The Standard Expression for the Party Algebra

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1 Introduction

In this note we explain about the party algebra $P_{n,r}(Q)$ and its standard expression. The party algebra is originally defined as the centralizer of the unitary reflection group of type $G(r,1,k)$ which diagonally acts on the n times tensor space $V^{\otimes n}$. This algebra is also defined as a diagram algebra like Temperly-Lieb's algebra, Brauer's centralizer algebra, the partition algebra, and so on. It is well known that these diagram algebras have the cellular structures.

In order to say that an algebra is cellular, we must make a cell datum of it. The cell datum describes that how the algebra's basis is decomposed into the cells, which may turn out to be the representatives of the irreducible representations.

When we trying to find a characterization for the party algebra by generators and relations, we found a good expression of the basis elements of the algebra, which naturally gives the cellular structure.

In this note, we present a candidate of the standard expression of the party algebra.

2 Definition of the party algebra

2.1 Define the party algebra as a tentalizer

[1] First we quickly review the definition of the unitary reflection group of type $G(r,1,k)$. The unitary reflection group $G(r,1,k)$ in Shephard-Todd's notation consists of all the monomial matrices of size k whose non-zero components are powers of r-th roots of unity and it is generated by all the permutation matrices and the identity matrix whose $(1,1)$ component was replaced by an r-th primitive root of unity ζ. Let V be a k-dimensional vector space on which the unitary reflection group $G(r,1,k)$ naturally acts. Consider the tensor space $V^{\otimes n}$ on which $G(r,1,k)$ acts diagonally. We assume that the dimension $k \geq n$ [resp. $k \geq 2n$] if $r > 1$ [resp. $r = 1$]. The party algebra $P_{n,r}(k)$ is defined as the

[1] This word was invented by A. Ram [11]. The meaning of it is "Centralizer of the tensor representation"

centralizer of $G(r,1,k)$ in $V^{\otimes n}$ with respect to the above action. Namely,

$$P_{n,r}(k) := \mathrm{End}_{G(r,1,k)} V^{\otimes n}.$$

It is well known that in the similar setting, we have the following correspondence:

$$\begin{array}{ccccc} GL_k(\mathbb{C}) & \supset & O_k(\mathbb{C}) \supset & \mathbb{C}\mathfrak{S}_n \\ \mathfrak{S}_k & \subset & B_n(k) \subset & A_n(k) = P_{n,1}(k) \end{array}$$

2.2 Find the basis

Since $G(r,1,k)$ contains $G(1,1,k) = \mathfrak{S}_k$, the party algebra $P_{n,r}(k)$ must be a subalgebra of the partition algebra $A_n(k)$. To find which element is in the party algebra precisely, we observe the actions of the generators of $G(r,1,k)$.

Let $e_1, \ldots, e_k$ be the natural basis of the vector space V. We assume that $G(r,1,k)$ acts naturally with respect to this basis.

Case $r = 1$

First consider the case $r = 1$, the partition algebra case. Suppose that an endomorphism map X moves one of the elements of the natural basis of the tensor space to a linear combination of the basis:

$$X(e_{m_1} \otimes \cdots \otimes e_{m_n}) = \sum_{f_1,\ldots,f_n} X^{f_1,\ldots,f_n}_{m_1,\ldots,m_n} e_{f_1} \otimes \cdots \otimes e_{f_n}. \tag{1}$$

Since X commutes with the diagonal action of the symmetric group, for an arbitrary element $\sigma \in \mathfrak{S}_k$ we have

$$\sigma^{-1} X \sigma(e_{m_1} \otimes \cdots \otimes e_{m_n}) = \sum_{f_1,\ldots,f_n} X^{\sigma(f_1),\ldots,\sigma(f_n)}_{\sigma(m_1),\ldots,\sigma(m_n)} e_{f_1} \otimes \cdots \otimes e_{f_n}.$$

Hence we have

$$X^{\sigma(f_1),\ldots,\sigma(f_n)}_{\sigma(m_1),\ldots,\sigma(m_n)} = X^{f_1,\ldots,f_n}_{m_1,\ldots,m_n}.$$

From this, Jones showed that the following transformations make a basis of $\mathrm{End}_{G(1,1,k)}(V^{\otimes n})$ [4].

$$\{T^{\sim} \mid \sim \text{ is an equivalent relation on } 2n, \text{ the number of classes} \leq k\} \tag{2}$$

$$\begin{aligned} (T^{\sim})^{m_{n+1},\ldots,m_{2n}}_{m_1,\ldots,m_n} &:= \begin{cases} 1 & \text{if } (m_i = m_j \Leftrightarrow i \sim j) \\ 0 & \text{otherwise,} \end{cases} \\ m_{n+j} &:= f_j \quad j = 1,2,\ldots,n. \end{aligned}$$

Since we assumed that the dimension k of the vector space is large enough, in the following we can omit the second condition of the expression 2.

