# Invariant theory of singular $\alpha$-determinants


Kazufumi KIMOTO

(University of the Ryukyus)

Joint work with Masato WAKAYAMA



Conference “Representation Theory, Systems of Differential Equations and their Related Topics” at Hokkaido University

July 5, 2007

# $\alpha$-determinant

For $\alpha \in \mathbb{C}$ and $A = (a_{ij})_{1 \le i,j \le n} \in \mathrm{Mat}_n = \mathrm{Mat}_n(\mathbb{C})$, the **$\alpha$-determinant of** $A$ is

$$\det^{(\alpha)}(A) = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\nu(\sigma)} a_{\sigma(1)1} \dots a_{\sigma(n)n}$$

where

$$\nu(\sigma) = \sum_{i \ge 1} (i-1) m_i$$

if the cycle-type of $\sigma$ is $1^{m_1} 2^{m_2} \dots$.

- $\det^{(-1)}(A) = \det(A) \quad (\because (-1)^{\nu(\sigma)} = \operatorname{sgn}\sigma)$

- $\det^{(1)}(A) = \operatorname{per}(A)$

- $\det^{(\alpha)}$ is multilinear in rows and columns

- $\det^{(\alpha)}({}^tA) = \det^{(\alpha)}(A) \quad (\because \nu(\sigma^{-1}) = \nu(\sigma))$

• $\alpha$-determinant is first introduced by Vere-Jones (1988) as coefficients in the expansion of $\det(I-\alpha A)^{-1/\alpha}$:

$$\det(I-\alpha A)^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1\le i_1,\dots,i_n\le d} \det{}^{(\alpha)} \begin{pmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ a_{i_n i_1} & \cdots & a_{i_n i_n} \end{pmatrix}$$

for $A = (a_{ij})_{1\le i,j\le d}$.

This is used to construct a certain point process.

- det is multiplicative : $\det(AB) = \det(A)\det(B)$

- $\det^{(\alpha)}(AB) \neq \det^{(\alpha)}(A)\det^{(\alpha)}(B)$ if $\alpha \neq -1$

$$\text{det is multiplicative} \longleftrightarrow GL_n(\mathbb{C}) \cdot \det(X) \subset \mathbb{C} \cdot \det(X)$$

$\longrightarrow$ Look at the smallest $GL_n(\mathbb{C})$-invariant subspace containing $\det^{(\alpha)}(X)$

Introduce a $\mathcal{U}(\mathfrak{gl}_n)$-module structure on the algebra $\mathcal{P}(\mathrm{Mat}_n)$ of polynomial functions on $\mathrm{Mat}_n$ by

$$E_{ij} = \sum_{p=1}^{n} x_{ip} \frac{\partial}{\partial x_{jp}}$$

$E_{ij}$ : standard basis of $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$

$x_{ij}$ : standard coordinate on $\mathrm{Mat}_n = \mathrm{Mat}_n(\mathbb{C})$

- $\mathcal{U}(\mathfrak{gl}_n) \cdot \det(X) = \mathbb{C} \cdot \det(X) \cong \mathcal{M}_n^{(1^n)}$
- $\mathcal{U}(\mathfrak{gl}_n) \cdot \mathrm{per}(X) \cong \mathcal{M}_n^{(n)}$

$\mathcal{M}_n^{\lambda}$ : irreducible highest weight module of $\mathcal{U}(\mathfrak{gl}_n)$ with highest weight $\lambda$ (we identify the highest weights and partitions and/or Young diagrams)

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det{}^{(\alpha)}(X) \cong \bigoplus_{\substack{\lambda \vdash n \\ f_\lambda(\alpha) \neq 0}} \left(\mathcal{M}_n^\lambda\right)^{\oplus f^\lambda}$$

where

$$f_\lambda(\alpha) = \prod_{(i,j) \in \lambda} (1 + (j - i)\alpha),$$

$$f^\lambda = K_{\lambda,(1^n)} = \# \text{ of standard tableaux of } \lambda,$$

(Matsumoto-Wakayama (2006))

