Laplacians and spectral zeta functions of totally ordered categories

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Abstract

In this paper we prove that the Laplacian $\Delta_{\mathbf{K}}$ attached to a totally ordered category $\mathbf{K}$ is positive definite in a framework of representation theory of ordered categories. We also study the spectra of the Laplacians for the categories **PB** and $\mathrm{Mod}(\mathbb{F}_q)$. We show that the corresponding spectral zeta functions exist and define holomorphic functions in a region $\mathrm{Re}(s) > 1-\varepsilon$ for these case.

1 Introduction

Let $\mathbf{K}$ be a given category. We denote by $\mathrm{Ob}(\mathbf{K})$ the set of objects of $\mathbf{K}$, $\mathrm{Mor}(\mathbf{K})$ the set of all morphisms in $\mathbf{K}$ and $\mathrm{Mor}_{\mathbf{K}}(X,Y)$ the set of all morphisms from X to Y. In [I, KuW], the following problem is proposed and several examples of categories are studied.

Problem 1.1 ([I, KuW])**.** Does a given 'nice' category $\mathbf{K}$ satisfy the **Cauchy-Schwarz type inequality**

$$\#\,\mathrm{Mor}_{\mathbf{K}}(X,Y)\#\,\mathrm{Mor}_{\mathbf{K}}(Y,X) \le \#\,\mathrm{Mor}_{\mathbf{K}}(X,X)\#\,\mathrm{Mor}_{\mathbf{K}}(Y,Y) \tag{1.1}$$

if all quantities in (1.1) are finite?

This is regarded as the special case of the following general problem. Analogous to the Laplacian (or the adjacency matrix) of a given oriented graph, the **Laplacian** $\Delta_{\mathbf{K}}$ of a given category $\mathbf{K}$ is introduced in [KuST] by

$$\Delta_{\mathbf{K}} \stackrel{\mathrm{def}}{=} (\#\,\mathrm{Mor}_{\mathbf{K}}(X,Y))_{X,Y\in\mathrm{Ob}_o(\mathbf{K})} \tag{1.2}$$

where $\mathrm{Ob}_o(\mathbf{K})$ denotes the subset of $\mathrm{Ob}(\mathbf{K})$ such that the number $\#\,\mathrm{Mor}_{\mathbf{K}}(X,Y)$ is finite for every pair (X,Y) of elements in $\mathrm{Ob}_o(\mathbf{K})$.

Problem 1.2 ([KuST, KuW])**.** Is the Laplacian $\Delta_{\mathbf{K}}$ of a given 'nice' category **K positive definite**?

Actually, we notice that the Cauchy-Schwarz inequality (1.1) is equivalent to the positive definiteness of a 2-minor

$$\begin{pmatrix} \#\operatorname{Mor}_{\mathbf{K}}(X,X) & \#\operatorname{Mor}_{\mathbf{K}}(X,Y) \\ \#\operatorname{Mor}_{\mathbf{K}}(Y,X) & \#\operatorname{Mor}_{\mathbf{K}}(Y,Y) \end{pmatrix}$$

of $\Delta_{\mathbf{K}}$. We expect that the positive definiteness of the Laplacian $\Delta_{\mathbf{K}}$ is true for 'good' categories —— categories whose objects have some algebraic structure, for instance —— but it seems difficult to treat this problem with full generality.

Remark 1.1 (Counterexample to Problem 1.1). It is easy to construct an artificial counterexample to our problem, which is suggested by Professor Anton Deitmar. Let $\mathbf{K}$ be a category which has only two objects, say X and Y, and the morphisms are given by

$$\begin{aligned} \operatorname{Mor}_{\mathbf{K}}(X,X) &= \{0,1\}, \quad \operatorname{Mor}_{\mathbf{K}}(Y,Y) = \{0,1\}, \\ \operatorname{Mor}_{\mathbf{K}}(X,Y) &= \{0\}, \quad \operatorname{Mor}_{\mathbf{K}}(Y,X) = \{a_1, a_2, \dots, a_n\} \end{aligned}$$

with the following composition rules:

$$0 \cdot p = 0, \quad p \cdot 0 = 0, \quad 1 \cdot p = 1, \quad p \cdot 1 = 1 \quad (p \in \operatorname{Mor}(\mathbf{K})).$$

Then we have

$$\begin{aligned} \#\operatorname{Mor}_{\mathbf{K}}(X,Y)\#\operatorname{Mor}_{\mathbf{K}}(Y,X) &= n, \\ \#\operatorname{Mor}_{\mathbf{K}}(X,X)\#\operatorname{Mor}_{\mathbf{K}}(Y,Y) &= 4. \end{aligned}$$

Therefore the inequality (1.1) does not hold if $n > 4$. □

One of the main purpose of this paper is to give an affirmative answer to this problem of positive definiteness of the Laplacian $\Delta_{\mathbf{K}}$ when $\mathbf{K}$ is a totally ordered category (Theorem 3.3). This is achieved by using representation theory of ordered categories. In fact, we will see that the key proposition in our description is an irreducible decomposition formula of certain modules (Proposition 3.1).

A study of the Laplacian $\Delta_{\mathbf{K}}$ of a given category $\mathbf{K}$ is originally motivated by the study of the zeta functions of categories introduced by Kurokawa for the sake of unifying various zeta functions [Ku]. Let us recall the definition of the zeta function of a category. Assume that $\mathbf{K}$ is a category with a **zero object**, that is, an object which is initial and terminal. An object X is called **simple** if the set $\operatorname{Mor}_{\mathbf{K}}(X,Y)$ of morphisms is consisting of monomorphisms for any object $Y \in \operatorname{Ob}(\mathbf{K})$. Denote by $\operatorname{Prim}(\mathbf{K})$ the set consisting of isomorphism classes of simple finite objects in $\mathbf{K}$. The **zeta function** of the category $\mathbf{K}$ is defined by the Euler product

$$\zeta(s,\mathbf{K}) \stackrel{\text{def}}{=} \prod_{P\in\mathrm{Prim}(\mathbf{K})} (1-N(P)^{-s})^{-1}, \tag{1.3}$$

where we put $N(P)=\#\,\mathrm{End}_{\mathbf{K}}(X)$ for $X\in P$.

For example, let us see the zeta function $\zeta(s,\mathbf{Ab})$ of the category $\mathbf{Ab}$ of abelian groups. A simple object of $\mathbf{Ab}$ is a cyclic group of prime order, and hence we have $\mathrm{Prim}(\mathbf{Ab})=\big\{\mathbb{Z}/p\mathbb{Z}\;\big|\;p:\ \text{prime}\big\}$ and $N(\mathbb{Z}/p\mathbb{Z})=p$. Therefore we have

$$\zeta(s,\mathbf{Ab}) = \prod_{P\in\mathrm{Prim}(\mathbf{Ab})} (1-N(P)^{-s})^{-1} = \prod_{p\,:\ \text{prime}} (1-p^{-s})^{-1} = \zeta(s),$$

which is nothing but the Riemann zeta function. Thus the Riemann zeta function $\zeta(s)$ permits us an interpretation as a zeta function of the category $\mathbf{Ab}$. Related to this fact, the spectrum of $\Delta_{\mathbf{Ab}}$ is studied experimentally in [KuST].

Another motivation to study the Laplacian $\Delta_{\mathbf{K}}$ of a category $\mathbf{K}$ is an appropriate formulation of Selberg-type zeta functions of infinite-dimensional groups. In particular, our main concern is a reasonable formulation of a Selberg zeta function of infinite symmetric groups; the group $\mathfrak{S}_\infty$ defined by the inductive limit of the finite symmetric groups $\mathfrak{S}_n$, and the group $\mathfrak{S}_\omega$ defined by the full permutation group of the set $\mathbb{Z}_+=\{1,2,3,\dots\}$.

According to Olshanskii and Neretin, there are two guiding principles in dealing with an infinite-dimensional group as follows [Ne].

Principle of semigroup extension (Olshanskii). Let G be an infinite-dimensional group. Then G is not so much a group, but rather the visible part of some semigroup $\Gamma\supset G$ which is not visible to the naked eye. Any unitary representation of G can be uniquely extended to that of Γ.

Principle of categorical extension (Neretin). An infinite-dimensional group G is merely the visible part of a certain category $\mathbf{K}$ which is not visible to the naked eye. The group G is the automorphism group $\mathrm{Aut}_{\mathbf{K}}(X)$ of a certain object $X\in\mathrm{Ob}(\mathbf{K})$, while the semigroup Γ appearing in the principle above is the semigroup $\mathrm{End}_{\mathbf{K}}(X)$ of endomorphisms of the same object X. Any unitary representation of G can be uniquely extended to that of $\mathbf{K}$.

