Zeta regularizations and q-analogue of ring sine functions

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1 Introduction

So called the *zeta regularization* is one of the most effective methods to carry out necessary renormalization calculations in a variety of situations such as the determinant expressions of elliptic operators [KV, Vo] and certain arithmetic applications [D] (see also [KKSW]). In the present paper we focus our interest on a particular class of functions which are defined in forms of the zeta regularized products. Let us recall first the formula essentially due to Lerch [L] as a typical example we deal with:

$$\frac{1}{\Gamma(x)} = \frac{1}{\sqrt{2\pi}} \prod_{n=0}^{\infty} (n+x). \tag{1.1}$$

Here the symbol $\prod$ denotes so called the *zeta regularized product*, as we explain in §2. It is well known that $1/\Gamma(x)$ is an entire function which has simple zeros at $x = 0, -1, -2, \dots$. The noteworthy point here is that the zeta regularized product in the left hand side of (1.1) may indicate the location $x = 0, -1, -2, \dots$ of zeros of $1/\Gamma(x)$ in a quite apparent way. In other words, this is interpreted as a kind of factorization formula, which is comparable with the Weierstrass canonical product expression:

$$\frac{1}{\Gamma(x)} = e^{\gamma x} x \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}. \tag{1.2}$$

With this example, we are naturally lead to study the general situation as follows. Suppose that a family of functions $\{F_n(x)\}_{n \in I}$ satisfying appropriate conditions is given. We hope to define a function $F(x)$ as

$$F(x) := \prod_{n \in I} F_n(x). \tag{1.3}$$

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The following two questions are basic here:

(i) When does the regularized product in (1.3) exist?

(ii) Suppose that the regularized product (1.3) exists. Can we conclude that $F(x)$ is a function whose zeros are exactly given by

$$Z = \coprod_{n\in I}\big\{a \in \mathbb{C} \;\big|\; F_n(a) = 0\big\}$$

within multiplicity?

The following (ii)' is equivalent to (ii) substantially, but slightly stronger.

(ii)' Assume that $F(x) := \displaystyle\prod_{n\in I} F_n(x)$ and $G(x) := \displaystyle\prod_{n\in I} G_n(x)$ exist. Can we conclude the multiplicativity $F(x)G(x) = \displaystyle\prod_{n\in I} F_n(x)G_n(x)$?

The first question (i) seems quite delicate. Actually, when we take the geometric progression $F_n(x) = q^{n+x}$ $(q > 1)$, then (1.3) does not exist. (See Example 2.2) Compared with the linear function $n+x$, it increases pretty too fast. We have hence in [KW2] introduced an extended notion called *a generalized zeta regularized product* (see Definition 2.3) in order to deal with a wider class of regularized products including the example $\displaystyle\prod\!\!\!\!\!\!\!\cdot\;\;_{n\geq 0} q^{n+x}$ above, where we express the generalized zeta regularized product by $\prod\!\!\!\!\!\!\!\cdot\;\;$ in stead of $\prod$. But there are, of course, a lot of curious and important examples of the sequences $\{F_n(x)\}_{n\in I}$ which do not have regularized products even in the sense of a generalized regularization. For instance,

$$\text{“}\prod\!\!\!\!\!\!\!\cdot\;\;_{n=1}^{\infty} \frac{\Gamma(n+x)}{\Gamma(x)}\text{”} \tag{1.4}$$

seems to give the double gamma function $\Gamma_2(x)$ (see [B]) but the product does not exist. The sequence $n!$ seems to increase too fast. However, even if $\boldsymbol{a} = \{a_n\}_{n\in I}$ is of moderate growth, we can not assure the existence of the regularized product $\displaystyle\prod\!\!\!\!\!\!\!\cdot\;\;_{n\in I} a_n$ of $\boldsymbol{a}$. For instance, let p_n be the n-th prime number and consider the sequence $\boldsymbol{p} = \{p_n\}_{n\geq 1}$. Though $p_n = o(n)$ as n tends to infinity, the regularized product $\displaystyle\prod\!\!\!\!\!\!\!\cdot\;\;_{n=1}^{\infty} p_n$ does not exist. In fact, $\zeta_{\boldsymbol{p}}(s) = \sum_{n=1}^{\infty} p_n^{-s}$ has a natural boundary $\mathrm{Re}(s) = 0$. Thus an extension of the notion of these zeta regularized products is also an interesting problem.

For the question (ii), Illies [I] deals with the case of linear factors $F_n(x) = a_n - x$ for a given sequence $\boldsymbol{a} = \{a_n\}_{n\in I}$, and gives an affirmative answer to (ii) whenever the generalized zeta regularized product of $\boldsymbol{a}$ exits. This is a generalization of Voros's result [Vo] for usual zeta regularizations. Related to (ii)', a multiplicative anomaly of zeta regularized products is studied in [KV].

In this paper we deal with the case of q-linear factors $f_n(x)=[a_n-x]_q$ $(q>1)$ for a given sequence $\boldsymbol{a}=\{a_n\}_{n\in I}$ and establish a relation between the function defined by a generalized zeta regularized product and the one defined by a Weierstrass canonical form (Theorem 3.1). Here we employ the following convention for q-numbers:

$$[a]_q := \frac{q^{a/2}-q^{-a/2}}{q^{1/2}-q^{-1/2}} \quad (a\in\mathbb{C})\,. \tag{1.5}$$

Moreover, using the idea similar to the proof of this relation, we also prove the same kind of the factorization for the case $F_n(x)$'s are polynomials whose degree equal d except a finite number of $n\in I$ (see Remark 3.2).

As an important example, we calculate a q-analogue of a ring sine function. A general notion of *a ring sine function* $S_A(x)$ of a commutative ring A has been introduced in [KMOW] as

$$S_A(x) := \prod_{a\in A}(a-x). \tag{1.6}$$

Here the product should be suitably interpreted. In the cases of the ring of rational integers $\mathbb{Z}$ and its imaginary quadratic extension $\mathbb{Z}[\tau]$ (τ is an imaginary quadratic integer), the corresponding ring sine functions $S_{\mathbb{Z}}(x)$ and $S_{\mathbb{Z}[\tau]}(x)$ are realized respectively by zeta regularized products as

$$S_{\mathbb{Z}}(x) := \prod_{m\in\mathbb{Z}}(m-x), \tag{1.7}$$

$$S_{\mathbb{Z}[\tau]}(x) := \prod_{m,n\in\mathbb{Z}}(m+n\tau-x), \tag{1.8}$$

and these are calculated explicitly; the former is the sine function and the latter is the elliptic theta function essentially.