It is easy to see that there is a one to one correspondence between the basis (2) and the following set-partition:

$$\begin{aligned} & F=\{f_1,\dots,f_n\}, \quad M=\{m_1,\dots,m_n\}, \quad |F\cup M|=2n, \\ \Sigma_n^1 \quad = \quad & \{\{T_1,\dots,T_s\} \mid s=1,2,\dots \\ & T_j(\neq\emptyset)\subset F\cup M \ (j=1,2,\dots,s), \\ & \cup T_j = F\cup M, \quad T_i\cap T_j=\emptyset \text{ if } i\neq j\}. \end{aligned} \tag{3}$$

Case $r > 1$

Next consider the case $r > 1$.

In this case we have to consider the action of $\xi = \mathrm{diag}(\zeta, 1, \dots, 1)$. Note that ξ multiplies $e_{m_1} \otimes \cdots \otimes e_{m_n}$ by ζ at each occurrence of e_1. Let $f_1, \dots, f_n$ and $m_1, \dots, m_n$ be the indices defined in the equation (1). Let p be the number of 1s in the array $(f_1, \dots, f_n)$ and q the number of 1s in the array $(m_1, \dots, m_n)$. Since

$$\begin{aligned} X\xi = \xi X \quad \Leftrightarrow \quad & \zeta^p X^{f_1,\dots,f_n}_{m_1,\dots,m_n} = \zeta^q X^{f_1,\dots,f_n}_{m_1,\dots,m_n} \\ & \text{for all possible } (f_1,\dots,f_n) \text{ and } (m_1,\dots,m_n), \end{aligned}$$

in order that X is an element of the centralizer $\mathrm{End}_{G(r,1,k)} V^{\otimes n}$, the coefficients $X^{f_1,\dots,f_n}_{m_1,\dots,m_n}$ must be 0 unless that the number of 1s in $(m_1, \dots, m_n)$ is equal to the number of 1s in $(f_1, \dots, f_n)$ modulo r. Further since $\sigma \in G(1, 1, k)$ runs all the permutations, the coefficients must be 0 unless that the number of is in $(m_1, \dots, m_n)$ is equal to that of is in $(f_1, \dots, f_n)$ modulo r for any letter i,

If we describe this in terms of the set-partitions, the ξ-action adds the following restriction to the basis of the partition algebra.

$$T_j \cap F \equiv T_j \cap M \pmod r.$$

In fact, Tanabe [14] showed that the following set becomes a basis of the party algebra $P_{n,r}(k)$.

$$\begin{aligned} & F=\{f_1,\dots,f_n\}, \quad M=\{m_1,\dots,m_n\}, \quad |F\cup M|=2n, \\ \Sigma_n^r \quad = \quad & \{\{T_1,\dots,T_s \mid s=1,2,\dots \\ & T_j(\neq\emptyset)\subset F\cup M \ (j=1,2,\dots,s), \\ & \cup T_j = F\cup M, \quad T_i\cap T_j=\emptyset \text{ if } i\neq j \\ & T_j\cap F \equiv T_j\cap M \pmod r\}. \end{aligned} \tag{4}$$

We call an element of Σ_n^r an *r-modular seat-plan.*

2.3 Define the party algebra as a diagram algebra

As we can see in Martin's papers [9, 10] the partition algebra $A_n(k)$ is defined as a diagram algebra imposing a product on the set (3). (See also the paper [2].)

In case that the partition $A_n(k)$ is defined as a diagram algebra, the parameter k does not have to be an integer any more.

Since the party algebra $P_{n,r}(k)$ is a subalgebra of the partition algebra (as a centralizer), it also becomes a diagram algebra using the same product. It is easily checked that the product is closed within the party algebra.

We explain this taking an example of the case $r = 2$. Let

$$w = \{\{f_1, m_1, m_2, m_4\}, \{f_2, m_5\}, \{f_3, f_4\}, \{f_5, m_3\}\} \in \Sigma_5^2.$$

The corresponding diagram of w will become the one in Fig. 1. In general, the diagram of a seat-plan is obtained as follows. Consider a rectangle with n marked points on the bottom and the same n on the top. The n marked

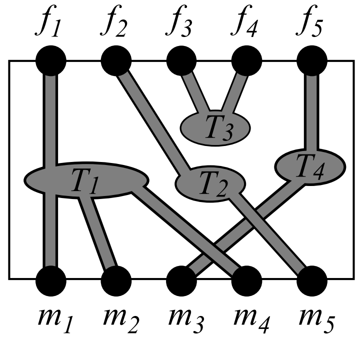

Figure 1: $w \in \Sigma_n^r$

points on the bottom are labeled by $f_1, f_2, \ldots f_n$ from left to right. Similarly, the n marked points on the top is labeled by $m_1, m_2, \ldots, m_n$ from left to right. If $w \in \Sigma_n^2$ has s parts, then put s shaded circles in the middle of the rectangle so that they have no intersections. Each of the circles corresponds to one of the non-empty T_js. Then we join the $2n$ marked points and the s circles with $2n$ shaded bands so that the marked points labeled by the elements of T_j are connected to the corresponding circle with $|T_j|$ bands. We call $T_j \cap F$ the *upper part* of T_j and $T_j \cap M$ the *lower part* of T_j

Define the composition product $w_1 \circ w_2$ of diagrams w_1 and w_2 to be the new diagram obtained by placing w_1 above w_2, gluing the corresponding boundaries and shrinking half along the vertical axis as in Fig. 2. We then have a new diagram possibly containing some islands and/or lakes. If there occur islands and/or lakes in the resulting diagram, then first bury the lakes and remove each island multiplying by Q (see Fig. 3). The product is the resulting diagram with the islands and lakes removed. It is easy to define this product in terms of set-partitions (See for example the papers [9, 10, 2].) In the following we will write $w_1 w_2 = w_1 \circ w_2$ for convenience.