These principles suggest that we should broaden our perspective from the (infinite-dimensional) group G itself to a certain category $\mathbf{K}_G$ which contains the group G as an automorphic group of some object. Hence one reasonable way to define a Selberg zeta function $Z(s;G)$ is taking it as an appropriate zeta function $Z(s;\mathbf{K}_G)$ defined for an associated category $\mathbf{K}_G$.

However, it is difficult to find such an adequate formulation of zeta functions. For instance, let us see the case of full symmetric group $\mathfrak{S}_\omega$. In this papar we take the category called **PB** (see Section 4 for definition) as an associated category of $\mathfrak{S}_\omega$. If we employ the zeta function of a category (1.3), then it is easy to see that $\mathrm{Prim}(\mathbf{PB}) = \{[1]\}$ and $N([1]) = 2$, and hence we have

$$\zeta(s, \mathbf{PB}) = (1 - 2^{-s})^{-1},$$

which is not very interesting. Thus this formulation of a zeta function does not seem appropriate in the present case.

In [Ki] we dealt with the case of $\mathfrak{S}_\infty$, and calculated a candidate of a Selberg-type zeta function of $\mathfrak{S}_\infty$ as a limit function of zeta functions defined for finite symmetric groups. In that case we have a clue to introduce a notion corresponding to the 'fundamental group', which enable us to formulate a considerably nice zeta function. But, in the categorical picture, it is not clear at this moment how to define such a fundamental group. This prevents us from the formulation of a zeta function of $\mathfrak{S}_\omega$ in a 'geometric' manner.

However, we have another strategy to introduce a zeta function of $\mathfrak{S}_\omega$. We recall that the Selberg zeta function $Z_X(s)$ of a locally symmetric space X has a determinant expression via the Laplacian Δ_X on X

$$Z_X(s) = \det(\Delta_X - s(2\rho - s)) \times (\text{some factor}).$$

Here ρ is a certain constant depending only on X. This fact suggests that we would define a Selberg-type zeta function $Z(s;\mathbf{K})$ of a category $\mathbf{K}$ in a 'spectral' manner, that is, by a determinant formula of the Laplacian $\Delta_{\mathbf{K}}$. This would also provide us a formulation of a fundamental group of a category $\mathbf{K}$ conversely. Thus it is important to study the spectrum of Laplacians of categories.

This paper is organized as follows. In Section 2 we recall necessary notions and facts on representation theory of ordered categories according to Neretin [Ne]. In Section 3 we prove the positive definiteness of Laplacians attached to totally ordered categories. This is achieved via irreducible decomposition of certain semigroup representations. In Sections 4 and 5 we study the categories **PB** and $\mathrm{Mod}(\mathbb{F}_q)$ respectively, which are totally ordered. We show that in these cases the spectral zeta function of the Laplacian exists and defines a holomorphic function in a region $\mathrm{Re}(s) > 1 - \varepsilon$ for some small $\varepsilon > 0$.

2 Representation theory of ordered categories

In this section we recall the fundamental notions and facts in representation theory of ordered categories according to Neretin [Ne].

For a given category $\mathbf{K}$, we denote by $\mathrm{Ob}(\mathbf{K})$ the set of all objects in $\mathbf{K}$, $\mathrm{Mor}(\mathbf{K})$ the set consisting of all morphisms of $\mathbf{K}$, and $\mathrm{Mor}_{\mathbf{K}}(X,Y)$ the set of the morphisms from X to Y for $X,Y \in \mathrm{Ob}(\mathbf{K})$. We may often write $P : X \to Y$ instead of $P \in \mathrm{Mor}_{\mathbf{K}}(X,Y)$ if the category $\mathbf{K}$ can be specified from the context.

We first recall the definition of an **ordered category**. Let $\Sigma = (\Sigma, \leq)$ be a partially ordered set (we often say poset briefly) such that every finite subset $S \subset \Sigma$ has an upper bound, i.e. there exists an element $\sigma \in \Sigma$ which satisfies $s \leq \sigma$ for all $s \in S$. Assume that the set $\mathrm{Ob}(\mathbf{K})$ is numbered by Σ, and write $\mathrm{Ob}(\mathbf{K}) = \{X_\sigma \mid \sigma \in \Sigma\}$. We say $\mathbf{K}$ is **purely ordered** if there exist distinguished morphisms $\lambda_{\beta\alpha} : X_\alpha \to X_\beta$ and $\mu_{\alpha\beta} : X_\beta \to X_\alpha$ for every ordered pair $\alpha \leq \beta$ in Σ, and satisfy the conditions

$$\mu_{\alpha\beta}\lambda_{\beta\alpha} = 1_\alpha \quad (\alpha \leq \beta)\,, \tag{2.1}$$

$$\lambda_{\gamma\beta}\lambda_{\beta\alpha} = \lambda_{\gamma\alpha} \quad (\alpha \leq \beta \leq \gamma)\,, \tag{2.2}$$

$$\mu_{\alpha\beta}\mu_{\beta\gamma} = \mu_{\alpha\gamma} \quad (\alpha \leq \beta \leq \gamma)\,. \tag{2.3}$$

Here 1_σ is the identity element in $\mathrm{End}_{\mathbf{K}}(X_\sigma)$. We also put $\theta^\alpha_\beta \stackrel{\mathrm{def}}{=} \lambda_{\beta\alpha}\mu_{\alpha\beta} \in \mathrm{End}_{\mathbf{K}}(X_\beta)$, which is an idempotent element. A category which is equivalent to a purely ordered category is called **ordered**. In particular, if $\mathbf{K}$ is equivalent to an ordered category whose index poset Σ of $\mathbf{K}$ is totally ordered, then we say $\mathbf{K}$ is a **totally ordered category**.

Next we give the definition of an **$*$-category**. If there is a map $\mathrm{Mor}(\mathbf{K}) \ni P \mapsto P^* \in \mathrm{Mor}(\mathbf{K})$ of morphisms such that

$$P : X \to Y \implies P^* : Y \to X,$$
$$P^{**} = P, \quad (PQ)^* = Q^*P^*,$$

we say $\mathbf{K}$ is a **$*$-category with the involution $*$**. Further, if $\mathbf{K}$ is ordered and the involution $*$ satisfies the additional condition

$$\lambda^*_{\beta\alpha} = \mu_{\alpha\beta} \quad (\alpha \leq \beta)\,,$$

we say $\mathbf{K}$ is an **ordered $*$-category with the involution $*$**.

In the sequel we fix an ordered $*$-category $\mathbf{K}$ with an index poset Σ and an involution $*$, and let $\mathrm{Ob}(\mathbf{K}) = \{X_\sigma \mid \sigma \in \Sigma\}$. For abbreviation we put $\Gamma_\sigma \stackrel{\mathrm{def}}{=} \mathrm{End}_{\mathbf{K}}(X_\sigma)$ for $\sigma \in \Sigma$.

Example 2.1. Let $\mathbf{A}$ be the category of Hilbert spaces over the complex number field $\mathbb{C}$ and bounded operators. Then the usual adjoint operation $*$ with respect to the inner products defines an involution on $\mathbf{A}$, and $\mathbf{A}$ becomes an (totally) ordered $*$-category with the involution $*$.

In fact, let $\mathbf{A}_0$ be the category whose objects are given by $\mathrm{Ob}(\mathbf{A}_0) = \{V_n = \mathbb{C}^n \mid n \in \mathbb{N}\} \coprod \{V_\infty = \ell^2\}$ ($\mathbb{N} = \{0, 1, 2, \dots\}$) and they are equipped with the standard inner product. We define the operator λ_{nm} to be the standard embedding and μ_{mn} the standard projection for $m \geq n$. Namely, we put

$$\begin{aligned}\lambda_{nm}(x_1, \dots, x_m) &= (x_1, \dots, x_m, 0, \dots, 0),\\ \mu_{mn}(x_1, \dots, x_n) &= (x_1, \dots, x_m).\end{aligned}$$

Then $\mathbf{A}_0$ becomes a purely ordered category, and hence $\mathbf{A}$ is ordered since $\mathbf{A}$ is equivalent to $\mathbf{A}_0$. □

We make use of representation theory of ordered categories to prove the positive definiteness of the Laplacian $\Delta_{\mathbf{K}}$. In fact, the desired result is obtained from the irreducible decomposition of certain modules. In the following subsections we give necessary results on the representations of ordered categories and semigroups respectively.