In Section 4 we introduce and study the q-ring sine function

$$S^q_{\mathbb{Z}}(x) := \coprod_{n\in\mathbb{Z}}[n-x]_q, \tag{1.9}$$

which is a q-analogue of $S_{\mathbb{Z}}(x)$ above. We calculate $S^q_{\mathbb{Z}}(x)$ explicitly by using a q-analogue of the Hurwitz zeta function (see [KW3]), and show that it essentially gives $S_{\mathbb{Z}[\tau]}(x)$ (see Remark 4.4).

2 Zeta regularizations

In this section we recall the usual notion of *the zeta regularization* and the genelarized regularization in order to deal with wider class of sequences.

Definition 2.1. Let $\boldsymbol{a} = \{a_n\}_{n\in I}$ be a divergent sequence of nonzero complex numbers. We define *the zeta function attached to* $\boldsymbol{a}$ by the Dirichlet series

$$\zeta_{\boldsymbol{a}}(s) := \sum_{n\in I} a_n^{-s}. \tag{2.1}$$

Throughout this paper we fix a log-branch by $-\pi \le \arg\log a < \pi$ for $a \in \mathbb{C}^\times$.

Assume that the series (2.1) converges absolutely if $\mathrm{Re}(s) > \mu$ for a sufficiently large real number μ. We take such a number μ to be the minimal one, and call it *the exponent of convergence of* $\boldsymbol{a}$.

If $\zeta_{\boldsymbol{a}}(s)$ has a meromorphic continuation to some region containing the origin $s = 0$, then we say $\boldsymbol{a}$ is *(meromorphically zeta-)regularizable*. We first recall the standard definition of zeta regularized products.

Definition 2.2 (Holomorphic regularization)**.** Let $\boldsymbol{a}$ be a regularizable sequence. If $\zeta_{\boldsymbol{a}}(s)$ is holomorphic at $s = 0$, then *the zeta regularized product* of $\boldsymbol{a}$ is defined by

$$\prod_{n\in I} a_n := \exp\left(-\zeta'_{\boldsymbol{a}}(0)\right). \tag{2.2}$$

This is a usual zeta regularization (see e.g. [D, Vo]).

Example 2.1 (Lerch's formula [L])**.** Let $x > 0$ and take $a_n = n + x$ for $n \ge 0$. The attached zeta function

$$\zeta(s,x) := \sum_{n=0}^{\infty} (n+x)^{-s}$$

is called *the Hurwitz zeta function*. This has a meromorphic continuation to the whole plane and holomorphic at $s = 0$. In fact, the regularized product of $(n + x)$'s is given by (1.1). □

Since the attached zeta function $\zeta_{\boldsymbol{a}}(s)$ of a simple geometric series $\boldsymbol{a} = \{q^n\}_{n\ge 0}$ $(q > 1)$ is given by

$$\zeta_{\boldsymbol{a}}(s) = \sum_{n=0}^{\infty} q^{-ns} = \frac{1}{1-q^{-s}} \tag{2.3}$$

and has a simple pole at $s = 0$, the zeta regularized product of $\boldsymbol{a}$ in the sense of (2.2) does not exist. Thus we needed an extended notion of the regularized product in [KW2] as follows.

Definition 2.3 (Meromorphic regularization [KW2])**.** If $\zeta_{\boldsymbol{a}}(s)$ has a pole at $s = 0$, then *the (generalized) zeta regularized product* of $\boldsymbol{a}$ is defined by

$$\dot{\prod_{n\in I}} a_n := \exp\left(-\operatorname*{Res}_{s=0} \frac{\zeta_{\boldsymbol{a}}(s)}{s^2}\right).$$

We use this dotted product symbol if $\zeta_{\boldsymbol{a}}(s)$ has a pole st $s = 0$ in order to distinguish this notion from the holomorphic regularization if necessary.

Remark 2.1. Since $\zeta'_{\boldsymbol{a}}(0) = \mathrm{Res}_{s=0}\,\zeta_{\boldsymbol{a}}(s)/s^2$ if $\zeta_{\boldsymbol{a}}(s)$ is holomorphic at $s=0$, it is obvious to see $\mathop{\prod\hspace{-0.85em}\cdot\hspace{0.3em}} = \prod$ in the holomorphic case.

Example 2.2 ([KKSW]). For any $q>1$, we have

$$\mathop{\prod\hspace{-0.85em}\cdot\hspace{0.3em}}_{n=0}^{\infty} q^{n+x} = q^{-B_2(x)/2}, \tag{2.4}$$

where $B_2(x)$ is the Bernoulli polynomial of degree 2. This follows from the Laurent expansion of the zeta function for $\boldsymbol{a} = \{q^{n+x}\}_{n\geq 0}$,

$$\zeta_{\boldsymbol{a}}(s,x) = \sum_{n=0}^{\infty} q^{-s(n+x)} = \frac{q^{-sx}}{1-q^{-s}} = \frac{1}{s\log q} + B_1(x) + \frac{s}{2}B_2(x)\log q + O(s^2). \tag{2.5}$$

□

Example 2.3 (q-Lerch's formula [KW2]). A q-analogue of Lerch's formula (1.1) is calculated as

$$\mathop{\prod\hspace{-0.85em}\cdot\hspace{0.3em}}_{n=0}^{\infty} [n+x]_q = \frac{[\infty]_q!}{\Gamma_q(x)}. \tag{2.6}$$

Here we denote by $\Gamma_q(x)$ the (modified) Jackson q-gamma function

$$\Gamma_q(x) := \frac{\prod_{n=1}^{\infty}(1-q^{-n})}{\prod_{n=0}^{\infty}(1-q^{-(n+x)})}(q^{1/2}-q^{-1/2})^{1-x}q^{x(x-1)/4}, \tag{2.7}$$

which satisfies the functional equation $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$ in our convention. The constant $[\infty]_q!$ is explicitly given by

$$[\infty]_q! := \mathop{\prod\hspace{-0.85em}\cdot\hspace{0.3em}}_{n=1}^{\infty} [n]_q = q^{-1/24}(q^{1/2}-q^{-1/2})^{-\log(1-q^{-1})/\log q}\prod_{n=1}^{\infty}\left(1-q^{-n}\right). \tag{2.8}$$

This follows from the calculation of the Laurent expansion of the q-Hurwitz zeta function

$$\zeta_q(s,x) := \sum_{n=0}^{\infty}[n+x]_q^{-s} \qquad (\mathrm{Re}(s)>0).$$

See Lemma 4.2 for the analytic continuation of $\zeta_q(s,x)$. □

3 Zeta regularizations and canonical forms

As we see typically in the case of Lerch's result, one of the important aspect of a regularized product is that the regularized product representation of a given function is useful to indicate the location of zeros. (For the other important aspect such as "transformation" properties of the regularized product representation, see [KKSW].) In this section we present a relation between a zeta regularization and a Weierstrass canonical form when a function is defined by a regularized product over q-linear factors.

