Elliptic curves arising from the spectral zeta function for non-commutative harmonic oscillators and $\Gamma_0(4)$-modular forms

Kazufumi KIMOTO* Masato WAKAYAMA†

Abstract

The Apéry-like numbers $J_2(n)$ associated to the special value $\zeta_Q(2)$ of the spectral zeta function $\zeta_Q(s)$ for the non-commutative harmonic oscillator Q have remarkable modular form interpretation. In fact, we show that the differential equation satisfied by the generating function $w_2(t)$ of $J_2(n)$ is the Picard-Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion. The parameter t of this family can be regarded as a modular function for the congruent subgroup $\Gamma_0(4)$. Further, we see that the function $w_2(t)$ is regarded as a $\Gamma_0(4)$-modular form of weight 1 in the variable τ by taking t as the classical Legendre modular function $\lambda(\tau)$.

Keywords: spectral zeta functions, non-commutative harmonic oscillators, Apéry numbers, elliptic curves, Picard-Fuchs equations, modular forms.

2000 Mathematics Subject Classification: 11M41, 10D12, 14K20, 81Q10.

1 Introduction

Let us consider the numbers a_n and b_n defined by the recurrence relation

$$n^2u_n - (11n^2 - 11n + 3)u_{n-1} - (n-1)^2u_{n-2} = 0 \tag{1.1}$$

together with the initial conditions $a_0 = 1$, $a_1 = 3$ and $b_0 = 0$, $b_1 = 5$. These numbers were introduced in 1978 by Apéry, who utilized them to prove the irrationality of the special value $\zeta(2) = \frac{\pi^2}{6}$ of the Riemann zeta function $\zeta(s)$. The important point is that a parallel method allows us to prove the irrationality of $\zeta(3)$ too (we refer to [11] for further information on his irrationality proofs).

Since then, several people have tried to understand the nature of the Apéry numbers and to generalize the theory to $\zeta(n)$ $(n > 3)$. For instance, there are many works on congruence and/or supercongruence properties, algebro-geometric interpretations, modular properties, etc. For detailed information, we refer to [1], [2], [3] and their references. Among them, in this note, we focus particularly the study of the relation between Apéry numbers and elliptic curves developed by Beukers [1].

Let us hence recall the result in [1] briefly. Consider the generating functions $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ of the Apéry numbers a_n and b_n. Then $A(t)$ and $B(t)$ satisfy the differential equation $L_2(A) = 0$ and $L_2(B) = -5$ respectively, where L_2 denotes the Fuchsian differential operator

$$L_2 = t(t^2 + 11t - 1)\frac{d^2}{dt^2} + (3t^2 + 22t - 1)\frac{d}{dt} + (t + 3).$$

The main result of [1] shows that the equation $L_2(Y) = 0$ is the Picard-Fuchs equation associated with the family of curves

$$y^2 = x^3 + \frac{1}{4}(t^2 + 6t + 1)x^2 + \frac{1}{2}t(t+1)x + \frac{1}{4}t^2. \tag{1.2}$$

*Partially supported by Grant-in-Aid for Scientific Research (B) No.16740021.
†Partially supported by Grant-in-Aid for Scientific Research (B) No.15340012.

We notice that any elliptic curve equipped with rational 5-torsion is birationally equivalent to (1.2) for a certain value of t [7]. By this fact, the function $A(t)$ is interpreted as the period of the holomorphic 1-form $\frac{\pi i dx}{y}$. Moreover, $A(t)$ is a $\Gamma_1(5)$-modular form of weight 1 in the variable τ ($\Im\tau > 0$), the ratio of the fundamental periods.