2.1 Representations of ordered categories

A **(linear) representation** of a category $\mathbf{K}$ is by definition a covariant functor ρ from $\mathbf{K}$ to $\mathbf{A}$, the category of Hilbert spaces and bounded operators as we defined in Example 2.1. If $\mathbf{K}$ is a $*$-category and ρ satisfies

$$\rho(P^*) = \rho(P)^*$$

for every morphism $P \in \mathrm{Mor}(\mathbf{K})$, we say that ρ is a **$*$-representation** of $\mathbf{K}$. For a representation ρ of an ordered category $\mathbf{K}$, we denote by ρ_σ the **subordinate representation** of Γ_σ on $\rho(X_\sigma)$ for $\sigma \in \Sigma$, which is a semigroup homomorphism defined by

$$\rho_\sigma : \Gamma_\sigma \ni P \mapsto \rho(P) \in \mathrm{End}_{\mathbf{A}}(\rho(X_\sigma)).$$

A representation τ of $\mathbf{K}$ is a **subrepresentation** of ρ if $\tau(X)$ is a closed subspace of $\rho(X)$ for every $X \in \mathrm{Ob}(\mathbf{K})$ and $\tau(P) = \rho(P)\big|_{\tau(X)}$ for every $P : X \to Y$. If ρ has no nontrivial subrepresentation, we say ρ is **irreducible**.

For a given representations ρ, ρ' of $\mathbf{K}$, a family

$$T = \{T_\sigma : \rho(X_\sigma) \to \rho'(X_\sigma) \mid \sigma \in \Sigma\} \subset \mathrm{Mor}(\mathbf{A})$$

of bounded operators is called an **intertwiner of ρ and ρ'** if

$$\rho'(P)T_\alpha = T_\beta \rho(P)$$

for all $\alpha, \beta \in \Sigma$ and all $P : X_\alpha \to X_\beta$. We say T is **invertible** if T_σ is invertible whenever $\rho(X_\sigma) \neq 0$. Two representations ρ, ρ' of $\mathbf{K}$ are said to be **equivalent** if there is an invertible intertwiner T between them. We denote by $\widehat{\mathbf{K}}$ the set of equivalence classes of irreducible $*$-representations of $\mathbf{K}$. The following fact is remarkable.

Proposition 2.1 ([Ne, Lemma 3.4.3])**.** *Let ρ be a representation of an ordered category $\mathbf{K}$. Then the following conditions are equivalent.*

(a) ρ is irreducible.

(b) Every nonzero subordinate representation ρ_σ ($\sigma \in \Sigma$) is irreducible.

□

For an ordered pair $\alpha \leq \beta$, we define an embedding

$$U_{\beta\alpha} : \Gamma_\alpha \ni P \mapsto \lambda_{\beta\alpha} P \mu_{\alpha\beta} \in \Gamma_\beta, \tag{2.4}$$

which is indeed a semigroup homomorphism because of the identity (2.1). For a representation π of Γ_β, we can define the representation $F_{\alpha\beta}\pi$ of Γ_α by

$$F_{\alpha\beta}\pi(P) \stackrel{\text{def}}{=} \pi(U_{\beta\alpha}(P)) \quad (P \in \Gamma_\alpha), \tag{2.5}$$

where the representation space of $F_{\alpha\beta}\pi$ is $\operatorname{im} \pi(\theta^\alpha_\beta)$.

Let $R = \{R_\sigma \in \widehat{\Gamma}_\sigma\}_{\sigma \in \Sigma}$ be a set of irreducible representations. We call R a **compatible system** if

$$R_\alpha \cong F_{\alpha\beta} R_\beta \tag{2.6}$$

for all ordered pair $\alpha \leq \beta$. The following fact is crucial.

Proposition 2.2 ([Ne, Proposition 3.4.11])**.** *For any irreducible representation $\rho \in \widehat{\mathbf{K}}$ of $\mathbf{K}$, the system $R_\rho = \{\rho_\sigma\}_{\sigma \in \Sigma}$ is a compatible system. Conversely, for any given compatible system $R = \{R_\sigma\}_{\sigma \in \Sigma}$, there is a unique irreducible representation $\rho \in \widehat{\mathbf{K}}$ such that $\rho_\sigma \cong R_\sigma$ for all $\sigma \in \Sigma$.* □

2.2 Representations of semigroups

Since we deal with the subordinate representations of a given representation of a category, we prepare several notions and summrize basic facts on this subject.

Let Γ be a finite semigroup with an involution $*$. A homomorphism $\pi : \Gamma \to \mathrm{End}(V)$ is called a **$*$-representation of Γ** if $\pi(\gamma^*) = \pi(\gamma)^*$ for every $\gamma \in \Gamma$. It is immediate to see that any finite dimensional $*$-representation of Γ is **completely reducible**.

Standard notion in representation theory of groups are naturally imported to that of $*$-semigroups with a slight modification, and the both theories are almost parallel to some extent. Roughly speaking, the operation $g \mapsto g^{-1}$ is replaced by $\gamma \mapsto \gamma^*$. For instance, when a representation (π, V) of Γ is given, the **contragradient representation** (π^*, V^*) of (π, V) is defined by

$$(\pi^*(\gamma)F)(v) \stackrel{\mathrm{def}}{=} F(\pi(\gamma^*)v) \quad (F \in V^*, v \in V). \tag{2.7}$$

Let (π_1, V), (π_2, W) be $*$-representations of Γ. A linear map $T : V \to W$ is called an (Γ-)**intertwiner** of (π_1, V) and (π_2, W) if the equality

$$T\pi_1(\gamma) = \pi_2(\gamma)T$$

holds for every $\gamma \in \Gamma$. We say π_1 and π_2 are **equivalent** and write $\pi_1 \cong \pi_2$ if there exists an invertible intertwiner of (π_1, V) and (π_2, W). We also denote by $\mathrm{Hom}_\Gamma(V, W)$ the set of Γ-intertwiners from V to W.

Remark 2.1. When two representations (π_1, V) and (π_2, W) of Γ are given, the space $\mathrm{Hom}(V, W)$ naturally becomes a Γ-module by

$$\varpi(\gamma)P \stackrel{\mathrm{def}}{=} \pi_1(\gamma)P\pi_2(\gamma)^*,$$

and $\mathrm{Hom}_\Gamma(V, W)$ is a submodule of $\mathrm{Hom}(V, W)$ as well as $\mathrm{Hom}(V, W)^\Gamma$, the subspace consisting of Γ-invariants. But these are not equal in general: $\mathrm{Hom}_\Gamma(V, W) \neq \mathrm{Hom}(V, W)^\Gamma$.

By a similar discussion as in the group case, we have the following **Schur's lemma** for $*$-semigroups.

Lemma 2.3 (Schur's lemma)**.** *Let (π_1, V), (π_2, W) be irreducible $*$-representations of a finite $*$-semigroup Γ. Then we have*

$$\mathrm{Hom}_\Gamma(V, W) \cong \begin{cases} \mathbb{C} & \pi_1 \cong \pi_2, \\ 0 & \pi_1 \not\cong \pi_2. \end{cases} \tag{2.8}$$

□

The following basic fact is a direct conclusion of the lemma above.

Proposition 2.4 (Irreducible decomposition theorem)**.** *Let (π, V) be a finite dimensional $*$-representation of a finite $*$-semigroup Γ. Then we have*

$$V \cong \sum_{\pi \in \widehat{\Gamma}}^{\oplus} \mathrm{Hom}_{\Gamma}(W_\pi, V) \otimes W_\pi, \tag{2.9}$$

where W_π is an irreducible Γ-module corresponding to $\pi \in \widehat{\Gamma}$. □

3 Positive definiteness of Laplacians

In this section we prove the positive definiteness of Laplacians attached to totally ordered categories. In the sequel we suppose that $\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$ is finite for any $\alpha, \beta \in \Sigma$, that is, $\mathrm{Ob}(\mathbf{K}) = \mathrm{Ob}_o(\mathbf{K})$. The semigroup $\Gamma_\alpha \times \Gamma_\beta$ naturally acts on the set $\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$ by

$$(a, b).P \stackrel{\text{def}}{=} aPb^*$$

for $a \in \Gamma_\alpha$, $b \in \Gamma_\beta$ and $P \in \mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$. We denote by $L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha))$ the induced $\Gamma_\alpha \times \Gamma_\beta$-module consisting of L^2-functions on $\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$ with respect to the counting measure on $\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$. Analogous to the well-known decomposition theorem

$$L(G) \cong \sum_{\pi \in \widehat{G}}^{\oplus} \pi^* \boxtimes \pi \quad \text{(as a } G \times G\text{-module)} \tag{3.1}$$

of regular representation $L(G)$ of a finite group G, we show a decomposition theorem for $L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha))$ when α and β are **comparable**.