3.1 A factorization theorem

Let $\boldsymbol{a}$ be a sequence of nonzero complex numbers. We denote by μ the exponent of convergence of the sequence $\boldsymbol{a}$, that is, the associated zeta function $\zeta_{\boldsymbol{a}}(s)=\sum_{n\in I}a_n^{-s}$ converges absolutely in the region $\mathrm{Re}(s)>\mu$, and hence defines a function which is holomorphic in the same region. We also denote by p the integer part of μ, or the minimum integer such that the series $\sum_{n\in I}\frac{1}{|a_n|^{1+p}}$ converges absolutely.

We are interested in the function defined by the zeta regularized product of $[\boldsymbol{a}-x]_q:=\{[a_n-x]_q\}_{n\in I}$, say,

$$D^q_{\boldsymbol{a}}(x):=\prod_{n\in I}\!\!\!\!\!\!\cdot\;\;[a_n-x]_q. \tag{3.1}$$

Since there is a trivial periodicity $q^{x+\tau}=q^x$ ($\tau:=2\pi i/\log q$), we may expect that (3.1) defines a function whose zeros are given by $\boldsymbol{a}(\tau):=\{a_n+k\tau\}_{n\in I,k\in\mathbb{Z}}$. In fact, our goal in this section is to show the following result.

Theorem 3.1. *Let $\boldsymbol{a}=\{a_n\}_{n\in I}$ be a regularizable sequence of real numbers (except a finite number of a_n's). Denote by μ the exponent of convergence of $\boldsymbol{a}$, and let p be the integer part of μ. Assume that there exists a certain connected domain $\mathbb{D}$ such that $\boldsymbol{a}(\tau)-x:=\{a_n+k\tau-x\}_{n\in I,k\in\mathbb{Z}}$ and $[\boldsymbol{a}-x]_q$ are both regularizable for any $x\in\mathbb{D}$. Then there exists a polynomial function $f_{\boldsymbol{a}}(x)$ defined on $\mathbb{D}$ such that*

$$\prod_{n\in I}\!\!\!\!\!\!\cdot\;\;[a_n-x]_q=\exp f_{\boldsymbol{a}}(x)\prod_{\substack{n\in I\\k\in\mathbb{Z}}}\left(1-\frac{x}{a_n+k\tau}\right)\exp\left(\sum_{j=1}^{p+1}\frac{1}{j}\left(\frac{x}{a_n+k\tau}\right)^j\right). \tag{3.2}$$

Remark 3.1. Theorem 3.1 is a preferable statement as a special case of the general *expectation*

$$\prod_{n\in I}\!\!\!\!\!\!\cdot\;\;F_n(x)=e^{f(x)}\prod_{a\in Z}\left(1-\frac{x}{a}\right)\exp\left(\sum_j\frac{1}{j}\left(\frac{x}{a}\right)^j\right), \tag{3.3}$$

where $Z=\coprod_n\{a\in\mathbb{C}\mid F_n(a)=0\}$ is the set of all zeros of $\{F_n(x)\}_{n\in I}$.

3.2 Proof of Theorem 3.1

We denote the attached zeta functions for $\boldsymbol{a}(\tau)-x$ and $[\boldsymbol{a}-x]_q$ by

$$\zeta_{\boldsymbol{a}(\tau)}(s,x):=\sum_{\substack{n\in I\\k\in\mathbb{Z}}}(a_n+k\tau-x)^{-s}, \tag{3.4}$$

$$\zeta^q_{\boldsymbol{a}}(s,x):=\sum_{n\in I}[a_n-x]_q^{-s}. \tag{3.5}$$

By the assumption of the theorem, $\zeta_{\boldsymbol{a}(\tau)}(s,x)$ converges absolutely in the region $\mathrm{Re}(s) > \mu + 1$. First we remark that $\zeta^q_{\boldsymbol{a}}(s,x)$ converges absolutely and defines a holomorphic function in the right half plane $\mathrm{Re}(s) > 0$ since the behavior of $\zeta^q_{\boldsymbol{a}}(s,x)$ is comparable with that of

$$\Phi_{\boldsymbol{a}}(s) = \sum_{n\in I} q^{-a_n s},$$

and we have assumed the positivity of $\boldsymbol{a}$.

We suppose that $\zeta^q_{\boldsymbol{a}}(s,x)$ has a pole of order N at $s=0$. Note that $\zeta^q_{\boldsymbol{a}}(s,x)$ satisfies the difference-differential equation

$$\frac{\partial^2}{\partial x^2}\zeta^q_{\boldsymbol{a}}(s,x) = -(\log q)^2 \left(s(s+1)\zeta^q_{\boldsymbol{a}}(s+2,x) + s^2\zeta^q_{\boldsymbol{a}}(s,x)\right). \tag{3.6}$$

By using (3.6) successively it follows that $\frac{\partial^{2n}}{\partial x^{2n}}\zeta^q_{\boldsymbol{a}}(s,x)$ is holomorphic at $s=0$ if $n \geq N/2$. It is convenient to introduce the function

$$\eta_{\boldsymbol{a}(\tau)}(s,x) := \Gamma(s)\zeta_{\boldsymbol{a}(\tau)}(s,x), \tag{3.7}$$

which is holomorphic if $\mathrm{Re}(s) \geq p+2$. We immediately check the functional equation

$$\frac{\partial}{\partial x}\eta_{\boldsymbol{a}(\tau)}(s,x) = \eta_{\boldsymbol{a}(\tau)}(s+1,x). \tag{3.8}$$

An entire function whose zeros are exactly given by $\boldsymbol{a}(\tau)$ is constructed by the Weierstrass canonical product as follows:

$$\Delta^q_{\boldsymbol{a}}(x) := \prod_{\substack{n\in I\\ k\in\mathbb{Z}}} \left(1 - \frac{x}{a_n + k\tau}\right) \exp\left(\sum_{j=1}^{p+1} \frac{1}{j}\left(\frac{x}{a_n+k\tau}\right)^j\right). \tag{3.9}$$

Our destination is to describe a relation between $D^q_{\boldsymbol{a}}(x)$ and $\Delta^q_{\boldsymbol{a}}(x)$, which assures that the generalized regularized product expression of a function indicates the location of its zeros.