In the present note, we deal with analogous objects of the Apéry numbers, say, the "Apéry-like numbers" $J_2(n)$ attached to the special value $\zeta_Q(2)$ introduced in [5], [6]. Here $\zeta_Q(s)$ is the spectral zeta function of a certain differential operator called the non-commutative harmonic oscillator Q. We recall shortly basic properties of the spectral zeta function $\zeta_Q(s)$ and how the numbers $J_2(n)$ arise in connection with the value $\zeta_Q(2)$. Let $Q = Q_{\alpha,\beta}$ be the ordinary differential operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ defined by

$$Q := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(x\frac{d}{dx} + \frac{1}{2} \right), \tag{1.3}$$

where the parameters $\alpha, \beta \in \mathbb{R}_{>0}$ satisfy $\alpha\beta > 1$. The system defined by the operator Q is called the *non-commutative harmonic oscillator* [10]. The operator Q is positive and self-adjoint, and has only a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to +\infty$. Then, the *spectral zeta function* $\zeta_Q(s)$ of Q is introduced as the Dirichlet series

$$\zeta_Q(s) := \sum_{n=1}^{\infty} \lambda_n^{-s} \tag{1.4}$$

in order to study the structure of the spectrum of Q. This series converges absolutely if $\Re s > 1$ [6]. In [5], it is shown that $\zeta_Q(s)$ is continued to the whole plane $\mathbb{C}$ as a meromorphic function which has a unique simple pole at $s = 1$, and trivial zeros at $s = 0, -2, -4, -6, \dots$ like the Riemann zeta function $\zeta(s)$. We note that the operator Q is unitarily equivalent to a pair of usual quantum harmonic oscillators when $\alpha = \beta$. In particular, we see that $\zeta_Q(s) = 2(2^s - 1)\zeta(s)$ when $\alpha = \beta = \sqrt{2}$.

The numbers $J_2(n)$ in question are defined by

$$J_2(n) := \int_0^\infty\!\!\int_0^\infty \frac{e^{-(t+s)/2}}{1 - e^{-(t+s)}} \left(\frac{(1 - e^{-2t})(1 - e^{-2s})}{(1 - e^{-(t+s)})^2} \right)^n dtds, \tag{1.5}$$

which arise in the expression of the special value of $\zeta_Q(s)$ at $s = 2$. Actually, we have

$$\zeta_Q(2) = \frac{(\alpha+\beta)^2}{2\alpha\beta(\alpha\beta - 1)} \left(3\zeta(2) + \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2 \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} J_2(n) \left(\frac{1}{\alpha\beta - 1} \right)^n \right). \tag{1.6}$$

In [6], a recurrence formula for $J_2(n)$ is obtained as

$$4n^2 J_2(n) - (8n^2 - 8n + 3) J_2(n-1) + 4(n-1)^2 J_2(n-2) = 0. \tag{1.7}$$

Introducing the generating function $w_2(t)$ of $J_2(n)$ by

$$w_2(t) = \sum_{n=0}^{\infty} J_2(n) t^n, \tag{1.8}$$

one finds that the recurrence formula (1.7) is equivalent to the singly confluent Heun differential equation

$$t(1-t)^2 w_2''(t) - (1-3t)(1-t) w_2'(t) + (t - \frac{3}{4}) w_2(t) = 0. \tag{1.9}$$

We show in [8] that the rational numbers $\tilde{J}_2(n) := J_2(n)/J_2(0)$ satisfy the congruence relation $\tilde{J}_2(mp^n) \equiv \tilde{J}_2(mp^{n-1}) \pmod{p^n}$ which is quite similar to the one for Apéry numbers; $a_{mp^n-1} \equiv a_{mp^{n-1}-1} \pmod{p^n}$. Here we note that the Apéry-like numbers $J_3(n)$ are introduced similarly to describe the value $\zeta_Q(3)$, and satisfy the recurrence formula

$$4n^2 J_3(n) - (8n^2 - 8n + 3)J_3(n-1) + 4(n-1)^2 J_3(n-2) = \frac{2^n (n-1)!}{(2n-1)!!}. \tag{1.10}$$

Remarkably, the only difference between the two recurrence equations (1.7) and (1.10) is the existence of the inhomogeneous term in (1.10). Therefore the normalized sequence $\tilde{J}_3(n) := J_3(n) - J_3(0)\tilde{J}_2(n) \in \mathbb{Q}$ also satisfies (1.10). As a result, the generating function $w_3(t)$ for $\tilde{J}_3(n)$ satisfies almost the same differential equation as (1.9) but with inhomogeneous term (see (6.7)). Furthermore, one can obtain a congruence formula for $\tilde{J}_3(n)$ too. See [8] for detail.