Proposition 3.1. *Let $\mathbf{K}$ be an ordered $*$-category. Take $\alpha, \beta \in \Sigma$ such that $\alpha \leq \beta$ and suppose that $\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$ is finite. Then we have the following decomposition*

$$L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)) \cong \sum_{\rho \in \widehat{\mathbf{K}}}^{\oplus} \rho_\alpha^* \boxtimes \rho_\beta \tag{3.2}$$

as a $\Gamma_\alpha \times \Gamma_\beta$-module.

Proof. Put $M \stackrel{\text{def}}{=} \mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$ for abbreviation. Let $(\tau, L(M))$ be the given representation

$$(\tau(a, b)F)(P) \stackrel{\text{def}}{=} F(a^* P b) \quad (a \in \Gamma_\alpha, b \in \Gamma_\beta, P \in M) \tag{3.3}$$

of the semigroup $\Gamma_\alpha \times \Gamma_\beta$. First we decompose $L(M)$ as a Γ_β-module by using (2.9):

$$L(M) \cong \sum_{\pi \in \widehat{\Gamma}_\beta}^{\oplus} \mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M)) \otimes W_\pi \tag{3.4}$$

where we denote by W_π the irreducible Γ_β-module corresponding to π. In the sequel we argue the equivalence for each component.

Fix $\pi \in \widehat{\Gamma}_\beta$. Since $\mathbf{K}$ is ordered, there exists a unique irreducible representation $\rho \in \widehat{\mathbf{K}}$ of $\mathbf{K}$ such that its subordinate representation satisfies the equivalence $\rho_\beta \cong \pi$ (Proposition 2.2). Thus it is enough to prove that $\mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$ is equivalent to ρ_α^* as a Γ_α-module. Remark that the subordinate representation ρ_α is equivalent to the representation $\sigma \stackrel{\mathrm{def}}{=} F_{\alpha\beta}\pi$ on the space $\operatorname{im}\pi(\theta_\beta^\alpha) \subset W_\pi$.

We show that the following map gives an invertible intertwiner of $\mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$ and $\operatorname{im}\pi(\theta_\beta^\alpha)^*$:

$$(T\psi)(x) \stackrel{\mathrm{def}}{=} (\psi x)(\mu_{\alpha\beta}) \quad \big(x \in \operatorname{im}\pi(\theta_\beta^\alpha) \subset W_\pi\big) \tag{3.5}$$

for $\psi \in \mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$. In fact, we have

$$\begin{aligned}
(\sigma^*(a)T\psi)(x) &= (T\psi)(\sigma(a^*)x) \\
&= (\psi\pi(\lambda_{\beta\alpha}a^*\mu_{\alpha\beta})x)(\mu_{\alpha\beta}) \\
&= \rho(1, \lambda_{\beta\alpha}a^*\mu_{\alpha\beta})(\psi x)(\mu_{\alpha\beta}) \quad (\because \psi \text{ is a } \Gamma_\beta\text{-intertwiner}) \\
&= (\psi x)(\mu_{\alpha\beta} \cdot \lambda_{\beta\alpha}a^*\mu_{\alpha\beta}) \\
&= (\psi x)(a^*\mu_{\alpha\beta}) \quad (\because \mu_{\alpha\beta}\lambda_{\beta\alpha} = 1_\alpha) \\
&= (\tau(a,1)\psi x)(\mu_{\alpha\beta}) \\
&= (T\tau(a,1)\psi)(x)
\end{aligned}$$

as we required.

At last we see that T is injective (which automatically implies that T is bijective since the finiteness of dimension). For $\psi \in \mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$, we have

$$\begin{aligned}
T\psi = 0 &\implies (\psi x)(\mu_{\alpha\beta}) = 0 \quad \big(\forall x \in \operatorname{im}\pi(\theta_\beta^\alpha)\big) \\
&\implies (\psi x)(\mu_{\alpha\beta}b) = 0 \quad \big(\forall x \in \operatorname{im}\pi(\theta_\beta^\alpha), \forall b \in \Gamma_\beta\big),
\end{aligned}$$

which implies that $\psi \equiv 0$ because it is easy to check that $\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha) = \mu_{\alpha\beta}\Gamma_\beta$.

Hence T is an invertible intertwiner and we have

$$\mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M)) \cong \operatorname{im}\pi(\theta_\beta^\alpha)^* = F_{\alpha\beta}\pi^* \cong F_{\alpha\beta}\rho_\beta^* \cong \rho_\alpha^*,$$

which implies the desired decomposition formula. □

As a corollary of Proposition 3.1, we have immediately the following key formula.

Theorem 3.2. *Let* $\mathbf{K}$ *be a totally ordered* $*$*-category. Suppose that every set of morphisms is finite. Then we have*

$$\#\,\mathrm{Mor}_{\mathbf{K}}(X_\alpha, X_\beta) = \sum_{\rho \in \widehat{\mathbf{K}}} \dim \rho_\alpha \dim \rho_\beta \tag{3.6}$$

for any $\alpha, \beta \in \Sigma$.

Proof. In fact, this formula is immediately obtained by looking at the dimensions of each modules in (3.2). □

Finally we have the desired conclusion.

Theorem 3.3 (Positive definiteness of the Laplacian)**.** *Let* $\mathbf{K}$ *be a totally ordered* $*$*-category. Suppose that every set of morphisms is finite. Then the Laplacian* $\Delta_{\mathbf{K}}$ *of the category* $\mathbf{K}$ *is positive definite.*

Proof. It is enough to prove the positivity of the principal minor determinants of $\Delta_{\mathbf{K}}$. This is immediate from the Cauchy-Lagrange identity

$$\det\,(\boldsymbol{a}_i \cdot \boldsymbol{a}_j)_{1 \le i,j \le m} = \sum_{\#\boldsymbol{k}=m} \det(a_{k_i,j})^2_{1 \le i,j \le m} \tag{3.7}$$

for n-dimensional vectors $\boldsymbol{a}_i = (a_{ij})_{1 \le j \le n} \in \mathbb{C}^n$ $(1 \le i \le m \le n)$. □

Now the Cauchy-Schwarz inequality is obtained by seeing the positivity of 2-minor determinant of $\Delta_{\mathbf{K}}$.

Corollary 3.4 (The Cauchy-Schwarz inequality)**.** *A totally ordered* $*$*-category* $\mathbf{K}$ *satisfies the Cauchy-Schwarz type inequality*

$$\#\,\mathrm{Mor}_{\mathbf{K}}(X_\alpha, X_\beta)\#\,\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha) \le \#\,\mathrm{End}_{\mathbf{K}}(X_\alpha)\#\,\mathrm{End}_{\mathbf{K}}(X_\beta) \tag{3.8}$$

for every $X_\alpha, X_\beta \in \mathrm{Ob}_o(\mathbf{K})$. □

Example 3.1. We consider the category $\mathrm{Mod}(\mathbb{F}_q)$ of finite dimensional $\mathbb{F}_q$-linear spaces and $\mathbb{F}_q$-linear maps. The structure of an ordered category is introduced just the same as $\mathbf{A}$ in Example 2.1. Thus we have Theorem 2 in [KuST] again: the Laplacian $\Delta_{\mathrm{Mod}(\mathbb{F}_q)}$ of $\mathrm{Mod}(\mathbb{F}_q)$ is positive definite. □

Example 3.2. Let $\mathbf{Ab}_0$ be the category consisting of finite abelian groups, which is a subcategory of $\mathbf{Ab}$. The set $\mathrm{Ob}(\mathbf{Ab}_0)$ itself naturally becomes a partially ordered set with respect to the inclusion map. For every pair $H < G$ of finite abelian groups, there exist an abelian group H' such that $G \cong H \times H'$. Thus $\lambda_{GH} : H \to G$ and $\mu_{HG} : G \to H$ are defined by

$$\lambda_{GH}(h) \stackrel{\mathrm{def}}{=} h \quad (\text{inclusion map}),$$
$$\mu_{HG}(g) \stackrel{\mathrm{def}}{=} h_g \quad \big(G \ni g \mapsto (h_g, h'_g) \in H \times H'\big).$$

For $G_1, G_2 \in \mathrm{Ob}(\mathbf{Ab}_0)$ and $f : G_1 \to G_2$, $f^* : \widehat{G}_2 \to \widehat{G}_1$ is defined by

$$(f^*(\lambda))(g_1) \stackrel{\mathrm{def}}{=} \lambda(f(g_1)) \quad \Big(\lambda \in \widehat{G}_2, g_1 \in G_1\Big).$$

By the duality of finite abelian groups, this map defines an involution. Thus the category $\mathbf{Ab}_0$ is an ordered $*$-category. Hence, if we take an inductive system $A = \{A_j\}_{j \geq 0}$ of finite abelian groups, then the Laplacian $\Delta_A \stackrel{\mathrm{def}}{=} (\#\,\mathrm{Mor}_{\mathbf{Ab}}(A_i, A_j))_{i,j}$ attached to A is positive definite.