We consider the log-derivatives of $\Delta^q_{\boldsymbol{a}}(x)$

$$R_k(x) := \frac{\partial^k}{\partial x^k}\log\Delta^q_{\boldsymbol{a}}(x) \qquad (k = 0,1,2,\dots). \tag{3.10}$$

They satisfies the initial condition $R_k(0) = 0$ for $k = 0,1,\dots,p+1$, and conversely, $\Delta^q_{\boldsymbol{a}}(x)$ is a unique entire function of order p determined by these conditions. The following equality is crucial:

$$R_n(x) = \frac{\partial^n}{\partial x^n}\log\Delta^q_{\boldsymbol{a}}(x) = \sum_{\substack{n\in I\\ k\in\mathbb{Z}}} \frac{(n-1)!}{(a_n+k\tau-x)^n} = \eta_{\boldsymbol{a}(\tau)}(n,x) \tag{3.11}$$

for any $n \geq p+2$.

To calculate the log-derivatives of $D^q_{\boldsymbol{a}}(x)$ in a desirable fashion, we need the following simple lemma.

Lemma 3.2. *For $a \neq 0$, we have*

$$[a-x]_q = [a]_q q^{-\frac{x}{2}\coth(\frac{a\log q}{2})} \prod_{k\in\mathbb{Z}} \left(1-\frac{x}{a+k\tau}\right)\exp\left(\frac{x}{a+k\tau}\right). \tag{3.12}$$

Proof. The set of zeros of the function

$$[a-x]_q = \frac{2}{q^{1/2}-q^{-1/2}} \sinh\left(\frac{(a-x)\log q}{2}\right)$$

is given by $\boldsymbol{a} = \{a+k\tau \mid k \in \mathbb{Z}\}$. Therefore it must have a canonical product expression of the form

$$[a-x]_q = e^{g(x;a)} \prod_{k\in\mathbb{Z}} \left(1-\frac{x}{a+k\tau}\right)\exp\left(\frac{x}{a+k\tau}\right)$$

for a suitable entire function $g(x;a)$. Taking the log-derivative of $[a-x]_q$ in two ways according to the two kinds of expressions above, we have

$$-\frac{\log q}{2}\coth\left(\frac{(a-x)\log q}{2}\right) = g'(x;a) - \sum_{k\in\mathbb{Z}}\left(\frac{1}{a+k\tau-x}-\frac{1}{a+k\tau}\right).$$

The fractional expansion of the hyperbolic cotangent function

$$\coth x = \frac{1}{x} + \sum_{k\neq 0}\left(\frac{1}{x-i\pi k}+\frac{1}{i\pi k}\right)$$

yields then $g'(x;a) = -\frac{\log q}{2}\coth\left(\frac{a\log q}{2}\right)$. Thus we have $g(x;a) = -\frac{x\log q}{2}\coth\left(\frac{a\log q}{2}\right) + \log[a]_q$ since $g(0;a) = \log[a]_q$. □

By using the lemma above, we have

$$\begin{aligned}
[a-x]_q^{-s} &= 1 - s\log[a-x]_q + O(s^2) \\
&= 1 - \left(g(0;a) + \sum_{k\in\mathbb{Z}}\left(\log\left(1-\frac{x}{a+k\tau}\right)+\frac{x}{a+k\tau}\right)\right)s + O(s^2).
\end{aligned}$$

Thus the zeta function attached to $[\boldsymbol{a}]_q$ is

$$\zeta_{\boldsymbol{a}}^q(s,x) = \sum_{n\in I}\left(1-\left(g(0;a_n)+\sum_{k\in\mathbb{Z}}\left(\log\left(1-\frac{x}{a_n+k\tau}\right)+\frac{x}{a_n+k\tau}\right)\right)s+O(s^2)\right). \tag{3.13}$$

The implied constant in $O(s^2)$ is depending on x. Differentiating repeatedly, it follows

$$\frac{\partial^m}{\partial x^m}\zeta_{\boldsymbol{a}}^q(s,x) = \sum_{n\in I}\left(-\sum_{k\in\mathbb{Z}}\frac{(m-1)!}{(a_n+k\tau-x)^m}s+O(s^2)\right) \tag{3.14}$$

if $m \geq p+2$. Since $\frac{\partial^m}{\partial x^m}\zeta_{\boldsymbol{a}}^q(s,x)$ is holomorphic at $s=0$ for $m \geq N$, the expression (3.14) gives the Taylor expansion of $\frac{\partial^m}{\partial x^m}\zeta_{\boldsymbol{a}}^q(s,x)$ around the origin $s=0$ when $m \geq \max\{p+2, N\}$. Hence we have

$$\frac{\partial^m}{\partial x^m} \operatorname*{Res}_{s=0} \frac{\zeta_{\boldsymbol{a}}^q(s,x)}{s^2} = -\eta_{\boldsymbol{a}(\tau)}(m,x). \tag{3.15}$$

From (3.11) and (3.15), we have

$$\frac{\partial^m}{\partial x^m}\left(\log \Delta_{\boldsymbol{a}}^q(x) + \operatorname*{Res}_{s=0} \frac{\zeta_{\boldsymbol{a}}^q(s,x)}{s^2}\right) = 0 \quad (m \geq \max\{p+2, N\}),$$

which implies that there exists a certain polynomial $f_{\boldsymbol{a}}(x)$ of degree at most $\max\{p+2, N\}$ such that

$$\log \Delta_{\boldsymbol{a}}^q(x) - \log D_{\boldsymbol{a}}^q(x) = f_{\boldsymbol{a}}(x).$$

This completes the proof of Theorem 3.1. □

By a similar discussion we have the following result for polynomial case.