In 2-modular seat-plan case, the set-partition of $2n$ elements has the constraint that if the cardinality of the upper part of a part is an even number, then

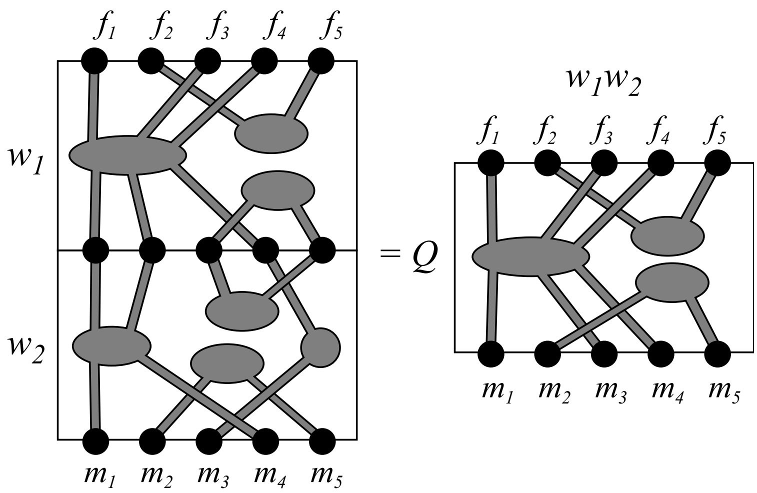

Figure 2: The product of seat-plans

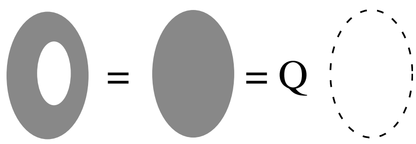

Figure 3: Remove islands multiplying by Q

that of the lower part of it also must be an even number. And if the cardinality of the upper is an odd number, then that of the lower must be an odd number. Note that this property is inherited when we define a product $w_1 w_2$ by placing the diagram of w_1 onto that of w_2.

If we specialize the parameter Q to an integer k which is the dimension of the fixed vector space, then this diagram algebra is surjectively mapped to the centralizer algebra in the endomorphism ring of the n times tensor space. If $k \geq n$ then the map will be injective. This condition is different from the partition algebra case. More precisely we have following proposition given by Tanabe [14].

Proposition 1. *(Tanabe [14, Theorem 3.1]) Let $G(r,1,k)$ be the group of all the monomial matrices of size n whose non-zero entries are r-th roots of unity. Let V be the $\mathbb{C}$-vector space of dimension k with the basis elements $e_1, e_2, \ldots, e_k$ on which $G(r,1,k)$ acts naturally. Let ϕ be the representation of the symmetric group $\mathfrak{S}_n$ on $V^{\otimes n}$ obtained by permuting the tensor product factors, i.e., for $v_1, v_2, \ldots, v_n \in V$ and for $w \in \mathfrak{S}_n$,*

$$\phi(w)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) := v_{w^{-1}(1)} \otimes v_{w^{-1}(2)} \otimes \cdots \otimes v_{w^{-1}(n)}.$$

Define further $\phi(f)$ *as follows:*

$$\phi(f)(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n}) := \begin{cases} e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_n} & \text{if } p_1 = p_2, \\ 0 & \text{otherwise.} \end{cases}$$

If $r > n$, *then* $\mathrm{End}_{G(r,1,k)}(V^{\otimes n})$ *is generated by* $\phi(\mathfrak{S}_n)$ *and* $\phi(f)$ *and* ϕ *defines a homomorphism from* $\mathcal{A}_n \otimes \mathbb{C}$ *to* $\mathrm{End}_{G(r,1,k)}(V^{\otimes n})$.

Proposition 2. *Let* ϕ *be the map previously defined. If* $k \geq n$, *then* ϕ *is injective.*

3 Bratteli diagram of $P_{n,r}(k)$

Thanks to the Schur-Weyl duality, we can obtain the Bratteli diagram of the party algebra observing how the tensor representation of $G(r, 1, k)$ is decomposed into irreducibles in accordance with the increase of the number of tensors. The irreducible representations of the unitary reflection group $G(r, 1, k)$ are indexed by the r-tuples of Young diagrams whose total number of the boxes is equal to k. As for the irreducible representation of $G(r, 1, k)$, we refer the paper [1].