We remark that Theorem 3.3 is not applicable to the category $\mathbf{Ab}_0$ since $\mathbf{Ab}_0$ is not **totally** ordered. □

4 Spectrum of the category PB

In this section we treat the category **PB** of partial bijections, which is originally introduced to study the representation of full symmetric group $\mathfrak{S}_\omega$ of all bijections on $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ (see [Ne]). Let us recall the definition of **PB**. An object in **PB** is a fintie set $[n] = \{1, 2, \ldots, n\}$. A morphism from $[m]$ to $[n]$ is given by a **partial bijection**, that is, the triplet $(\varphi, D_\varphi, R_\varphi)$, where $D_\varphi \subset [m]$ and $R_\varphi \subset [n]$ have the same cardinality and $\varphi : D_\varphi \to R_\varphi$ is a bijection. For given two morphisms $\varphi : [l] \to [m]$ and $\psi : [m] \to [n]$, the composition $\psi\varphi : [l] \to [n]$ of φ and ψ is defined to be a partial bijection from $D_{\psi\varphi} \stackrel{\mathrm{def}}{=} \varphi^{-1}(R_\varphi \cap D_\psi)$ to $R_{\psi\varphi} \stackrel{\mathrm{def}}{=} \psi(R_\varphi \cap D_\psi)$. The maps $\lambda_{nm}[m] \to [n]$ and $\mu_{mn} : [n] \to [m]$ are defined by

$$\lambda_{nm} : [m] \ni x \mapsto x \in [m] \subset [n],$$
$$\mu_{mn} : [n] \supset [m] \ni x \mapsto x \in [m],$$

for $n \leq m$. For a given partial bijection $\varphi : D_\varphi \to R_\varphi$, φ^* is defined to be the partial bijection $\varphi^* : R_\varphi \ni x \mapsto \varphi^{-1}(x) \in D_\varphi$.

Proposition 4.1. *The category* **PB** *is a totally ordered* $*$*-category, and hence, the Laplacian* $\Delta_{\mathbf{PB}}$ *is positive definite.* □

By an elementary combinatorial calculation we see that the number of morphisms are given by

$$d_{mn} \overset{\text{def}}{=} \#\,\mathrm{Mor}_{\mathbf{PB}}([m],[n]) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!. \tag{4.1}$$

Irreducible representations of **PB** are labeled by **Young diagrams**. Denote by ρ^λ the attached irreducible representation of **PB**, and by ρ_n^λ the subordinate representation which is the restriction of ρ^λ to Γ_n.

Proposition 4.2 ([Ne])**.** *We have*

$$\dim \rho_n^\lambda = \binom{n}{|\lambda|} \dim \lambda \tag{4.2}$$

for any $\lambda \in \mathbb{Y}$*. Here we denote by* $\dim \lambda$ *the dimension of the irreducible* $\mathfrak{S}_{|\lambda|}$*-module corresponding to* λ*. We remark that* $\binom{n}{k} = 0$ *if* $k > n$*.*

By using Theorem 3.2 and the well-known fact

$$\sum_{|\lambda|=k} (\dim \lambda)^2 = k!,$$

we have in fact

$$\begin{aligned} \#\,\mathrm{Mor}_{\mathbf{PB}}([m],[n]) &= \sum_{\lambda \in \mathbb{Y}} \left\{ \binom{m}{|\lambda|} \dim \lambda \times \binom{n}{|\lambda|} \dim \lambda \right\} \\ &= \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k! \end{aligned} \tag{4.3}$$

for $m, n \in \mathbb{N}$, which (of course) coincides with the result (4.1) of a combinatorial calculation.

We write the Laplacian $\Delta = \Delta_{\mathbf{PB}}$ briefly. The main concern of this section is a study of the spectrum of the Laplacian Δ and attached spectral zeta function $\zeta_\Delta(s)$. We put $\Delta_N = (d_{ij})_{0 \le i,j \le N}$, the principal N-minor of the Laplacian Δ. Let us denote by $\lambda_{N,j}$ $(0 \le j \le N)$ the $(j+1)$-th eigenvalue of Δ_N, that is,

$$0 < \lambda_{N,0} \le \lambda_{N,1} \le \cdots \le \lambda_{N,N}.$$

One of the main purpose is to prove the following theorem concerning the existence of the spectrum of Δ.

Theorem 4.3 (Existence of the spectrum)**.** *For every* $k \ge 0$*, there exists the limit* $\lambda_k \overset{\text{def}}{=} \lim_{N\to\infty} \lambda_{N,k} > 0$*.*

4.1 Preliminary calculations

Proposition 4.4. *The inverse Δ_N^{-1} of the principal N-minor Δ_N is given by*

$$\Delta_N^{-1} = \left((-1)^{i+j} \sum_{k=0}^{N} \binom{k}{i} \binom{k}{j} \frac{1}{k!} \right)_{0 \le i,j \le N}. \tag{4.4}$$

Proof. Denote by $f_{ml}(N)$ the cofactors of Δ_N. Let

$$\begin{aligned} &i_{m,1} < i_{m,2} < \cdots < i_{m,N}, \quad \boldsymbol{i}_m = \{i_{m,1}, i_{m,2}, \ldots, i_{m,N}\} = [0,N] \setminus \{m\}, \\ &j_{l,1} < j_{l,2} < \cdots < j_{l,N}, \quad \boldsymbol{j}_l = \{j_{l,1}, j_{l,2}, \ldots, j_{l,N}\} = [0,N] \setminus \{l\}. \end{aligned}$$

Here we put $[0,N] \stackrel{\text{def}}{=} \{0,1,2,\ldots,N\}$. Remarking that the number d_{ij} of morphisms is given by

$$d_{ij} = \sum_{k=0}^{N} \frac{i^{\underline{k}} j^{\underline{k}}}{k!},$$

we have

$$\begin{aligned} f_{ml}(N) &= \det(d_{i_{m,\alpha} j_{l,\beta}})_{1 \le \alpha,\beta \le N} \\ &= \sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma d_{i_{m,\sigma(1)} j_{l,1}} d_{i_{m,\sigma(2)} j_{l,2}} \cdots d_{i_{m,\sigma(N)} j_{l,N}} \\ &= \sum_{0 \le k_1,\ldots,k_N \le N} \sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma \frac{i_{m,\sigma(1)}{}^{\underline{k_1}} j_{l,1}{}^{\underline{k_1}}}{k_1!} \cdots \frac{i_{m,\sigma(N)}{}^{\underline{k_N}} j_{l,N}{}^{\underline{k_N}}}{k_N!} \\ &= \sum_{0 \le k_1,\ldots,k_N \le N} \frac{j_{l,1}{}^{\underline{k_1}}}{k_1!} \cdots \frac{j_{l,N}{}^{\underline{k_N}}}{k_N!} \sum_{\sigma \in \mathfrak{S}_N} (-1)^\sigma i_{m,\sigma(1)}{}^{\underline{k_1}} \cdots i_{m,\sigma(N)}{}^{\underline{k_N}} \\ &= \sum_{0 \le k_1,\ldots,k_N \le N} \frac{j_{m,1}{}^{\underline{k_1}}}{k_1!} \cdots \frac{j_{l,N}{}^{\underline{k_N}}}{k_N!} \det(i_{m,\alpha}{}^{\underline{k_{l,\beta}}})_{1 \le \alpha,\beta \le N}. \end{aligned}$$

Since the factor $\det(i_{m,\alpha}{}^{\underline{k_\beta}})_{1 \le \alpha,\beta \le N}$ does not vanish only if $k_1, \ldots, k_N$ are distinct members, we have

$$f_{ml}(N) = \sum_{p=0}^{N} \sum_{\sigma \in \mathfrak{S}_N} \frac{j_{l,1}{}^{\underline{k_{p,\sigma(1)}}}}{k_{p,\sigma(1)}!} \cdots \frac{j_{l,N}{}^{\underline{k_{p,\sigma(N)}}}}{k_{p,\sigma(N)}!} \det(i_{m,\alpha}{}^{\underline{k_{p,\sigma(\beta)}}})_{1 \le \alpha,\beta \le N}$$

where $k_{p,j}$'s are given by the condition

$$k_{p,1} < k_{p,2} < \cdots < k_{p,N}, \quad \boldsymbol{k} = \{k_{p,1}, k_{p,2}, \ldots, k_{p,N}\} = [N] \setminus \{p\}.$$

Notice that $\det(i_{m,\alpha}{}^{\underline{k_{p,\sigma(\beta)}}})_{1\leq\alpha,\beta\leq N} = (-1)^{\sigma}\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}$. Thus it follows that

$$\begin{aligned}
f_{ml}(N) &= \sum_{p=0}^{N}\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}\sum_{\sigma\in\mathfrak{S}_N}(-1)^{\sigma}\frac{j_{l,1}{}^{\underline{k_{p,\sigma(1)}}}}{k_{p,\sigma(1)}!}\cdots\frac{j_{l,N}{}^{\underline{k_{p,\sigma(N)}}}}{k_{p,\sigma(N)}!}\\
&= \sum_{p=0}^{N}\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}\det\left(\frac{j_{l,\alpha}{}^{\underline{k_{p,\beta}}}}{k_{\beta}!}\right)_{1\leq\alpha,\beta\leq N}\\
&= \frac{1}{0!1!\cdots N!}\sum_{p=0}^{N}\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}\det(j_{l,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}\times p!.
\end{aligned}$$

Therefore it is enough to calculate the determinant $\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}$ for each m, p.