Theorem 3.3. *For $j = 1, 2, \ldots, d$, let $\boldsymbol{a}^{(j)} = \{a_{j,n}\}_{n\in I}$ be regularizable sequences of positive numbers, and suppose that the $\sum_{n\in I} a_{j,n}^{-(p+1)}$ converges absolutely for every j. There exists a polynomial function $F(x)$ defined on a certain domain $\mathbb{D}$ such that*

$$\begin{aligned} &\coprod_{n\in I}(a_{1,n}-x)(a_{2,n}-x)\cdots(a_{d,n}-x) \\ &= \exp F(x) \prod_{\substack{n\in I \\ 1\leq j\leq d}} \left(1 - \frac{x}{a_{j,n}}\right) \exp\left(\sum_{k=1}^{p} \frac{1}{k}\left(\frac{x}{a_{j,n}}\right)^k\right) \end{aligned} \tag{3.16}$$

for any $x \in \mathbb{D}$. In particular, the following two regularized products

$$\coprod_{n\in I}\left(\prod_{j=1}^{d}(a_{j,n}-x)\right), \quad \prod_{j=1}^{d}\left(\coprod_{n\in I}(a_{j,n}-x)\right)$$

are equal up to a nonzero elementary factor.

Proof. Denote by $\Delta(x)$ the canonical product appearing in the right hand side of (3.16). The $(p+1)$-th log-derivative if $\Delta(x)$ is given by

$$\frac{\partial^{p+1}}{\partial x^{p+1}} \log \Delta(x) = \sum_{j=1}^{d}\sum_{n\in I} \frac{\Gamma(p+1)}{(a_{j,n}-x)^{p+1}}. \tag{3.17}$$

The attached zeta function $\varphi(s,x)$ for $\{(a_{1,n}-x)(a_{2,n}-x)\cdots(a_{d,n}-x)\}_{n\in I}$ is

$$\begin{aligned} \varphi(s,x) &= \sum_{n\in I} \left((a_{1,n}-x)\ldots(a_{d,n}-x)\right)^{-s} \\ &= \sum_{n\in I}\left(1 - s\log(a_{1,n}-x)\ldots(a_{d,n}-x) + O(s^2)\right). \end{aligned}$$

Differentiation with respect to x successively yields

$$\frac{\partial^{p+1}}{\partial x^{p+1}}\varphi(s,x) = \sum_{n\in I}\left(-s\sum_{j=1}^{d}\frac{p!}{(a_{j,n}-x)^{p+1}} + O(s^2)\right),$$

which implies

$$\frac{\partial^{p+1}}{\partial x^{p+1}}\operatorname*{Res}_{s=0}\frac{\varphi(s,x)}{s^2} = -\sum_{j=1}^{d}\sum_{n\in I}\frac{\Gamma(p+1)}{(a_{j,n}-x)^{p+1}} = -\frac{\partial^{p+1}}{\partial x^{p+1}}\log\Delta(x) \tag{3.18}$$

in view of (3.17). Thus we have (3.16) by a similar argument of the proof of Theorem 3.1. The latter statement follows immediately. □

Remark 3.2. Theorem 3.3 insists that the basic questions proposed in §1 is affirmative in the case of polynomial functions satisfying certain conditions: Assume that all but finite exception of the functions $F_n(x)$ are polynomial functions of degree d such that the sequence consisting of their roots is regularizable. Then the reguralized product $\displaystyle\coprod_{n\in I} F_n(x)$ exists, and it gives a function which exhibits the information of the location of zeros.

Example 3.1 (Generalized Lerch's formula [L]: see also [KW1])**.**

$$\prod_{n=0}^{\infty}\left((n+x)^2+y^2\right) = \frac{2\pi}{\Gamma(x+iy)\Gamma(x-iy)} = \prod_{n=0}^{\infty}(n+x+iy)\prod_{n=0}^{\infty}(n+x-iy). \tag{3.19}$$

□

The following is a example which does not satisfy the required condition of Theorem 3.3.

Example 3.2 ([KKW])**.** For $n \geq 3$, we have

$$\prod_{a\in \mathrm{Sym}_n(\mathbb{Z})}\det(a-x) = 1 \tag{3.20}$$

for $x \in \mathrm{Sym}_n(\mathbb{C})\setminus\mathrm{Sym}_n(\mathbb{Z})$. Here we denote by $\mathrm{Sym}_n(R)$ $(R=\mathbb{Z},\mathbb{C})$ the set of $n\times n$ symmetric matrices whose entries belong to R. □

4 Example: q-sine and theta functions

The ring sine function for a commutative ring A is defined by

$$S_A(x) := \prod_{a\in A}(a-x),$$

where the product "$\prod_{a\in A}$" over A should be, of course, suitably interpreted like zeta regularized product (see [KMOW]). For example, in the cases of the ring of rational integers $\mathbb{Z}$ and its imaginary quadratic extension $\mathbb{Z}[\tau]$ (τ is an imaginary quadratic integer), the corresponding ring sine functions $S_{\mathbb{Z}}(x)$ and $S_{\mathbb{Z}[\tau]}(x)$ are realized by zeta regularized products and calculated as follows.

Theorem 4.1 ([KMOW]). *We have*

$$S_{\mathbb{Z}}(x) := \prod_{m\in\mathbb{Z}}(m-x) = 1-e^{2\pi i x} \quad (0<x<1)\,, \tag{4.1}$$

$$\begin{aligned} S_{\mathbb{Z}[\tau]}(x) &:= \prod_{m,n\in\mathbb{Z}}(m+n\tau-x) \\ &= (1-q^{-x/\tau})\prod_{n=1}^{\infty}(1-q^{-(n+x/\tau)})(1-q^{-(n-x/\tau)}) \quad (0<\operatorname{Im}x<\operatorname{Im}\tau)\,, \end{aligned} \tag{4.2}$$

which are essentially the sine function and the elliptic theta function respectively.

Note that for $(A,K)=(\mathbb{Z},\mathbb{Q}),(\mathbb{Z}[\tau],\mathbb{Q}(\tau))$, $S_A(x)$ generates the maximal abelian extension K^{ab} of K, that is, $K^{\mathrm{ab}}=K(S_A(K))$ ($(\mathbb{Z},\mathbb{Q})$ case is due to Kronecker [Kr], and $(\mathbb{Z}[\tau],\mathbb{Q}(\tau))$ case is due to Takagi [T]).

In this section we introduce a q-analogue $S^q_{\mathbb{Z}}(x)$ of the ring sine function $S_{\mathbb{Z}}(x)$ of $\mathbb{Z}$ by

$$S^q_{\mathbb{Z}}(x) := \prod_{n\in\mathbb{Z}}\!\!\!\!\cdot\;\,[n-x]_q, \tag{4.3}$$

and calculate this explicitly.

Remark 4.1. It is essential to our argument to use the normalization (1.5) of q-numbers. In fact, if we take another convention $\{a\}_q=(q^a-1)/(q-1)$, the attached zeta function

$$\xi_q(s,x) := \sum_{n\in\mathbb{Z}}\{n-x\}_q^{-s} \tag{4.4}$$

for $S^q_{\mathbb{Z}}(x)$ *does diverge* since the summation is taken over the lattice $\mathbb{Z}$ (not the semi-lattice $\mathbb{Z}_{\geq 0}$ like Example 2.3).