- If $\alpha = -1/k$ $(k = 1, 2, \dots, n-1)$,

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det{}^{(\alpha)}(X) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda_1 \le k}} \left(\mathcal{M}_n^{\lambda}\right)^{\oplus f^{\lambda}}$$

- If $\alpha = 1/k$ $(k = 1, 2, \dots, n-1)$,

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det{}^{(\alpha)}(X) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda'_1 \le k}} \left(\mathcal{M}_n^{\lambda}\right)^{\oplus f^{\lambda}}$$

- Otherwise,

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det{}^{(\alpha)}(X) \cong (\mathbb{C}^n)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \left(\mathcal{M}_n^{\lambda}\right)^{\oplus f^{\lambda}}$$

$$\alpha = \pm 1, \pm\frac{1}{2}, \dots, \pm\frac{1}{n-1} \text{ : singular values}$$

# Singular $\alpha$-determinants

Look at the case where $\alpha = -1/k$ $(k = 1, 2, \ldots, n-1)$.

$$\det^{(\alpha)} \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\nu(\sigma)} = \prod_{1 \le i < n} (1 + i\alpha)$$

For $I \subset [n] = \{1, 2, \dots, n\}$,

$$\mathfrak{S}_n(I) = \{\sigma \in \mathfrak{S}_n \,;\, \sigma(x) = x,\ x \notin I\}\,.$$

$$\sigma \cdot (a_{ij}) = (a_{\sigma^{-1}(i)j}), \quad (a_{ij}) \cdot \tau = (a_{i\tau(j)})$$

$$((a_{ij}) \in \mathrm{Mat}_{m,n},\ \sigma \in \mathfrak{S}_m, \tau \in \mathfrak{S}_n)$$

$$\sum_{\sigma \in \mathfrak{S}_n(I)} \det{}^{(\alpha)}(X \cdot \sigma) = \prod_{1 \le i < k} (1 + i\alpha) \sum_{g \in \mathfrak{S}_n} \alpha^{m(g,I)} x_{g(1)1} \dots x_{g(n)n}$$

($m(g, I)$: some nonnegative integer)

"$-1/k$-analogue" of the alternating property:

$$I \subset [n],\ \#I > k \implies \sum_{\sigma \in \mathfrak{S}_n(I)} \det{}^{(-1/k)}(X \cdot \sigma) = 0.$$

In particular,

$(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) \in \mathrm{Mat}_n$,

$\boldsymbol{a}_{i_1} = \cdots = \boldsymbol{a}_{i_k} = \boldsymbol{b}$ $(1 \le i_1 < \cdots < i_k \le n)$,

$j \ne i_1, \dots, i_k$

$$\implies \quad \det{}^{(-1/k)}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_j + \boldsymbol{b}, \dots, \boldsymbol{a}_n) = \det{}^{(-1/k)}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n)$$

- The ordinary determinant is characterized (up to constant) by the multilinearity and alternating property with respect to column vectors.

- How about the $-1/k$-determinant?

$$\mathrm{ML}_n = \bigoplus_{1 \le i_1, \dots, i_n \le n} \mathbb{C} \cdot x_{i_1 1} \dots x_{i_n n}$$
$$= \left\{ f(X) \in \mathcal{P}(\mathrm{Mat}_n) \,;\, f(X) \text{ is multilinear in columns} \right\}.$$

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det{}^{(-1/k)}(X)$$
$$= \left\{ f(X) \in \mathrm{ML}_n \,;\, \#I > k \implies \sum_{\sigma \in \mathfrak{S}_n(I)} f(X \cdot \sigma) = 0 \right\}$$

# Wreath determinant

For $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) \in \mathrm{Mat}_{m,n}$,

$$A^{[k]} = A \otimes (\overbrace{1, \dots, 1}^{k}) = (\overbrace{\boldsymbol{a}_1, \dots, \boldsymbol{a}_1}^{k}, \dots, \overbrace{\boldsymbol{a}_n, \dots, \boldsymbol{a}_n}^{k}) \in \mathrm{Mat}_{m,kn}$$