When $m > p$, it is easy to see that $\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N} = 0$. When $m = p$, the matrix $(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}$ becomes triangular and we immediately have $\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N} = (0!1!\cdots N!)/m!$. Finally, if $m < p$ then it follows from the block decomposition that

$$\begin{aligned}
&\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N}\\
&=\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq m}\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{m<\alpha,\beta\leq p}\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{p<\alpha,\beta\leq N}\\
&=\det((\alpha-1)^{\underline{\beta-1}})_{1\leq\alpha,\beta\leq m}\det(\alpha^{\underline{\beta-1}})_{m<\alpha,\beta\leq p}\det(\alpha^{\underline{\beta}})_{p<\alpha,\beta\leq N}.
\end{aligned}$$

The matrices in the first and the third determinants are triangular, and we see

$$\det((\alpha-1)^{\underline{\beta-1}})_{1\leq\alpha,\beta\leq m} = 0!1!\cdots(m-1)!,\quad \det(\alpha^{\underline{\beta}})_{p<\alpha,\beta\leq N} = (p+1)!\cdots N!.$$

The second determinant is given by

$$\det(\alpha^{\underline{\beta-1}})_{m<\alpha,\beta\leq p} = (m+1)!\cdots p!\times\det(1/(\alpha-\beta+1)!)_{1\leq\alpha,\beta\leq p-m}$$

where we put $1/k! = 0$ for negative integer k (interpreted as the values of $1/\Gamma(z)$). Notice that the determinant $\det(1/(\alpha-\beta+1)!)_{1\leq\alpha,\beta\leq p-m}$ depends only on the difference $p-m$, and it is proved by induction that $\det(1/(\alpha-\beta+1)!)_{1\leq\alpha,\beta\leq r} = 1/r!$. Therefore we have

$$\det(i_{m,\alpha}{}^{\underline{k_{p,\beta}}})_{1\leq\alpha,\beta\leq N} = \frac{0!1!\cdots N!}{m!(p-m)!},$$

which is also true when $p \leq m$. Thus we have

$$\begin{aligned}
f_{ml}(N) &= \frac{1}{0!1!\cdots N!}\sum_{p=0}^{N}\frac{0!1!\cdots N!}{m!(p-m)!}\times\frac{0!1!\cdots N!}{l!(p-l)!}\times p!\\
&= 0!1!\cdots N!\sum_{p=0}^{N}\frac{p!}{m!(p-m)!l!(p-l)!}\\
&= 0!1!\cdots N!\sum_{p=0}^{N}\frac{1}{p!}\binom{p}{m}\binom{p}{l}.
\end{aligned}$$

Since we have in particular $\det \Delta_N = f_{N+1,N+1}(N+1) = 0!1!\cdots N!$, we conclude that

$$\Delta_N^{-1} = \left((-1)^{i+j} \frac{f_{ji}(N)}{\det \Delta_N} \right)_{i,j} = \left((-1)^{i+j} \sum_{p=0}^{N} \frac{1}{p!} \binom{p}{i} \binom{p}{j} \right)_{i,j}.$$

□

Theorem 4.5. *We have*

$$\lim_{N\to\infty} \sum_{k=0}^{N} \binom{k}{i} \binom{k}{j} \frac{1}{k!} = \frac{d_{ij}e}{i!j!}, \tag{4.5}$$

or equivalently, the inverse of the Laplacian Δ *is given by*

$$\Delta^{-1} = \left(\frac{(-1)^{i+j} d_{ij} e}{i!j!} \right)_{i,j\geq 0}. \tag{4.6}$$

Proof. We should check the equality

$$(-1)^{i+j} \sum_{k=0}^{\infty} \binom{k}{i} \binom{k}{j} \frac{1}{k!} = \frac{(-1)^{i+j} d_{ij} e}{i!j!}. \tag{4.7}$$

In order to achieve this, we introduce the following generating functions

$$\mathcal{F}_N(x,y) \stackrel{\text{def}}{=} \sum_{i,j=0}^{N} \left\{ (-1)^{i+j} \sum_{k=0}^{N} \binom{k}{i} \binom{k}{j} \frac{1}{k!} \right\} x^i y^j,$$
$$\mathcal{F}(x,y) \stackrel{\text{def}}{=} \sum_{i,j\geq 0} \left\{ \frac{(-1)^{i+j} d_{ij} e}{i!j!} \right\} x^i y^j.$$

Then it is elementary to see that

$$\mathcal{F}_N(x,y) = \sum_{k=0}^{N} \frac{(1-x)^k (1-y)^k}{k!},$$
$$\mathcal{F}(x,y) = e^{(1-x)(1-y)}.$$

Thus we have

$$\lim_{N\to\infty} \mathcal{F}_N(x,y) = \mathcal{F}(x,y),$$

which converges absolutely and uniformly on any compact subset of $\mathbb{C}$. The conclusion is obtained by the comparison of coefficients. □

We show that the limit value $\operatorname{tr} \Delta^{-m} = \lim_{N\to\infty} \operatorname{tr} \Delta_N^{-m}$, which is regarded as a special value of the spectral zeta function $\zeta_\Delta(s)$ of Δ, exists for every $m \geq 1$ as follows.

Theorem 4.6. *We have*

$$\operatorname{tr}\Delta^{-m} = \sum_{k_1,\dots,k_m\geq 0} \frac{1}{k_1!\dots k_m!}\binom{k_1+k_2}{k_1}\binom{k_2+k_3}{k_2}\cdots\binom{k_m+k_1}{k_m}. \tag{4.8}$$

Proof. In fact, for any $N \geq 0$ we have

$$\begin{aligned}
\operatorname{tr}\Delta_N^{-m} &= \sum_{i_1,\dots,i_m=0}^{N}\left(\sum_{k_1=0}^{N}\binom{k_1}{i_1}\binom{k_1}{i_2}\frac{1}{k_1!}\right)\cdots\left(\sum_{k_m=0}^{N}\binom{k_m}{i_m}\binom{k_1}{i_1}\frac{1}{k_m!}\right)\\
&= \sum_{k_1,\dots,k_m=0}^{N}\frac{1}{k_1!\cdots k_m!}\left(\sum_{i_1=0}^{N}\binom{k_m}{i_1}\binom{k_1}{i_i}\right)\cdots\left(\sum_{i_m=0}^{N}\binom{k_{m-1}}{i_m}\binom{k_m}{i_m}\right)\\
&= \sum_{k_1,\dots,k_m=0}^{N}\frac{1}{k_1!\cdots k_m!}\binom{k_m+k_1}{k_1}\cdots\binom{k_{m-1}+k_m}{k_m},
\end{aligned}$$

where the last equality holds by the convolution formula

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}.$$

□

In particular, we have

$$\operatorname{tr}\Delta^{-1} = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}^2 = \sum_{k=0}^{\infty}\frac{1}{k!}\binom{2k}{k}, \tag{4.9}$$

$$\operatorname{tr}\Delta^{-2} = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}^3. \tag{4.10}$$

Since every power sum $\sum_{j=0}^{N}\lambda_{N,j}^{-m} = \operatorname{tr}\Delta_N^{-m}$ converges to a certain finite value as $N\to\infty$, the value

$$e_d(\lambda_N) \overset{\text{def}}{=} \sum_{0\leq i_1<\cdots<i_d\leq N}\lambda_{N,i_1}^{-1}\cdots\lambda_{N,i_d}^{-1}$$

also converges as $N\to\infty$ because this is expressible by $\operatorname{tr}\Delta_N^{-k}$'s.