We are going to explain this, taking the case $r = 3$ (Fig. 4). In this example

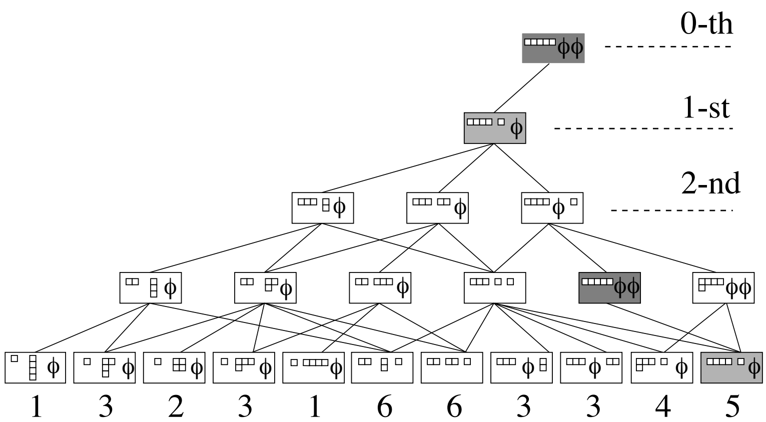

Figure 4: Bratteli diagram of $P_{4,3}(5)$

we set $k = 5$. Since we set $r = 3$, the irreducible components are indexed by 3-tuples of Young diagrams of total size 5. First consider the case $n = 1$. In this case we have the natural representation. We know that it is irreducible and indexed by the 3-tuple of Young diagrams on the vertex on the 1-st floor in Fig. 4. If the number of tensors increases by one, then the irreducible components will be branched obeying the following rule:

- one of the boxes in one coordinate is moved to the right so that all the coordinates have again Young diagrams,

- in case the box to be removed is in the last (r-th) coordinate, this box is moved to the first coordinate.

Note that in this picture, the vertex on the 1-st floor appears on the 4-th floor.

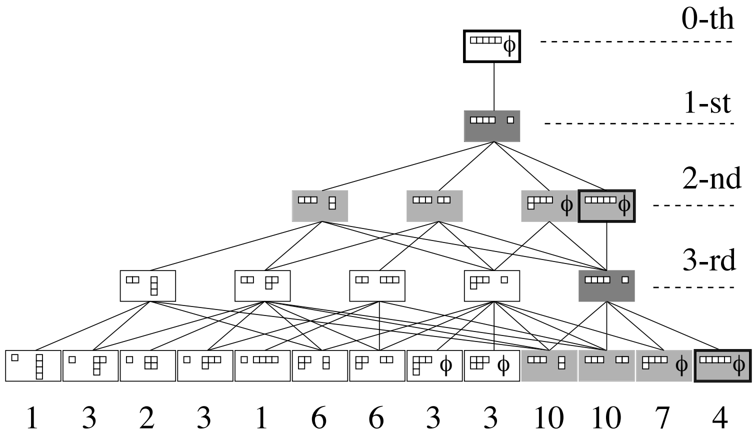

Figure 5: Bratteli diagram of $P_{4,2}(5)$

The Fig. 5 is another example of the Bratteli diagram of the party algebra in case $r = 2$, $k = 5$. In this example, we note that vertices on the 2-nd floor again appear on the 4-th floor.

In case $r = 1$, or the partition algebra case, the situation is slightly different, however, our argument still applies for the case $r = 1$ with slight modifications. Hereafter, we assume that $r > 1$.

Schur-Weyl's duality asserts that the multiplicity of each irreducible component becomes the degree of the corresponding irreducible of the centralizer. In this Bratteli diagram, each vertex on the bottom expresses an irreducible component of the party algebra as well as the corresponding irreducibles of the unitary reflection group and the number of the paths from the top vertex to a vertex on the bottom becomes the size of the irreducible of the party algebra.

4 Irreducible components of $P_{n,r}(Q)$

If we define the party algebra as a diagram algebra, the parameter Q does not necessarily have to be a large integer. Although the Bratteli diagrams in the previous section seem to be made depending on the integer k, it is natural to guess that there must exist a description which does not depend on the choice of k.

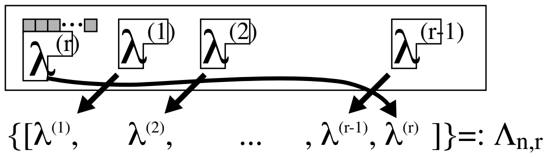

Figure 6: Irreducibles of $P_{n,r}(Q)$

Fig. 6 is such a description. We shift an r-tuple of Young diagrams on the n-th floor to the left and the left most Young diagram to the right most removing the 1-st row.

Let $\Lambda_{n,r}$ be the index set obtained from such operations. We find that $\Lambda_{n,r}$ is equal to the following set:

$$\Lambda_{n,r} = \{[\lambda^{(1)}, \ldots, \lambda^{(r)}] \ ; \ \sum_{j}^{r} j|\lambda^{(j)}| = n, n-r, n-2r, ...\}.$$

Let $\ell_j = |\lambda^{(j)}|$ be the size of λ^j and $\ell = (\ell_1, \ldots, \ell_r)$ the array of ℓ_js. In the following section this array ℓ will play an important role.

As for the previous examples, Fig. 4 and Fig. 5, we obtain the parametrizations Fig. 7 and Fig. 8 respectively which do not depend on the choice of k.