**Example.** If $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \in \mathrm{Mat}_{3,2}$,

$$A^{[3]} = \begin{pmatrix} a_1 & a_1 & a_1 & b_1 & b_1 & b_1 \\ a_2 & a_2 & a_2 & b_2 & b_2 & b_2 \\ a_3 & a_3 & a_3 & b_3 & b_3 & b_3 \end{pmatrix} \in \mathrm{Mat}_{3,6}\,.$$

For $A \in \mathrm{Mat}_{kn,n}$, the $k$-**wreath determinant** of $A$ is

$$\begin{aligned}\mathrm{wrdet}_k(A) &= \det{}^{(-1/k)}(A^{[k]})\\ &= \sum_{\sigma \in \mathfrak{S}_{kn}} \left(-\frac{1}{k}\right)^{\nu(\sigma)} \prod_{i=1}^{n} \prod_{j=1}^{k} a_{\sigma((i-1)k+j),i}\end{aligned}$$

- By the "$-1/k$-alternating property" of $\det^{(-1/k)}$,

$$\begin{aligned}&\mathrm{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_i+c\boldsymbol{a}_j,\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)\\&\qquad=\mathrm{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_i,\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)\end{aligned}$$

- By the column-multilinearity of $\det^{(-1/k)}$,

$$\begin{aligned}&\mathrm{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},c\boldsymbol{a}_i,\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)\\&\qquad=c^k\,\mathrm{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_i,\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)\end{aligned}$$

For $A\in\mathrm{Mat}_{kn,n}$ and $P\in GL_n(\mathbb{C})$,

$$\mathrm{wrdet}_k(AP)=\det(P)^k\,\mathrm{wrdet}_k(A)$$

For $A \in \mathrm{Mat}_{kn,n}$ and $g \in \mathfrak{S}_k \wr \mathfrak{S}_n = \mathfrak{S}_k^n \rtimes \mathfrak{S}_n$,

$$\mathrm{wrdet}_k(g \cdot A) = \chi_{n,k}(g)^k \,\mathrm{wrdet}_k(A)$$

where

$$\chi_{n,k}(g) = \operatorname{sgn} \tau, \qquad g = (\sigma, \tau) \in \mathfrak{S}_k^n \rtimes \mathfrak{S}_n$$

We regard $\mathfrak{S}_k^n \rtimes \mathfrak{S}_n \subset \mathfrak{S}_{kn}$ by

$$\sigma((i-1)k + j) = (i-1)k + \sigma_i(j),$$
$$\tau((i-1)k + j) = (\tau(i)-1)k + j$$

$$(1 \le i \le n,\ 1 \le j \le k)$$

for $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathfrak{S}_k^n$ and $\tau \in \mathfrak{S}_n$.

# Determinant expression of $\mathrm{wrdet}_k$

Introduce a $GL_{kn} \times GL_n$-module structure on $\mathcal{P}(\mathrm{Mat}_{kn,n})$ by

$$((g,h).f)(A) := f({}^t gAh)$$

$(g \in GL_{kn},\ h \in GL_n,\ A \in \mathrm{Mat}_{kn,n})$

We have the multiplicity-free decomposition

$$\mathcal{P}(\mathrm{Mat}_{kn,n}) \cong \bigoplus_{\ell(\lambda) \leq n} \mathcal{M}_{kn}^{\lambda} \boxtimes \mathcal{M}_{n}^{\lambda}$$

by $(GL_{kn}, GL_n)$-**duality**.

Look at the det-eigenspace for the diagonal torus $\mathbb{T} = \mathbb{T}_{kn} \cong (\mathbb{C}^\times)^{kn}$ of $GL_{kn}$:

$$\mathcal{P}(\mathrm{Mat}_{kn,n})^{\mathbb{T},\det} \cong \bigoplus_{\ell(\lambda)\leq n} \left(\mathcal{M}^\lambda_{kn}\right)^{\mathbb{T},\det} \boxtimes \mathcal{M}^\lambda_n$$

where

$$V^{\mathbb{T},\det} = \{v \in V\,;\, t.v = (\det t)v,\ t \in \mathbb{T}\}$$

- $\left(\mathcal{M}^\lambda_{kn}\right)^{\mathbb{T},\det}$ becomes a $\mathfrak{S}_{kn}$-module ($\mathfrak{S}_{kn}$ is the normalizer of $\mathbb{T}$ in $GL_{kn}$).
- It is known that $\left(\mathcal{M}^\lambda_{kn}\right)^{\mathbb{T},\det}$ is irreducible $\mathfrak{S}_{kn}$-module corresponding to $\lambda$ if $\lambda \vdash kn$.

