{A}$ is equivalent to the set $\Pi_{2n}(r,1,k)$ if $k\ge n$ and $r>n$. The set Σ^B_n of seat-plans of type B is equivalent to the case $r=2$ and $k\ge n$. In this way we can obtain a basis of the party algebra $P_{n,r}(k)$ and its geometrical presentation. Moreover, replacing k with the parameter Q in case $k\ge n$ in the geometrical definition of the product, we obtain the party algebra $P_{n,r}(Q)$.

We further know the generator of the party algebra $P_{n,r}(Q)$ by Tanabe's paper [11].

Proposition 6. *(Tanabe [11, Theorem 3.1]) The party algebra $P_{n,r}(Q)$ is generated by the symmetric group $\langle s_1,s_2,\ldots,s_{n-1}\rangle$ together with f and e_r as in Figure 5.*

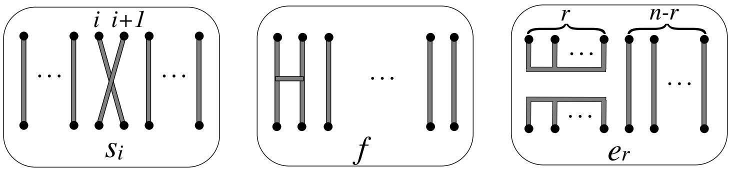

Figure 5: The generators of $P_{n,r}(Q)$

5 $\mathrm{End}_{G(2,1,3)} V^{\otimes n}$

So far, we have assumed that the left coordinate of the top vertex $\boldsymbol{\beta}^{(0)} = [(k), \emptyset]$ has k boxes such that $k \geq n$. It is easy to see that the same diagram will appear even if we begin with $\boldsymbol{\beta}^{(0)} = [(k_1), \emptyset]$ such that $k_1 \geq n$ and $k_1 \neq k$. On the other hand, in case $k_1 < n$, the resulting diagram vary. We mention what happens if we draw a diagram under the condition that $\boldsymbol{\beta}^{(0)} = [(3), \emptyset]$ according to the same recipe. In this situation, we have Figure 6. This corresponds to the centralizer algebra $\mathrm{End}_{G(2,1,3)} V^{\otimes n}$, which is a quotient of the party algebra $P_{n,2}(3)$.

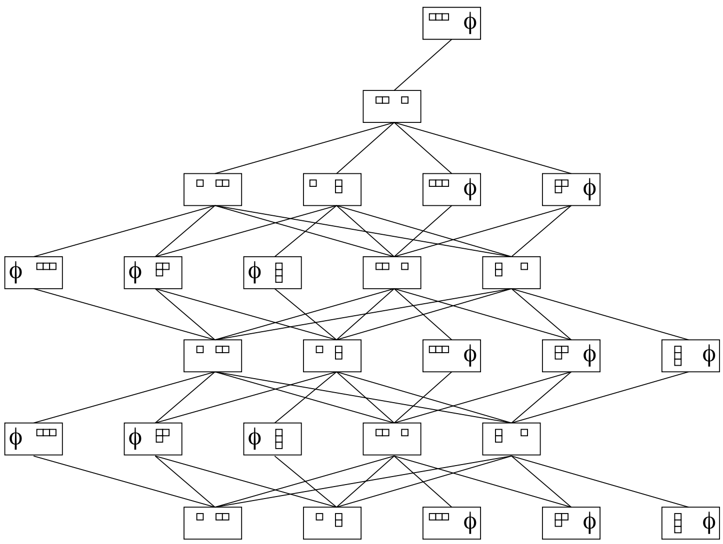

Figure 6: The Bratteli diagram of $\mathrm{End}_{G(2,1,3)} V^{\otimes n}$

This diagram periodically grows in higher levels. This indicates that this centralizer may give an example of subfactors. Hence we can expect that using this algebra the Turaev-Viro-Ocneanu invariants of 3-dimensional manifolds will be calculated in the same way as in the papers [8, 9].

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