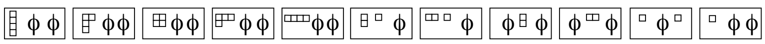

Figure 7: Irreducibles of $P_{4,3}(Q)$

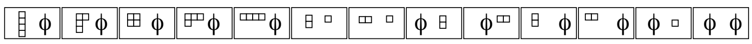

Figure 8: Irreducibles of $P_{4,2}(Q)$

The weight sums of the parameters in Fig. 7 are 4 and $4-3=1$. Those of the parameters in Fig. 7 are 4 and $4-2=2$ and $4-2-2=0$.

5 Standard expression of $P_{n,r}(Q)$

Keeping the facts presented in the previous sections in mind, we now try to define the standard expression of the party algebra by the generators showing how each r-modular seat-plan is presented as a product of the generators.

5.1 Generators of $P_{n,r}(Q)$

As for the centralizer $P_{n,r}(k) = \mathrm{End}_{G(r,1,k)} V^{\otimes n}$, Tanabe showed that the diagrams in Fig. 9 are the generators of it [14]. He showed this using a diagram

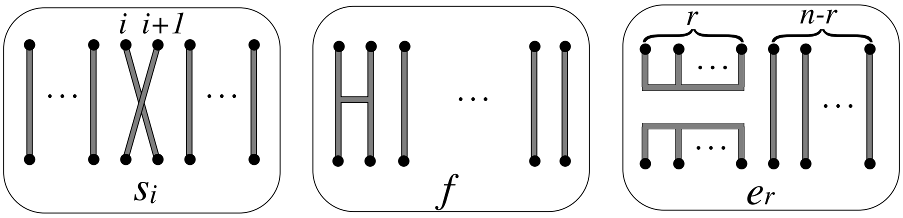

Figure 9: Generators of $P_{n,r}(Q)$

calculation. Accordingly, the diagrams in Fig. 9 generates also the party algebra $P_{n,r}(Q)$ (even if we define it as a diagram algebra). In Fig. 9, since s_is make

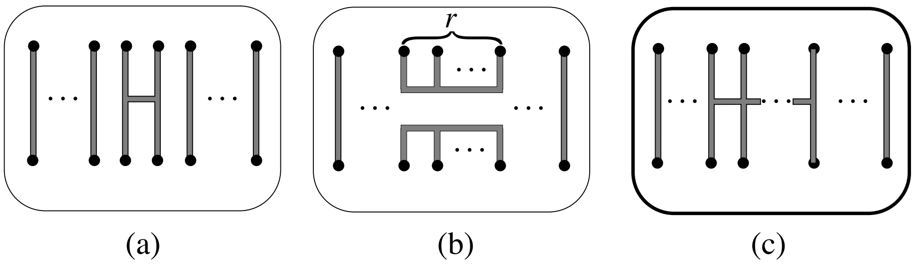

Figure 10: Special elements of $P_{n,r}(Q)$

a symmetric group, we can shift f and e_r to the right. Further, it is easy to check that the diagram (c) in Fig. 10 is also obtained from some fs and s_is in Fig. 9. Hence we understand the diagrams (a),(b) and (c) in Fig. 10 are also generators of the party algebra $P_{n,r}(Q)$.

5.2 Propagating number

To define the standard expression, we introduce the notion of the *thickness* for the propagating parts and classify the propagating parts by the thickness. Now we quickly review the definition of propagating parts. Then we define the thickness of a part of an r-modular seat plan.

For a part T of an r-modular seat-plan, if $T \cap F \neq \emptyset$ and $T \cap M \neq \emptyset$, we call T *propagating.*

For an r-modular seat-plan $w \in \Sigma_n^r$, let $\pi(w) = \{T \in w \mid T : \text{propagating}\}$ be the set of propagating parts. If $T \in w \backslash \pi(w)$, then we call T non-propagating.

The number of propagating parts $|\pi(w)|$ of w is called the *propagating number* (of w).

For example, in Fig. 1, $\pi(w) = \{T_1, T_2, T_4\}$. Hence $|\pi(w)| = 3$. On the other hand T_3 is non-propagating. Note that the following remark holds.

Remark 3. *The number of elements contained in a non-propagating part is an integer multiple of r. Namely, if $w \in \Sigma_n^r$ and $T_i \in w$ is non-propagating, then there exists an integer d such that*

$$|T_i| = dr.$$

5.3 Thickness

For a propagating part of a seat-plan, we define its thickness. The notion of the thickness will also be used to define the conjugacy classes of the party algebra. As for the conjugacy classes and characters of $P_{n,r}(Q)$, it is now being studied by Naruse [12].

Suppose that $w \in \Sigma_n^r$ and $T_i \in \pi(w)$. We define the *thickness* $t(T_i)$ of T_i as the least positive integer which is equal to the number of elements contained in its upper part by modulo r:

$$\begin{aligned} t(T_i) &\in \{1, 2, \ldots, r\}, \\ t(T_i) &\equiv |T_i \cap F| \pmod{r}. \end{aligned}$$

Since $|T_i \cap F| \equiv |T_i \cap M| \pmod{r}$ for any part $T_i \in w$, we can also define the thickness using its lower part.