In order to carry out the calculation of $S^q_{\mathbb{Z}}(x)$, it is necessary to have an explicit form of the previously defined q-Hurwitz zeta function.

Lemma 4.2. *Assume that* $0<\operatorname{Re}x<1$ *and* $-\frac{2\pi}{\log q}\leq \operatorname{Im}x<\frac{2\pi}{\log q}$. *The Laurent expansion of the q-Hurwitz zeta function* $\zeta_q(s,x)=\sum_{n=0}^{\infty}[n+x]_q^{-s}$ *around the origin* $s=0$ *is given by*

$$\zeta_q(s,x) = \frac{2}{\log q}\frac{1}{s}+\frac{2\log\left(q^{1/2}-q^{-1/2}\right)}{\log q}-\frac{1}{2}(2x-1)+s\log\frac{\Gamma_q(x)}{[\infty]_q!}+O(s^2). \tag{4.5}$$

Proof. First we remark that

$$(q^{(n+x)/2}(1-q^{-n-x}))^{-s} = q^{-s(n+x)/2}(1-q^{-n-x})^{-s}$$

for any $n \geq 0$ under the hypothesis of the lemma. It follows hence that

$$\begin{aligned}
\zeta_q(s,x) &= \sum_{n=0}^{\infty}[n+x]_q^{-s} = (q^{1/2}-q^{-1/2})^s \sum_{n=0}^{\infty}(q^{(n+x)/2}(1-q^{-n-x}))^{-s} \\
&= (q^{1/2}-q^{-1/2})^s \sum_{n=0}^{\infty} q^{-s(n+x)/2} \sum_{k=0}^{\infty}\binom{-s}{k}(-1)^k q^{-(n+x)k} \\
&= (q^{1/2}-q^{-1/2})^s \sum_{k=0}^{\infty}\binom{s+k-1}{k}\frac{q^{(k+s/2)(1-x)}}{q^{k+s/2}-1}.
\end{aligned}$$

Since $\binom{s+k-1}{k} = \frac{s}{k} + O(s^2)$ if $k \geq 1$, we get

$$\begin{aligned}
\sum_{k=0}^{\infty}\binom{s+k-1}{k}\frac{q^{(k+s/2)(1-x)}}{q^{k+s/2}-1} &= \frac{q^{s(1-x)/2}}{q^{s/2}-1} + s\sum_{k=1}^{\infty}\frac{1}{k}\frac{q^{k(1-x)}}{q^k-1} + O(s^2) \\
= \frac{2}{\log q}\frac{1}{s} - \frac{1}{2}(2x-1) &+ \left\{\left(\frac{x^2}{4}-\frac{x}{4}+\frac{1}{24}\right)\log q + \sum_{k=1}^{\infty}\frac{1}{k}\frac{q^{k(1-x)}}{q^k-1}\right\} s + O(s^2).
\end{aligned}$$

Hence we obtain the desired expansion of $\zeta_q(s,x)$ around $s=0$ as follows:

$$\begin{aligned}
\zeta_q(s,x) = {} & \frac{2}{\log q}\frac{1}{s} - \frac{1}{2}(2x-1) + \frac{2}{\log q}\log(q^{1/2}-q^{-1/2}) \\
& + \left\{\left(\frac{x^2}{4}-\frac{x}{4}+\frac{1}{24}\right)\log q + \sum_{k=1}^{\infty}\frac{1}{k}\frac{q^{k(1-x)}}{q^k-1}\right. \\
& \left. - \frac{1}{2}(2x-1)\log(q^{1/2}-q^{-1/2}) + \frac{1}{\log q}\left(\log(q^{1/2}-q^{-1/2})\right)^2\right\} s + O(s^2).
\end{aligned} \tag{4.6}$$

It is straightforward to check the coefficient of s is equal to $\log \frac{\Gamma_q(x)}{[\infty]_q!}$. This shows the proof. □

Using the lemma above, we can calculate the q-analogue of the ring sine function $S_{\mathbb{Z}}^q(x)$.

Theorem 4.3. *Let $q > 1$ be a fixed parameter and suppose that x lies in the region $0 < \operatorname{Re} x < 1$, $-\frac{2\pi}{\log q} \leq \operatorname{Im} x < \frac{2\pi}{\log q}$. The q-analogue of the ring sine function $S_{\mathbb{Z}}^q(x)$ of $\mathbb{Z}$ is given as follows:*

(i) If $-\frac{2\pi}{\log q} \leq \operatorname{Im} x < 0$, then

$$S_{\mathbb{Z}}^q(x) = ie^{-\pi i x + \frac{\pi^2}{\log q}}\left(q^{1/2}-q^{-1/2}\right)^{\frac{2\pi i}{\log q}}\frac{([\infty]_q!)^2}{\Gamma_q(x)\Gamma_q(1-x)}. \tag{4.7}$$

(ii) If $0 \leq \operatorname{Im} x < \frac{2\pi}{\log q}$, then

$$S_{\mathbb{Z}}^q(x) = -ie^{\pi i x + \frac{\pi^2}{\log q}}\left(q^{1/2}-q^{-1/2}\right)^{-\frac{2\pi i}{\log q}}\frac{([\infty]_q!)^2}{\Gamma_q(x)\Gamma_q(1-x)}. \tag{4.8}$$

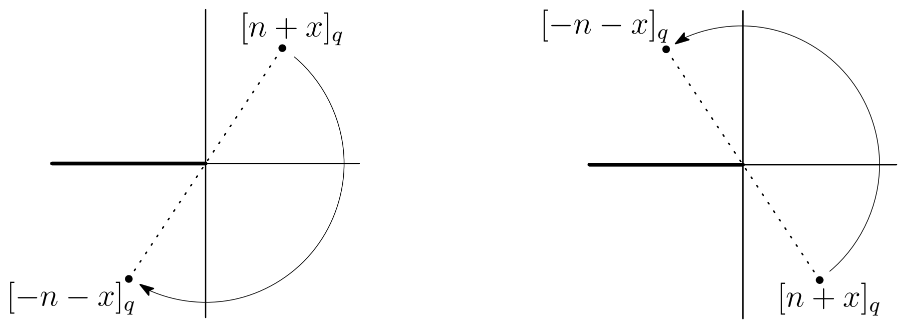
Proof. Let $\xi_q(s,x)$ be the zeta function attached to the sequence $\{[n-x]_q\}$. We divide the sum in $\xi_q(s,x)$ into two parts:

$$\xi_q(s,x) = \sum_{n\in\mathbb{Z}} [n-x]_q^{-s} = \sum_{n\geq 0} [n+(1-x)]^{-s} + \sum_{n\geq 0} [-n-x]^{-s}.$$