Based on these similarities between the Apéry numbers and our Apéry-like numbers $\tilde{J}_2(n)$, $\tilde{J}_3(n)$, in the talk at the *Conference on L-functions*, the second author gave a rather vague conjecture; like the results in [1] for the Apéry numbers, the differential equation for $w_2(t)$ (resp. $w_3(t)$) would be understood as a Picard-Fuchs differential equation attached to a certain family of elliptic curves (resp. a certain family of $K3$ surfaces; see [2]). At that time, he presented some numerical data for $\tilde{J}_2(n)$, from which Zagier immediately showed his interest and indicated that the numbers $\tilde{J}_2(n)$ would be coming from a certain modular form. In fact, the next day he kindly suggested the precise formula from his study in [14].

The aim of this note is to establish an analogue of the results in [1] for the Apéry-like numbers $J_2(n)$. Precisely, we show that the differential equation satisfied by the generating function $w_2(t)$ of $J_2(n)$ is the Picard-Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion. The parameter t for the family of such elliptic curves can be considered as a modular function for the congruent subgroup $\Gamma_0(4)$. Moreover, we show that the function $w_2(t)$ is regarded as a $\Gamma_0(4)$-modular form of weight 1 in the variable τ by taking t as the classical Legendre modular function $\lambda(\tau)$. Our strategy is almost parallel to the discussion in [1].

2 Setting the stage

Changing the variables of integral, we have

$$\begin{aligned} J_2(n) &= \int_0^\infty\int_0^\infty \frac{e^{-(t+s)/2}}{1-e^{-(t+s)}}\left(\frac{(1-e^{-2t})(1-e^{-2s})}{(1-e^{-(t+s)})^2}\right)^n dtds \\ &= 4\int_0^1\int_0^1 \left(\frac{(1-X^4)(1-Y^4)}{(1-X^2Y^2)^2}\right)^n \frac{dXdY}{1-X^2Y^2}. \end{aligned} \tag{2.1}$$

Thus we obtain the expression

$$w_2(t) = \sum_{n=0}^\infty J_2(n)t^n = 4\int_0^1\int_0^1 \frac{1-X^2Y^2}{(1-X^2Y^2)^2 - t(1-X^4)(1-Y^4)}dXdY \tag{2.2}$$

of $w_2(t)$ as an integration of a rational function. This integral expression is calculated as

$$\begin{aligned} \frac{1}{4}w_2(t) &= \int_0^1\int_0^1 \frac{1-X^2Y^2}{(1-t)(1-X^2Y^2)^2 + t(X^2-Y^2)^2}dXdY \\ &= \frac{1}{2(1-t)}\left(\int_0^1\int_0^1 \frac{dXdY}{(1-X^2Y^2)+T(X^2-Y^2)} + \int_0^1\int_0^1 \frac{dXdY}{(1-X^2Y^2)-T(X^2-Y^2)}\right) \\ &= \frac{1}{1-t}\int_0^1\int_0^1 \frac{dXdY}{(1-X^2Y^2)+T(X^2-Y^2)}, \end{aligned} \tag{2.3}$$

where we put $T = \sqrt{\frac{t}{t-1}}$ or $t = \frac{T^2}{T^2-1}$. Set $Q_T(X,Y) = (1-X^2Y^2) + T(X^2-Y^2)$. Then we define $\mathcal{W}_2(T)$ by

$$\mathcal{W}_2(T) = \frac{1-t}{4}w_2(t) = \int_\square \frac{dXdY}{Q_T(X,Y)}, \tag{2.4}$$