Let $M_{n,k} \subset \mathcal{P}(\mathrm{Mat}_n)$ be the irreducible $GL_{kn} \times GL_n$-submodule corresponding to $(k^n)$:

$$M_{n,k} \cong \mathcal{M}_{kn}^{(k^n)} \boxtimes \mathcal{M}_n^{(k^n)}.$$

As a $\mathfrak{S}_{kn}$-module, $M_{n,k}^{\mathbb{T},\det}$ is irreducible (corresponding to $(k^n)$) since $\dim \mathcal{M}_n^{(k^n)} = 1$.

In particular, $\dim M_{n,k}^{\mathbb{T},\det} = f^{(k^n)}$.

Since

$$\mathrm{wrdet}_k(tAP) = (\det t)(\det P)^k \, \mathrm{wrdet}_k(A) \quad (t \in \mathbb{T},\ P \in GL_n),$$

it follows that $\mathrm{wrdet}_k(X) \in M_{n,k}^{\mathbb{T},\det}$.

Let $T = (t_{ij})_{\substack{1\le i\le n\\ 1\le j\le k}}$ be a standard tableau of $(k^n)$.

- For $A \in \mathrm{Mat}_{kn,n}$,

$$\det_T(A) := \prod_{p=1}^{k} \det(a_{t_{ip},j})_{1\le i,j\le n}$$

- Let $I(T) \in \mathrm{Mat}_{kn,n}$ be a matrix whose $t_{ij}$-th row vector is

$$(0,\ldots,0,\overset{i\text{-th}}{1},0,\ldots,0)$$

**Example.** If $n = 3$, $k = 2$ and $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$,

$$\det_T(A) = \begin{vmatrix} a_{\mathbf{1}1} & a_{\mathbf{1}2} & a_{\mathbf{1}3} \\ a_{\mathbf{3}1} & a_{\mathbf{3}2} & a_{\mathbf{3}3} \\ a_{\mathbf{4}1} & a_{\mathbf{4}2} & a_{\mathbf{4}3} \end{vmatrix} \begin{vmatrix} a_{\mathbf{2}1} & a_{\mathbf{2}2} & a_{\mathbf{2}3} \\ a_{\mathbf{5}1} & a_{\mathbf{5}2} & a_{\mathbf{5}3} \\ a_{\mathbf{6}1} & a_{\mathbf{6}2} & a_{\mathbf{6}3} \end{vmatrix}$$

$$I(T) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Since

$$\det_T(tAP) = (\det t)(\det P)^k \det_T(A) \quad (t \in \mathbb{T},\ P \in GL_n),$$

it follows that $\det_T(X) \in M_{n,k}^{\mathbb{T},\det}$.

- For standard tableaux $T, U$ of $(k^n)$,

$$\det_T(I(U)) = \delta_{T,U}$$

In particular, $\{\det_T(X)\}_T$ are linearly independent.

Since $\dim M_{n,k}^{\mathbb{T},\det} = f^{(k^n)}$,

$$M_{n,k}^{\mathbb{T},\det} = \bigoplus_{T} \mathbb{C} \cdot \det_T(X)$$

Thus we have

$$\mathrm{wrdet}_k(X) = \sum_{T} \mathrm{wrdet}_k(I(T)) \det_T(X)$$

**Example.** The standard tableaux of $(2^3)$ are

$$U_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array},\ U_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline\end{array},\ U_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array},\ U_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array},\ U_5 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array},$$

and the corresponding matrices $I(U_p)$ are

$$\begin{pmatrix}1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\1&0&0\\0&1&0\\0&0&1\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&1&0\\1&0&0\\0&1&0\\0&0&1\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&1&0\\1&0&0\\0&0&1\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