We observe that

$$\arg[-n-x]_q = \arg[n+x]_q \pm \pi, \tag{4.9}$$

where the upper (resp. lower) sign is taken in the case $-\frac{2\pi}{\log q} \leq \operatorname{Im} x < 0$ (resp. $0 \leq \operatorname{Im} x < \frac{2\pi}{\log q}$). In fact, $\operatorname{Im}[-n-x]_q = -\frac{2}{q^{1/2}-q^{-1/2}} \cosh(\frac{n+\operatorname{Re}(x)}{2} \log q) \sin(\frac{\operatorname{Im}(x)}{2} \log q)$, and it is clear that $\cosh(\frac{n+\operatorname{Re}(x)}{2} \log q) > 0$ for any n.



$-\pi \leq \frac{\log q}{2} \operatorname{Im} x < 0$ $\qquad$ $0 \leq \frac{\log q}{2} \operatorname{Im} x < \pi$

It follows that

$$[-n-x]_q^{-s} = \begin{cases} e^{-\pi i s}[n+x]_q & -\frac{2\pi}{\log q} \leq \operatorname{Im} x < 0, \\ e^{\pi i s}[n+x]_q & 0 \leq \operatorname{Im} x < \frac{2\pi}{\log q}. \end{cases}$$

Thus we have

$$\xi_q(s,x) = \begin{cases} \zeta_q(s,1-x) + e^{-\pi i s}\zeta_q(s,x) & -\frac{2\pi}{\log q} \leq \operatorname{Im} x < 0, \\ \zeta_q(s,1-x) + e^{\pi i s}\zeta_q(s,x) & 0 \leq \operatorname{Im} x < \frac{2\pi}{\log q}. \end{cases}$$

By Lemma 4.2 we have

$$\operatorname*{Res}_{s=0} \frac{e^{\mp\pi i s}\zeta_q(s,x)}{s^2} = \log \frac{\Gamma_q(x)}{[\infty]_q!} \pm \frac{1}{2} i\pi(2x-1) \mp 2\pi i \frac{\log(q^{1/2}-q^{-1/2})}{\log q} - \frac{\pi^2}{\log q},$$

and hence we obtain

$$\operatorname*{Res}_{s=0} \frac{\xi_q(s,x)}{s^2} = \log \frac{\Gamma_q(x)\Gamma_q(1-x)}{([\infty]_q!)^2} (q^{1/2}-q^{-1/2})^{\frac{\mp 2\pi i}{\log q}} \pm \frac{1}{2} i\pi(2x-1) - \frac{\pi^2}{\log q},$$

where the upper (resp. lower) sign is taken in the case (i) (resp. (ii)). This completes the proof of the theorem. $\square$

Corollary 4.4. *The function $\xi_q(s,x)$ satisfies the difference-differential equation*

$$\frac{\partial^2}{\partial x^2}\xi_q(s,x) = \left(\frac{s\log q}{2}\right)^2 \xi_q(s,x) + s(s+1)\left(\frac{\log q}{q^{1/2}-q^{-1/2}}\right)^2 \xi_q(s+2,x). \tag{4.10}$$

In particular, we have

$$\frac{\partial^2}{\partial x^2}\operatorname*{Res}_{s=0}\frac{\xi_q(s,x)}{s^2} = \left(\frac{\log q}{2}\right)^2 \operatorname*{Res}_{s=0}\xi_q(s,x) + \left(\frac{\log q}{q^{1/2}-q^{-1/2}}\right)^2 \xi_q(2,x) \tag{4.11}$$

and $\xi_q(2,x)$ essentially gives the Weierstrass $\wp$-function (see [KW4]). □

Remark 4.2. It follows also from (4.11) that our q-ring sine function $S^q_{\mathbb{Z}}(x)$ is determined up to "linear factor", that is,

$$S^q_{\mathbb{Z}}(x) = e^{\alpha x+\beta}\Gamma_q(x)^{-1}\Gamma_q(1-x)^{-1}$$

for some $\alpha,\beta\in\mathbb{C}$ in view of Theorem 3.1.

We give a q-analogue of Kronecker's limit formula. (See Remark 5 in [KMOW])

Theorem 4.5. *We have*

$$\prod_{n\in\mathbb{Z}}\!\!\!\!\!\!\cdot\;\;\; |[n-x]_q| = q^{(x-\overline{x})^2/8}\frac{([\infty]_q!)^2}{|\Gamma_q(x)\Gamma_q(1-x)|} \tag{4.12}$$

for $0<\mathrm{Re}(x)<1$.

Remark 4.3. The two theorems above show that

$$\prod_{n\in\mathbb{Z}}\!\!\!\!\!\!\cdot\;\;\; |[n-x]_q| = e^{-\frac{\pi^2}{\log q}} q^{(x-\overline{x})^2/8}\left|\prod_{n\in\mathbb{Z}}\!\!\!\!\!\!\cdot\;\;\; [n-x]_q\right|. \tag{4.13}$$

Proof of Theorem 4.5. We should study the attached zeta function

$$\tilde{\zeta}_q(s,x) := \sum_{n\in\mathbb{Z}}|[n-x]_q|^{-s} = \sum_{n>0}|[n-x]_q|^{-s} + \sum_{n\le 0}|[n-x]_q|^{-s}.$$

First we look at

$$\sum_{n>0}|[n-x]_q|^{-s} = \sum_{n>0}\left(q^{1/2}-q^{-1/2}\right)^s q^{-\frac{n-\mathrm{Re}\,x}{2}s}\left|1-q^{-n+x}\right|^{-s}.$$