where $\square$ denotes the domain $[0,1]\times[0,1]$. The denominator $Q_T(X,Y)$ of the integrand in $\mathcal{W}_2(T)$ defines an algebraic curve $Q_T\,:\,Q_T(X,Y)=0$ in $\mathbb{C}^2$. From (1.9), it is easy to see that the function $\mathcal{W}_2(T)$ satisfies the differential equation $\mathcal{L}(\mathcal{W}_2)=0$. Here the differential operator $\mathcal{L}$ is given by

$$\mathcal{L}=(T^3-T)\frac{d^2}{dT^2}+(3T^2-1)\frac{d}{dT}+T. \tag{2.5}$$

In the sequel, we explain that the differential equation $\mathcal{L}(\mathcal{W}_2)=0$ has a geometric origin, that is,

(1) the algebraic curve Q_T is birationally equivalent to a certain elliptic curve C_T for all but finite values of T, and $\{C_T\}_{T\in\sqrt{-1}\mathbb{Q}^\times}$ gives the family of elliptic curves having rational 4-torsion.

(2) the differential equation $\mathcal{L}(\mathcal{W}_2)=0$ is the Picard-Fuchs equation corresponding to the family $\{C_T\}_T$.

Remark 2.1. In [9], the differential equation (1.9) is solved so that we have

$$w_2(t)=\frac{J_2(0)}{1-t}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{t}{t-1}\right), \tag{2.6}$$

where ${}_2F_1\,(a,b;c;z)$ denotes the Gaussian hypergeometric function. As a corollary, we obtain the hypergeometric expression

$$\mathcal{W}_2(T)={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;T^2\right) \tag{2.7}$$

for $\mathcal{W}_2(T)$. This yields the binomial expression for $J_2(n)$

$$J_2(n)=\frac{\pi^2}{2}\sum_{k=0}^{n}(-1)^k\binom{-\frac{1}{2}}{k}^2\binom{n}{k}. \tag{2.8}$$

■

3 Elliptic curves associated to $J_2(n)$

Let us consider the series of birational transformations

$$X=\frac{1}{1+x_1},\quad Y=\frac{x_1+y_1+1}{\sqrt{T}(x_1+y_1)+1} \tag{3.1a}$$

$$x_1=\frac{(1+\sqrt{T})y_2}{(1+T)x_2-(1+\sqrt{T})y_2-1},\quad y_1=\frac{(1+\sqrt{T})(1+\sqrt{T}(1-\sqrt{T})y_2)}{(1+T)((1+T)x_2-(1+\sqrt{T})y_2-1)} \tag{3.1b}$$

$$x_2=x_3,\quad y_2=\frac{y_3}{x_3} \tag{3.1c}$$

$$x_3=\frac{-1+6\sqrt{T}-6T+6\sqrt{T}T-T^2+12x}{12\sqrt{T}(1+T)^2},\quad y_3=\frac{1-5T-5T^2+T^3-12(1+T)x+24y}{24T(1+\sqrt{T})(1+T)^2}. \tag{3.1d}$$

The first transformation (3.1a) reduces the curve Q_T to the cubic curve

$$\begin{aligned}2(1+T)(1-\sqrt{T})y_1+(1+T)(1-T)y_1^2-2\sqrt{T}(1-\sqrt{T})^2x_1+2(1-\sqrt{T})(1-\sqrt{T}+T+T\sqrt{T})x_1y_1\\+\sqrt{T}(\sqrt{T}-1)(T+\sqrt{T}-4)x_1^2-2\sqrt{T}(1-\sqrt{T})x_1^2y_1-2\sqrt{T}(1-\sqrt{T})x_1^3=0,\end{aligned} \tag{3.2}$$

and the remaining three steps (3.1b), (3.1c), (3.1d) are the regular procedure to obtaining the standard form of an elliptic curve (see, e.g. [12]).

By a tedious step-by-step calculation, we obtain the following key lemma.