The 2-wreath determinants $\mathrm{wrdet}_2(I(U_p))$ are

$$\mathrm{wrdet}_2(I(U_1)) = \frac{1}{8},$$
$$\mathrm{wrdet}_2(I(U_2)) = \mathrm{wrdet}_2(I(U_3)) = -\frac{1}{16},$$
$$\mathrm{wrdet}_2(I(U_4)) = \mathrm{wrdet}_2(I(U_5)) = \frac{1}{32}.$$

Hence, for $A \in \mathrm{Mat}_{6,3}$,

$$\mathrm{wrdet}_2(A) = \frac{1}{8}\det_{U_1}(A) - \frac{1}{16}\det_{U_2}(A) - \frac{1}{16}\det_{U_3}(A) + \frac{1}{32}\det_{U_4}(A) + \frac{1}{32}\det_{U_5}(A).$$

By the Frobenius reciprocity,

$$\dim\left(M_{n,k}^{\mathbb{T},\det}\right)^{\mathfrak{S}_k^n} = K_{(k^n),(k^n)} = 1.$$

Since $\mathrm{wrdet}_k(X)$ is $\mathfrak{S}_k^n$-invariant, we have

$$\left(M_{n,k}^{\mathbb{T},\det}\right)^{\mathfrak{S}_k^n} = \mathbb{C}\cdot \mathrm{wrdet}_k(X).$$

Consequently, by determining the proportional constant,

$$\mathrm{wrdet}_k(X) = \frac{1}{k^{kn}} \sum_{\sigma\in\mathfrak{S}_k^n} \det_{T_0}(\sigma\cdot X)$$

where $T_0$ is a standard tableau of $(k^n)$ whose $(i,j)$-entry is $(i-1)k+j$.

As a corollary,

Let $\mathcal{P}(\mathrm{Mat}_{kn,n})^{\chi_{n,k},\det^k}$ be the subspace consisting of the functions satisfying

$$f(gAP) = \chi_{n,k}(g)^k(\det P)^k f(A)$$

for $g \in \mathfrak{S}_k \wr \mathfrak{S}_n$, $P \in GL_n$, $A \in \mathrm{Mat}_{kn,n}$. Then

$$\mathcal{P}(\mathrm{Mat}_{kn,n})^{\chi_{n,k},\det^k} = \mathbb{C} \cdot \mathrm{wrdet}_k(X)$$

• The "$k$-ply multiplicativity"

$$\det{}^{(\alpha)}(AP^{[k]}) = (\det P)^k \det{}^{(\alpha)}(A^{[k]}) \quad (A \in \mathrm{Mat}_{kn,n}, P \in GL_n)$$

holds only when $\alpha = -1/k$.

• Then it is natural to look at the cyclic module $\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \det^{(\alpha)}(X^{[k]})$. It is not difficult to see

$$\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \det{}^{(\alpha)}(X^{[k]}) \cong \bigoplus_{\substack{\lambda \vdash kn \\ f_\lambda(\alpha) \neq 0}} \left(\mathcal{M}_{kn}^{\lambda}\right)^{K_{\lambda,(k^n)}}$$

• It is much harder to describe the irreducible decomposition (i.e. determine explicitly the 'singular values') of the cyclic module

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^l$$

for $l > 1$.

• When $n = 2$, the 'singular values' are given as roots of the Jacobi polynomials:

$$\mathcal{U}(\mathfrak{gl}_2) \cdot \det^{(\alpha)}(X)^l \cong \bigoplus_{\substack{0 \le i \le l \\ F_i(\alpha) \ne 0}} \mathcal{M}_2^{(2l-i,i)}$$

where

$$F_i(\alpha) = (1+\alpha)^{l-i} \times (\text{Jacobi polynomial})$$

(K.-Matsumoto-Wakayama)

# Expansion of $\mathrm{wrdet}_k$

For $k, n \in \mathbb{N}$, put

$$\mathfrak{R}_{n,k} := \left\{ f : [kn] \to [n] \,;\, \# f^{-1}(j) = k,\ \forall j \in [n] \right\}.$$

(Notice that $\mathfrak{R}_{n,1} = \mathfrak{S}_n$)