Let $t = t(T_i)$. Then there exist at least t elements both in the upper and the lower parts of T_i. The number of other elements in T_i must be an integer multiple of r. Hence there exist permutations $w_1, w_2 \in \mathfrak{S}_n$ so that the diagram of $w_1 T_i w_2$ does not contain any crossing as in Fig. 11.

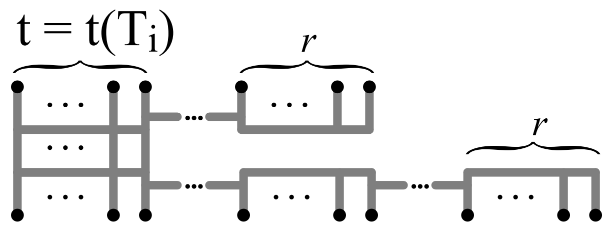

Figure 11: $w_1 T_i w_2$

Conversely, every propagating part is obtained from a seat-plan as in Fig. 11. by attaching permutations to its lower and/or upper part.

The *thickness array* of a seat-plan $\mathbf{t}(w) = (\ell_1, \ldots, \ell_r)$ is defined as the list of the numbers of the parts whose thickness values are 1, 2, 3 to r:

$$\begin{aligned} \mathbf{t}(w) &:= (\mathbf{t}(w)_1, \ldots, \mathbf{t}(w)_r) \\ &:= (\sharp\{T_i \in w \ ; \ t(T_i) = 1\}, \ldots, \sharp\{T_i \in w \ ; \ t(T_i) = r\}). \end{aligned}$$

Note that we are abusing the same notation ℓ_i which we have used to measure the sizes of Young diagrams for indexing the irreducibles.

For example, in Fig. 12 if we regard w_1, w_2 as 3-modular seat-plans, then $\mathbf{t}(w_1) = (2, 0, 1)$ and $\mathbf{t}(w_2) = (3, 1, 0)$. On the other hand, if we regard w_1, w_2 as 2-modular seat-plans, then $\mathbf{t}(w_1) = (3, 0)$ and $\mathbf{t}(w_2) = (3, 1)$.

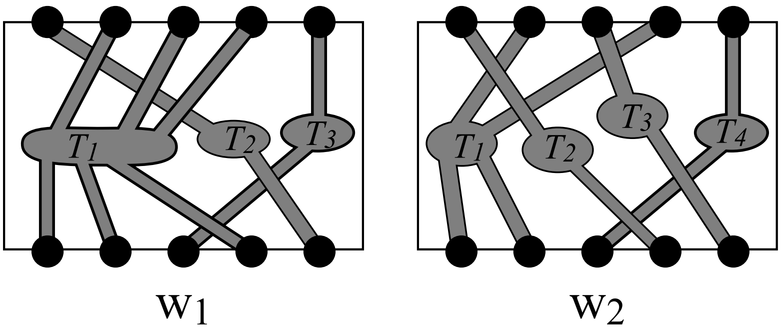

Figure 12:

Note that

$$|\mathbf{t}(w)| = \mathbf{t}(w)_1 + \cdots + \mathbf{t}(w)_r = \ell_1 + \cdots + \ell_r = \pi(w) \text{ (propageting number).}$$

5.4 Standard expression

To obtain the standard expression, first we rename all the parts of the seat-plan so that

$$\begin{aligned}
&t(T_1) = t(T_2) = \cdots t(T_{\ell_1}) = 1,\\
&t(T_{\ell_1+1}) = t(T_{\ell_1+2}) = \cdots t(T_{\ell_1+\ell_2}) = 2,\\
&\quad\vdots\\
&t(T_{\ell_1+\ell_2+\cdots+\ell_{r-1}+1}) = t(T_{\ell_1+\ell_2+\cdots+\ell_{r-1}+2}) = \cdots = t(T_{\ell_1+\ell_2+\cdots+\ell_{r-1}+\ell_r}) = r.
\end{aligned}$$

Then we *twist* the parts which have the same thickness as follows.

Let

$$T_{\ell_1+\ell_2+\cdots+\ell_{j-1}+1}, T_{\ell_1+\ell_2+\cdots+\ell_{j-1}+2}, \ldots, T_{\ell_1+\ell_2+\cdots+\ell_{j-1}+\ell_j}$$

be all the parts whose thickness is j. First we divide each of them into the upper and the lower parts. Then we sort the upper parts of them so that the minimum elements of the upper parts become increasing order. (Here we assumed that the elements of F have an order, $f_1 < f_2 < \cdots < f_n$.) Next we sort the lower parts of them so that the minimum elements of the lower parts become increasing order. (Here we assumed that the elements of M have an order, $m_1 < m_2 < \cdots < m_n$.) In order to restore the original parts whose thickness is j, join the upper and the lower parts of them. In this process, we have a permutation $v_j \in \mathfrak{S}_{\ell_j}$.