Notice that $|q^{-n+x}|<1$ since $n>0$ and $0<\mathrm{Re}(x)<1$. By the binomial expansion we have

$$\begin{aligned}\left|1-q^{-n+x}\right|^{-s} &= \left(1-q^{-n+x}\right)^{-s/2}\left(1-q^{-n+\bar{x}}\right)^{-s/2}\\ &= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\binom{-s/2}{\ell}\binom{-s/2}{m}(-1)^{\ell+m}q^{(-n+\mathrm{Re}\,x)(\ell+m)}q^{i\,\mathrm{Im}\,x(\ell-m)}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n>0}|[n-x]_q|^{-s} &= \left(q^{1/2}-q^{-1/2}\right)^s \\ &\times\left\{\frac{q^{\frac{1}{2}s\operatorname{Re}x}}{q^{\frac{1}{2}s}-1}+\sum_{\ell+m>0}\binom{-s/2}{\ell}\binom{-s/2}{m}(-1)^{\ell+m}\frac{q^{\operatorname{Re}x(\ell+m+\frac{1}{2}s)+i\operatorname{Im}x(\ell-m)}}{q^{\ell+m+\frac{1}{2}s}-1}\right\}.\end{aligned} \tag{4.14}$$

In order to observe the behavior of the zeta function around $s=0$, we calculate the Laurent expansions of the casts in (4.14):

$$\left(q^{1/2}-q^{-1/2}\right)^s = 1+\left\{\log\left(q^{1/2}-q^{-1/2}\right)\right\}s+\frac{1}{2}\left\{\log\left(q^{1/2}-q^{-1/2}\right)\right\}^2 s^2+O(s^3), \tag{4.15}$$

$$\frac{q^{\frac{1}{2}s\operatorname{Re}x}}{q^{\frac{1}{2}s}-1} = \frac{2}{\log q}\frac{1}{s}+\frac{1}{2}(2\operatorname{Re}x-1)+\frac{1}{24}\left(6(\operatorname{Re}x)^2-6\operatorname{Re}x+1\right)(\log q)\,s+O(s^2), \tag{4.16}$$

$$\begin{aligned}&\sum_{\ell+m>0}\binom{-s/2}{\ell}\binom{-s/2}{m}(-1)^{\ell+m}\frac{q^{\operatorname{Re}x(\ell+m+\frac{1}{2}s)+i\operatorname{Im}x(\ell-m)}}{q^{\ell+m+\frac{1}{2}s}-1} \\ &\qquad=\left\{\sum_{m>0}\frac{q^{m\bar{x}}}{2m(q^m-1)}+\sum_{\ell>0}\frac{q^{\ell x}}{2\ell(q^\ell-1)}\right\}s+O(s^2).\end{aligned} \tag{4.17}$$

Combining these calculation we have

$$\begin{aligned}\sum_{n>0}|[n-x]_q|^{-s} &= \frac{2}{\log q}\frac{1}{s}+\left(\frac{2\operatorname{Re}x-1}{2}+\frac{2}{\log q}\log\left(q^{1/2}-q^{-1/2}\right)\right) \\ &+\left\{\frac{\left\{\log\left(q^{1/2}-q^{-1/2}\right)\right\}^2}{\log q}+\frac{(2\operatorname{Re}x-1)\log\left(q^{1/2}-q^{-1/2}\right)}{2}\right. \\ &\left.+\frac{(6(\operatorname{Re}x)^2-6\operatorname{Re}x+1)\log q}{24}+\sum_{m>0}\frac{q^{m\bar{x}}}{2m(q^m-1)}+\sum_{\ell>0}\frac{q^{\ell x}}{2\ell(q^\ell-1)}\right\}s+O(s^2).\end{aligned} \tag{4.18}$$

Hence we have

$$\begin{aligned}&\operatorname*{Res}_{s=0}\frac{\sum_{n>0}|[n-x]_q|^{-s}}{s^2} \\ =&\frac{\left\{\log\left(q^{1/2}-q^{-1/2}\right)\right\}^2}{\log q}+\frac{(2\operatorname{Re}x-1)\log\left(q^{1/2}-q^{-1/2}\right)}{2} \\ +&\frac{(6(\operatorname{Re}x)^2-6\operatorname{Re}x+1)\log q}{24}+\sum_{m>0}\frac{q^{m\bar{x}}}{2m(q^m-1)}+\sum_{\ell>0}\frac{q^{\ell x}}{2\ell(q^\ell-1)}.\end{aligned} \tag{4.19}$$

By the discussion similar to the above, we also have

$$\begin{aligned}&\operatorname*{Res}_{s=0}\frac{\sum_{n\le 0}|[n-x]_q|^{-s}}{s^2} \\ =&\frac{\left\{\log\left(q^{1/2}-q^{-1/2}\right)\right\}^2}{\log q}-\frac{(2\operatorname{Re}x-1)\log\left(q^{1/2}-q^{-1/2}\right)}{2} \\ +&\frac{(6(\operatorname{Re}x)^2-6\operatorname{Re}x+1)\log q}{24}+\sum_{m>0}\frac{q^{m\bar{x}}}{2m(q^m-1)}+\sum_{\ell>0}\frac{q^{\ell x}}{2\ell(q^\ell-1)}.\end{aligned} \tag{4.20}$$

Therefore, we obtain

$$\operatorname*{Res}_{s=0} \frac{\tilde{\zeta}_q(s,x)}{s^2} = \log|\Gamma_q(1-x)\Gamma_q(x)| - \log\left([\infty]!\right)^2 + \frac{1}{2}(\operatorname{Im} x)^2 \log q, \tag{4.21}$$

which is the desired conclusion. □

Remark 4.4. If we put $\tau = -\log q/2\pi i$, then we see that

$$S_{\mathbb{Z}}^q(x/\tau) = \left(\frac{q^{-1/2\tau^2 - 1/4\tau} \left(q^{1/2} - q^{-1/2}\right)^{1+1/\tau} \left([\infty]_q!\right)^2}{\prod_{n=1}^{\infty}(1-q^{-n})^2} \right) \times q^{\frac{x(x-\tau-1)}{2\tau^2}} S_{\mathbb{Z}[\tau]}(x). \tag{4.22}$$

Namely, our q-deformation $S_{\mathbb{Z}}^q(x)$ of $S_{\mathbb{Z}}(x)$ essentially gives the ring sine function $S_{\mathbb{Z}[\tau]}(x)$ for the ring $\mathbb{Z}[\tau]$.

Remark 4.5. Since the function $S_{\mathbb{Z}}^q(x)$ has an imaginary period $2\pi i/\log q$, classical limit "$q \to 1$" is corresponding to the limit "(imaginary period) $\to \infty$". On the other hand, we can interpret that $S_{\mathbb{Z}[\tau]}(x)$ tends to $S_{\mathbb{Z}}(x)$ by taking a formal limit $\tau \to \infty$, where τ is an imaginary period of $S_{\mathbb{Z}[\tau]}(x)$.

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