Remark 4.1. The value $\operatorname{tr}\Delta^{-1}$ is also given by

$$\begin{aligned}
\operatorname{tr}\Delta^{-1} &= e\sum_{k=0}^{\infty}\frac{d_{kk}}{(k!)^2} = e\sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{k}{l}\binom{k}{l}l!\frac{1}{(k!)^2}\\
&= e\sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\frac{1}{l!((k-l)!)^2} = e\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\\
&= \sum_{k=0}^{\infty}\left(\frac{e}{k!}\right)^2.
\end{aligned}$$

Hence we have an identity

$$\sum_{k=0}^{\infty}\left(\frac{e}{k!}\right)^2=\sum_{k=0}^{\infty}\frac{1}{k!}\binom{2k}{k},$$

which is also given as a special value of the Bessel function

$$J_0(z)={}_0F_1(1;-z^2/4)=e^{-iz}{}_1F_1(1/2;1;2iz) \tag{4.11}$$

at $z=2i$. Here ${}_1F_1(a;c;z)$ denotes the confluent hypergeometric function of Kummer's type. It is interesting to note that special values of ${}_1F_1(a;c;z)$ also appear in the calculations of zeta functions for $\mathfrak{S}_\infty$ [Ki].

We also note the numerical estimation of $\operatorname{tr}\Delta^{-1}$:

$$0<\operatorname{tr}\Delta_N^{-1}<\operatorname{tr}\Delta^{-1}=16.8439836812589880674...<17. \tag{4.12}$$

4.2 Existence of the spectral zeta function

We put

$$P_N(x)\overset{\text{def}}{=}\frac{\det(\Delta_N-x)}{\det\Delta_N}=\prod_{j=0}^{N}\left(1-\frac{x}{\lambda_{N,j}}\right)=\sum_{d=0}^{N+1}(-1)^d e_d(\lambda_N)x^d,$$

which is a normalized characteristic polynomial of Δ_N.

Proposition 4.7. *The sequence $\{P_N(x)\}_{N\geq 0}$ of polynomials converges absolutely and uniformly to a certain holomorphic function $P(x)$ on any compact subset of $\mathbb{C}$. Hence $P(x)$ gives an entire function.*

Proof. Let us prove that $\{P_N(x)\}_{N\geq 0}$ converges absolutely and uniformly on the disk $\{z\in\mathbb{C}\mid |z|\leq R\}$ for any $R>0$. For any $\varepsilon>0$, if we take $N>M\gg 1$ then we have

$$|e_d(\lambda_N)-e_d(\lambda_M)|<\frac{\varepsilon}{2R^d}$$

because $\{e_d(\lambda_N)\}_{N\geq 0}$ converges to a certain finite value. On the other hand each $e_d(\lambda_N)$ is roughly evaluated as

$$0<e_d(\lambda_N)=\sum_{i_1<\cdots<i_d}\lambda_{N,i_1}^{-1}\cdots\lambda_{N,i_d}^{-1}\leq\frac{1}{d!}\sum_{i_1,\ldots,i_d}\lambda_{N,i_1}^{-1}\cdots\lambda_{N,i_d}^{-1}\leq\frac{17^d}{d!}.$$

Here we use a rough estimation (4.12) of $\sum_i\lambda_{N,i}^{-1}<17$. It is elementary to see that

$$\sum_{d=M+1}^{N}\frac{(17R)^d}{d!}<\frac{\varepsilon}{2}$$

when M is large enough. Therefore we have

$$|P_N(x) - P_M(x)| \leq \sum_{d=0}^{M} |e_d(\lambda_N) - e_d(\lambda_M)| R^d + \sum_{d=M+1}^{N} |e_d(\lambda_N)| R^d < \varepsilon$$

for any $|x| < R$. $\square$

By approximating $P(x)$ by the polynomial $P_N(x)$ on the interval $[0, 1]$, we can check that $P_N(1) < 0$ for $N \gg 1$, which means that 0 is not an exceptional value of $P(x)$. Hence $P(x)$ has infinitely many zeros.

Proposition 4.8. *Let $\boldsymbol{a} = \{a_n\}_{n \geq 0}$ be the zeros of $P(x)$ such that $0 < |a_0| \leq |a_1| \leq \cdots$. Put $\Omega(N, R) := \big\{ |z| < R \;\big|\; P_N(z) = 0 \big\} \subset \mathbb{C}$ for $R > 0$. Then the number $\#\Omega(N, R)$ is independent of N for every $N \gg 1$ and $R \notin \boldsymbol{a}$.*

Proof. For any $R \notin \boldsymbol{a}$, put $\varepsilon = \min_{|z|=R} |P(z)| > 0$. By the uniform convergence of $\{P_N(z)\}$, we have $|P_N(x) - P(x)| < \varepsilon \leq |P(z)|$ for $N \gg 1$. The conclusion is now clear by Rouché's theorem. $\square$

Now we prove Theorem 4.3, the existence of the spectrum of Δ.

Proof of Theorem 4.3. Put $r_j = |a_j|$ for $j \geq 0$. For any $\varepsilon > 0$ and each $j \geq 0$, we have

$$\# \left(\Omega(N, r_j + \varepsilon) \setminus \Omega(N, r_j - \varepsilon) \right) \geq 1$$

for $N \gg 1$, which implies that $\{\lambda_{N,j}\}_{N > M}$ is monotone decreasing. Hence we have $\lim_{N \to \infty} \lambda_{N,j} = r_j$. Since $\lim_{N \to \infty} P(\lambda_{N,j}) = 0$ for every $j \geq 0$, we also see that every r_j is a zero of $P(x)$, which implies $r_j = a_j$. $\square$

We show the numerical estimation of first 10 eigenvalues up to 10 digits (Table 1). These values are calculated as limits of $\lambda_{N,k}$'s.

Since the series $\sum_{k=0}^{\infty} \lambda_k^{-1}$ converges, we have the

Theorem 4.9. *The spectral zeta function $\zeta_\Delta(s)$ of $\Delta = \Delta_{\mathbf{PB}}$ is well-defined, that is, there exists some small $\varepsilon > 0$ such that*

$$\zeta_\Delta(s) \stackrel{\text{def}}{=} \operatorname{tr} \Delta^{-s} = \sum_{k=0}^{\infty} \lambda_k^{-s}$$

converges absolutely and hence defines a holomorphic function in the region $\operatorname{Re}(s) > 1 - \varepsilon$. $\square$

$$
\begin{aligned}
\lambda_0 &= 0.08487190949\dots\\
\lambda_1 &= 0.2919019234\dots\\
\lambda_2 &= 0.8906738137\dots\\
\lambda_3 &= 2.607762169\dots\\
\lambda_4 &= 9.640545861\dots\\
\lambda_5 &= 46.47152499\dots\\
\lambda_6 &= 273.9773421\dots\\
\lambda_7 &= 1899.150590\dots\\
\lambda_8 &= 15101.52483\dots\\
\lambda_9 &= 135369.6103\dots
\end{aligned}
$$

Table 1: First 10 eigenvalues of $\Delta_{\mathbf{PB}}$

Now it also follows immediately the

Corollary 4.10. *The canonical product expression of $P(x)$ is given by*

$$P(x) = \prod_{k=0}^{\infty}\left(1 - \frac{x}{\lambda_k}\right). \tag{4.13}$$

□

5 Spectrum of the category $\mathrm{Mod}(\mathbb{F}_q)$

In this section we treat another interesting case of $\mathbf{K} = \mathrm{Mod}(\mathbb{F}_q)$, the category of $\mathbb{F}_q$-modules. We write the Laplacian $\Delta = \Delta_{\mathbf{K}}$ in short.

An object of $\mathrm{Mod}(\mathbb{F}_q)$ is a n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$. The number of morphisms from $\mathbb{F}_q^m$ to $\mathbb{F}_q^n$ is given by q^{mn}. We put $\Delta_N = (q^{ij})_{0\le i,j\le N}$, the principal N-minor of the Laplacian Δ. Let us denote by $\lambda_{N,j}$ $(0 \le j \le N)$ the $(j+1)$-th eigenvalue of Δ_N, that is,

$$0 < \lambda_{N,0} \le \lambda_{N,1} \le \cdots \le \lambda_{N,N}.$$

By a similar argument as in Section 4, we can prove the following existence theorem of spectrum.

Theorem 5.1 (Existence of the spectrum)**.** *For every $k \ge 0$, there exists the limit $\lambda_k \stackrel{\mathrm{def}}{=} \lim_{N\to\infty} \lambda_{N,k} > 0$.*

Thus we only give a necessary calculation concerning the inverse of the N-minor Δ_N of the Laplacian Δ.