We explain this process using Fig. 13. In this picture,

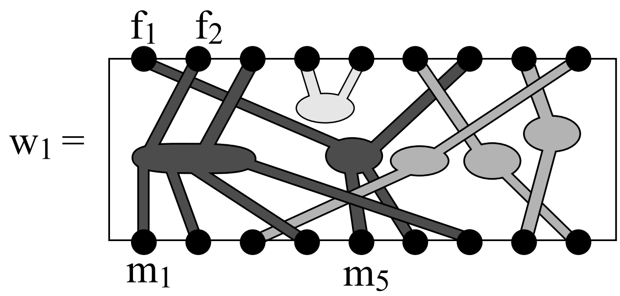

Figure 13: w_1

$$w_1 = \begin{array}{llll} \{ & \{f_2, f_3, m_1, m_2, m_4, m_7\}, & \{f_1, f_7, m_5, m_6\}, & \{f_4, f_5\}, \\ & \{f_9, m_3\}, & \{f_6, m_9\}, & \{f_8, m_3\} \quad \}. \end{array}$$

The thickness of the three gray parts of w_1 is 1. So $\ell_1 = 3$. And the thickness of the two black pats is two. So $\ell_2 = 2$.

Consider the gray parts first. The minimum of the upper parts is joined to the maximum of the lower parts. And the maximum of the upper parts is joined to the minimum of the lower parts. So we have a permutation $\sigma_1 = (13)(2) \in \mathfrak{S}_3$ (See the left figure of Fig. 14).

Next consider the black parts. If we sort these parts in accordance with the minimum elements of the upper parts, then the right island comes first. On the other hand, if we sort them in accordance with the minimum elements of the lower parts, then the left island comes first. Hence in order to restore the original parts, we have to join the upper parts and the lower parts with a crossing.

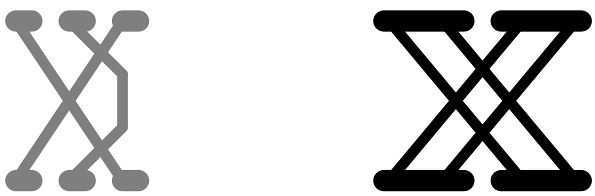

Figure 14:

Note that if there exist ℓ_t parts whose thickness is t, then the permutation obtained in this way is an element of the symmetric group of degree ℓ_t, however, to present this permutation by the generators we need t-parallel strings at each crossing as in Fig. 15. So the permutation is realized in the symmetric group of degree $t \times \ell_t$.

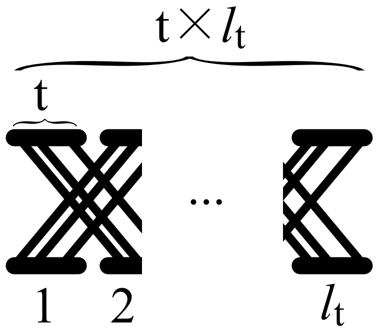

Figure 15:

As for the black parts of w_1 in Fig. 13 we have a transposition $\sigma_2 = (12) \in \mathfrak{S}_2$. However this is presented as an element of $\mathfrak{S}_{2\times 2}$ as in the right figure of Fig. 14.

The standard expression of w_1 in Fig. 13 is obtained from the expression in Fig. 16 by attaching the following two permutations w_1 and w_2 on the top and bottom respectively:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 8 & 9 & 1 & 7 & 2 & 3 & 4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 9 & 1 & 2 & 4 & 7 & 5 & 6 \end{pmatrix}^{-1}.$$

Fig. 17 is another example of a standard expression. Since w_2 is a 3-modular seat-plan, the thickness array is a 3-tuple of permutations. In this case we have a thickness array

$$\ell = \mathbf{t}(w_2) = (\ell_1, \ell_2, \ell_3) = (4, 2, 2)$$

and a permutation array

$$(v_1, v_2, v_3) \in \mathfrak{S}_{l_1} \times \mathfrak{S}_{l_2} \times \mathfrak{S}_{l_3},$$

The permutation array is uniquely determined by the given r-modular seat-plan.

6 Application

Using the standard expression above, we can obtain the defining relation of the party algebra $P_{n,r}(Q)$ and a cell datum for $P_{n,r}(Q)$.

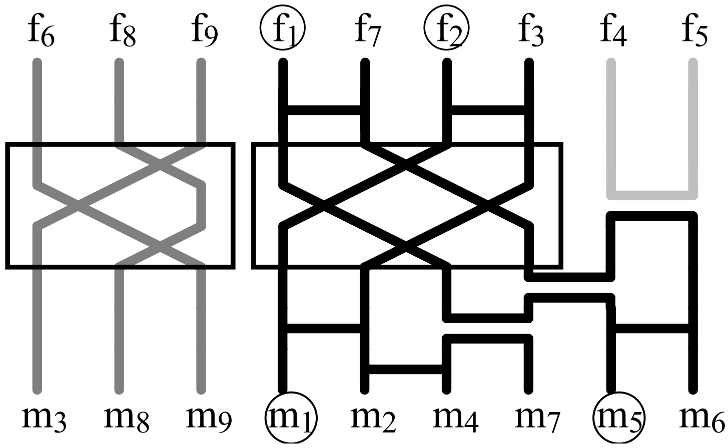

Figure 16:

6.1 Defining relation

We can find the defining relation of the algebra by the following try-and-error method: First guess the relations. Then try to show that a multiple of a generator and a standard expression will be transformed again a scalar multiple of a standard expression, using the guessed relations only.