We define the sign of $f \in \mathfrak{R}_{n,k}$ by

$$\mathrm{sgn}_{n,k}(f) := \mathrm{wrdet}_k (\delta_{f(i),j})_{\substack{1 \le i \le kn \\ 1 \le j \le n}}$$

$$\operatorname{wrdet}_k A = \sum_{f \in \mathfrak{R}_{n,k}} \operatorname{sgn}_{n,k}(f) \prod_{i \in [kn]} a_{if(i)}$$
$$= \sum_T \operatorname{sgn}_{n,k}(T) \det_T(A)$$

where we regard a standard tableau $T = (t_{ij})$ of $(k^n)$ as an element in $\mathfrak{R}_{n,k}$ by

$$T : [kn] \ni t_{ij} \longmapsto i \in [n]$$

- $\operatorname{sgn}_{n,k}(T) = \operatorname{wrdet}_k I(T)$

Define the injection

$$\omega : \mathfrak{S}_n^k \ni (w_1, \ldots, w_k) \longmapsto \Big((i-1)k + j \mapsto w_j(i)\Big) \in \mathfrak{R}_{n,k}$$

For $f \in \mathfrak{R}_{n,k}$,

$$\mathrm{sgn}_{n,k}(f) = \mathrm{sgn}(w) \left(\frac{k!}{k^k}\right)^n \frac{\#\left(f \cdot \mathfrak{S}_k^n \cap \omega(\mathfrak{S}_n^k)\right)}{\#\left(f \cdot \mathfrak{S}_k^n\right)}$$

where $w \in \mathfrak{S}_n^k$ such that $\omega(w) \in f \cdot \mathfrak{S}_k^n$.

For $f \in \mathfrak{R}_{n,k}$, define

$$P_f(x_{11}, \ldots, x_{nk}) := \frac{1}{\#\mathfrak{S}_k^n} \sum_{\sigma_1,\ldots,\sigma_n \in \mathfrak{S}_k^n} \prod_{i=1}^{n} \prod_{j=1}^{k} x_{f((i-1)k+j),\sigma_i(j)}.$$

Then

$$\frac{\#\left(f \cdot \mathfrak{S}_k^n \cap \omega(\mathfrak{S}_n^k)\right)}{\#\left(f \cdot \mathfrak{S}_k^n\right)} = \text{the coefficient of } \prod_{i=1}^{n} \prod_{j=1}^{k} x_{ij} \text{ in } P_f(x_{11}, \ldots, x_{nk})$$

**Example.** If $n=3$, $k=2$ and $T = U_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$, then

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 3 & 2 & 3 \end{pmatrix} : [6] \to [3],$$

$$\begin{aligned} & P_T(x_{11}, \dots, x_{32}) \\ & \quad = \frac{1}{8}(x_{11}x_{22} + x_{12}x_{21})(x_{11}x_{32} + x_{12}x_{31})(x_{21}x_{32} + x_{22}x_{31}). \end{aligned}$$

The coefficient of $x_{11}x_{21}x_{31}x_{12}x_{22}x_{32}$ is $\dfrac{1}{4}$.

If we take $w = ((23),(12)) \in \mathfrak{S}_3^2$, then $\omega(w) = T \cdot (34) \in T \cdot \mathfrak{S}_2^3$ and $\operatorname{sgn} w = 1$. Therefore

$$\operatorname{sgn}_{n,k}(T) = \operatorname{wrdet}_2(U_4) = 1 \cdot \left(\frac{2!}{2^2}\right)^3 \cdot \frac{1}{4} = \frac{1}{32}.$$

# References

• K.-Wakayama,
Invariant theory of singular $\alpha$-determinants,
math.RT/0603699

• Matsumoto-Wakayama,
Alpha-determinant cyclic modules of $\mathfrak{gl}_n(\mathbb{C})$,
J. Lie Theory **16** (2006), 393-405.

• K.-Wakayama,
Quantum $\alpha$-determinant cyclic modules of $\mathcal{U}_q(\mathfrak{gl}_n)$,
doi:10.1016/j.jalgebra.2006.12.015