Proposition 5.2. *Let Δ_N be the principal N-minor of the Laplacian Δ. Then we have*

$$\Delta_N^{-1} = \left((-1)^{i+j} \prod_{k=1}^{N} (1-q^{-k})^{-1} e_i^{(N)}(q;j) e_j^{(N)}(q;0) \right)_{0 \le i,j \le N}. \tag{5.1}$$

Here we define

$$e_n^{(N)}(q;i) \overset{\text{def}}{=} \sum_{\substack{S \subset [0,N] \\ \#S=n \\ i \notin S}} q^{-S}$$

where $q^{-S} \overset{\text{def}}{=} q^{-(s_1+\cdots+s_n)}$ if $S = \{s_1, \ldots, s_n\}$, and we put $[0,N] = \{0,1,2,\ldots,N\}$.

Remark 5.1. It is easy to see that

$$e_n^{(N)}(q;i) = \sum_{k=0}^{n} (-q^i)^k e_{n-k}^{(N)}(q)$$

where we put

$$e_n^{(N)}(q) \overset{\text{def}}{=} \sum_{\substack{S \subset [0,N] \\ \#S=n}} q^{-S}.$$

To calculate the cofactor of Δ_N, we need the following lemma concerning the specialization of symmetric functions.

Lemma 5.3 ([Ma1, p.44])**.** *For a Young diagram $\lambda \in \mathbb{Y}$, we denote by $s_\lambda(z_1, \ldots, z_N)$ the Schur function attached to λ defined by the Jacobi-Trudi type identity*

$$s_\lambda(z_1, \ldots, z_N) = \frac{\det(z_i^{\lambda_j+N-j})_{1\le i,j \le N}}{\det(z_i^{N-j})_{1 \le i,j \le N}}. \tag{5.2}$$

If we substitute $z_j = q^{j-1}$ for every j, then we have the following formula

$$s_\lambda(1, q, \ldots, q^{N-1}) = q^{n(\lambda)} \prod_{x \in \lambda'} \frac{1-q^{N-c(x)}}{1-q^{h(x)}}. \tag{5.3}$$

Here λ' is the conjugate diagram of λ defined by flipping λ with respect to the diagonal line. The number $n(\lambda)$ is defined by

$$n(\lambda) = \sum_{i \ge 1} (i-1)\lambda_i.$$

For each entry $x=(i,j)\in\lambda$ of i-th row and j-th column, the content $c(x)$ of x and the hook length $h(x)$ of x are defined respectively by

$$c(x)=j-i,\quad h(x)=\lambda_i+\lambda'_j-i-j+1.$$

$\square$

Proof of Proposition 5.2. Denote by $f_{ml}(N)$ the cofactors of Δ_N. For simplicity we put $x_j \stackrel{\text{def}}{=} q^j$ for $0\le j\le N$. We also let

$$i_{m,1}<i_{m,2}<\cdots<i_{m,N},\quad \boldsymbol{i}_m=\{i_{m,1},i_{m,2},\dots,i_{m,N}\}=[0,N]\setminus\{m\}.$$

First we remark that

$$(N,N-1,\dots,l+1,l-1,\dots,1,0)=(N-1,N-2,\dots,1,0)+(\overbrace{1,\dots,1}^{N-l},0,\dots,0).$$

Therefore, if we put $\lambda=(1^{N-l})$, then we have

$$\begin{aligned} f_{ml}(N)&=\det\left(x_{i_\alpha}^{\lambda_\beta+N-\beta}\right)_{1\le\alpha,\beta\le N}\\ &=s_\lambda(x_{i_1},\dots,x_{i_N})\det\left(x_{i_\alpha}^{N-\beta}\right)_{1\le\alpha,\beta\le N}\\ &=e_{N-l}(x_{i_1},\dots,x_{i_N})\prod_{1\le\alpha<\beta\le N}(x_{i_\beta}-x_{i_\alpha}) \end{aligned}$$

by the Jacobi-Trudi identity (5.2). Here we denote by $e_j(z_1,\dots,z_N)$ the j-th elementary symmetric function of variables $z_1,\dots,z_N$. Since it follows that

$$\begin{aligned} &\prod_{1\le\alpha<\beta\le N}(x_{i_\beta}-x_{i_\alpha})\\ &=\prod_{0\le\alpha<\beta\le N}(x_\beta-x_\alpha)\times\{(x_i-x_0)\cdots(x_i-x_{i-1})\cdot(x_{i+1}-x_i)\cdots(x_N-x_i)\}^{-1}\\ &=\det\Delta_N\times(-1)^m\prod_{0\le k\ne m\le N}(x_k-x_m)^{-1}, \end{aligned}$$

we have

$$\begin{aligned} \frac{f_{ml}(N)}{\det\Delta_N}&=e_{N-l}(x_{i_1},\dots,x_{i_N})\times(-1)^m\prod_{0\le k\ne m\le N}(x_k-x_m)^{-1}\\ &=e_l(x_{i_1}^{-1},\dots,x_{i_N}^{-1})\times x_{i_1}\cdots x_{i_N}\times(-1)^m\prod_{0\le k\ne m\le N}(x_k-x_m)^{-1}\\ &=e_l(x_{i_1}^{-1},\dots,x_{i_N}^{-1})\times(-1)^m\prod_{0\le k\ne m\le N}\left(1-\frac{x_m}{x_k}\right)^{-1}. \end{aligned}$$

Now we replace x_j by q^j. Then it follows that

$$\begin{aligned}\frac{f_{ml}(N)}{\det \Delta_N} &= e_l^{(N)}(q;m) \times (-1)^m \prod_{0\le k\ne m\le N} \left(1-q^{m-k}\right)^{-1} \\ &= e_l^{(N)}(q;m) \times (-1)^m \prod_{k=0}^{m-1} \left(1-q^{m-k}\right)^{-1} \prod_{k=m+1}^{N} \left(1-q^{m-k}\right)^{-1} \\ &= \prod_{k=1}^{N} \left(1-q^{-k}\right)^{-1} e_l^{(N)}(q;m) q^{-m(m+1)/2} \prod_{k=1}^{m} \frac{1-q^{-(N-k+1)}}{1-q^{-k}}.\end{aligned}$$

By the lemma above, we have

$$e_m^{(N)}(q;N) = e_k^{(N)}(1, q^{-1}, \dots, q^{-(N-1)}) = q^{-m(m-1)/2} \prod_{j=1}^{m} \frac{1-q^{-N+j-1}}{1-q^{-j}}.$$

Thus we have

$$\frac{f_{ml}(N)}{\det \Delta_N} = \prod_{k=1}^{N} \left(1-q^{-k}\right)^{-1} e_l^{(N)}(q;m) e_m^{(N)}(q;N) q^{-m}.$$

It is immediate to see by definition that the identity $e_m^{(N)}(q;N)q^{-m} = e_m^{(N)}(q;0)$ holds. Thus we have the conclusion. □

Remark 5.2. We know that the quantity $e_l^{(N)}(q;m)e_m^{(N)}(q;0)$ appearing in the cofactor $f_{ml}(N)$ is symmetric in m and l because of the symmetry of Δ_N. But this expression does not exhibit the symmetry in apparent manner. At this moment it does not seem a very easy question to find an another expression of $e_l^{(N)}(q;m)e_m^{(N)}(q;0)$ such that the symmetry in m and l is obviously seen. This is an interesting problem of its own right.

By taking a limit $N \to \infty$ in Theorem 5.2, we have

Theorem 5.4. *The inverse Δ^{-1} of the Laplacian Δ attached to* $\mathrm{Mod}(\mathbb{F}_q)$ *is given by*

$$\Delta^{-1} = \left((-1)^{i+j} \prod_{k=1}^{\infty}(1-q^{-k})^{-1} \prod_{k=1}^{j}(q^k-1)^{-1} e_i(q;j)\right)_{i,j\ge 0}. \tag{5.4}$$

Here we define

$$e_n(q;i) \overset{\text{def}}{=} \sum_{\substack{S\subset\mathbb{N} \\ \#S=n \\ i\notin S}} q^{-S}.$$

By a similar discussion as in the case of the category **PB**, we can prove the existence of the spectral zeta function $\zeta_\Delta(s)$ of the Laplacian $\Delta = \Delta_{\mathrm{Mod}(\mathbb{F}_q)}$ for $\mathrm{Mod}(\mathbb{F}_q)$.

Theorem 5.5. *The spectral zeta function $\zeta_\Delta(s)$ of $\Delta = \Delta_{\mathrm{Mod}(\mathbb{F}_q)}$ is well-defined, that is, there exists some small $\varepsilon > 0$ such that*

$$\zeta_\Delta(s) \stackrel{\mathrm{def}}{=} \mathrm{tr}\,\Delta^{-s} = \sum_{k=0}^{\infty} \lambda_k^{-s}$$

converges absolutely and hence defines a holomorphic function in the region $\mathrm{Re}(s) > 1 - \varepsilon$. □

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