For example, 2-modular party algebra $P_{n,2}(Q)$ has the following relations:

$$\begin{aligned}
&s_i^2 = 1 \quad (1 \leq i \leq n-1),\\
&s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \leq i \leq n-2),\\
&s_i s_j = s_j s_i \quad (|i-j| \geq 2),\\
&e^2 = Qe, \quad f^2 = f,\\
&ef = fe = e, \quad es_1 = s_1 e = e, \quad fs_1 = s_1 f = f,\\
&es_i = s_i e, \quad fs_i = s_i f \quad (i \geq 3),\\
&es_2 e = e, \quad fs_2 fs_2 = s_2 fs_2 f, \quad fs_2 es_2 f = fs_2 f,\\
&xs_2 s_1 s_3 s_2 y s_2 s_1 s_3 s_2 = s_2 s_1 s_3 s_2 y s_2 s_1 s_3 s_2 x \quad (x, y \in \{e, f\}).
\end{aligned}$$

Here we put $e = e_2$ in Fig. 9.

6.2 Cell datum

As another application of the standard expression, we can obtain a cell datum of $P_{n,r}(Q)$. The precise description for the cell datum of $P_{n,r}(Q)$ is now in preparation [8]. The following is the outline of the recipe.

The standard expression of an r-modular seat-plan w is divided into three parts, the upper half, the lower half, and a permutation array in a direct product of the symmetric groups (See Fig. 17).

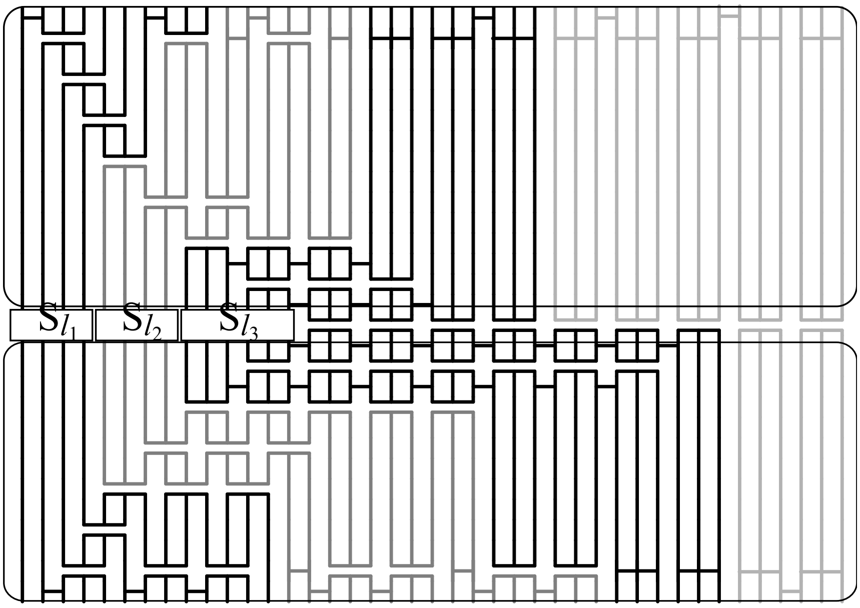

Figure 17: $w_2 \in \Sigma^3_{41}$

More precisely, we can decompose the basis of the party algebra as follows. For an r-modular seat-plan $w \in \Sigma_n^r$, we obtain the thickness array

$$\mathbf{t}(w) = (\ell_1, \ell_2, \cdots, \ell_r)$$

and the set-partitions of F and M into $F \cap T_i$ $(i = 1, 2, \ldots, s)$ and $M \cap T_i$ $(i = 1, 2, \ldots, s)$ respectively. Further we obtain an r-tuple of permutations v_1 to v_r. Here permutation v_t is an element of $\mathfrak{S}_{\ell_t}$. By RS correspondence [3], we have a pair of standard tableaux (P_t, Q_t) of shape $\lambda^{(t)}$ whose size is equal to ℓ_t. Hence for each seat-plan, we can determine an r-tuple of Young diagrams in $\Lambda_{n,r}$.

In order to show that the decomposition above makes a cell datum, we have only to define a partial order on $\Lambda_{n,r}$.

The partial order is easily defined combining the propagating number and the partial order which is introduced to define the Kazhdan-Lusztig's cell representation of the Iwahori-Hecke algebra of type A [5] as follows. For r-tuples of Young diagrams $\lambda, \mu \in \Lambda_{n,r}$, define an order as follows. First measure the total size of λ and μ, and define the order by

$$|\lambda| > |\mu| \Leftrightarrow \lambda > \mu.$$

If $|\lambda| = |\mu|$, only the case where each Young diagram has the same size is comparable. If $|\lambda^{(t)}| = |\mu^{(t)}|$ for $t = 1, 2, \ldots, r$, compare $\lambda^{(t)}$ and $\mu^{(t)}$ one by one by Kazhdan-Lusztig order, and compare the r-tuples of them by lexicographical order.

By observing the actions of the generators of the party algebra, we can check that this order satisfies the condition of the cell datum. Thus, the party algebra becomes cellular.

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