Lemma 3.1. *By the transformation $X = X(x, y)$ and $Y = Y(x, y)$ given in* (3.1)*, the curve Q_T is birationally equivalent to the elliptic curve $C_T : C_T(x, y) = 0$ defined by*

$$C_T(x,y) = \left(x^3 - \frac{T^4+14T^2+1}{48}x + \frac{T^6-33T^4-33T^2+1}{864}\right) - y^2. \tag{3.3}$$

Furthermore, the equality

$$\frac{dXdY}{Q_T(X,Y)} = \frac{dxdy}{2C_T(x,y)} \tag{3.4}$$

holds. □

Let us look at the resulting elliptic curve

$$\begin{aligned} C_T : y^2 &= x^3 - \frac{T^4+14T^2+1}{48}x + \frac{T^6-33T^4-33T^2+1}{864} \\ &= \frac{1}{864}(1-6T+T^2-12x)(1+6T+T^2-12x)(1+T^2+6x). \end{aligned} \tag{3.5}$$

The points of order 2 on C_T are

$$\left(\frac{1-6T+T^2}{12}, 0\right), \left(\frac{1+6T+T^2}{12}, 0\right), \left(-\frac{1+T^2}{6}, 0\right). \tag{3.6}$$

For any $T \in \mathbb{C}$, these three points together with the point of infinity (the identity element of the group $C_T(\mathbb{C})$) form a finite subgroup of $C_T(\mathbb{C})$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We also conclude that the curve C_T is singular if and only if $T = 0, \pm 1, \infty$ since the discriminant of C_T is equal to $\frac{1}{256}T^2(1-T^2)^4$.

When $T = \sqrt{-1}u \in \sqrt{-1}\mathbb{Q}$, C_T has only one *rational* 2-torsion $(\frac{u^2-1}{6}, 0)$. In this case, the points

$$\left(\frac{1+5u^2}{12}, \pm\frac{u(1+u^2)}{4}\right) \tag{3.7}$$

are the all of the rational points on C_T of order 4, and hence the torsion part of $C_T(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ by Mazur's theorem on the structure of torsion subgroups.

In [7], it is shown that the universal family of elliptic curves with rational 4-torsion is given by

$$E_b : y^2 + xy - by = x^3 - bx^2, \qquad \Delta = b^4(1+16b) \neq 0. \tag{3.8}$$

In this form, the origin $(0, 0)$ is a generator of the 4-torsion. If we set

$$b = \frac{\sqrt{T}(T+1)}{2(\sqrt{T}-1)^4}, \tag{3.9}$$

then we see that E_b and C_T are birationally equivalent, and the family $\{C_T\}$ is the universal family of the elliptic curves equipped with rational 4-torsion described by another form.

4 Geometric interpretation of the differential equation for $\mathcal{W}_2(T)$

We follow the discussion expanded in [1] to see that the differential equation (1.9) is the Picard-Fuchs equation associated with the family of elliptic curves $\{C_T\}$.

We recall that the function $\mathcal{W}_2(T)$ is the unique (up to constant) holomorphic solution of the differential equation $\mathcal{L}(\mathcal{W}_2) = 0$ ($\mathcal{L}$ is given in (2.5)) around $T = 0$.

Fix a point $T_0 \in \mathbb{C}$ near the origin and take a closed curve γ through T_0 such that γ does not contain any singularity of the form $\frac{dXdY}{Q_T(X,Y)}$. Then, following the idea in [1], we consider the analytic continuation of $\mathcal{W}_2(T)$ along γ, which is given by

$$\widetilde{\mathcal{W}_2}(T) := \int_D \frac{dXdY}{Q_T(X,Y)} \tag{4.1}$$

where $D \subset \mathbb{C}^2$ is a suitable domain such that $\partial D = \partial\square$. Notice that $\widetilde{\mathcal{W}_2}(T)$ also satisfies the differential equation $\mathcal{L}(\widetilde{\mathcal{W}_2}) = 0$, and the integral

$$\widetilde{\mathcal{W}_2}(T) - \mathcal{W}_2(T) = \int_{D-\square} \frac{dXdY}{Q_T(X,Y)} \tag{4.2}$$

is over the closed and oriented surface (2-cycle) $D - \square$. Namely, this integral can be written as the form

$$I_\Gamma(T) := \int_\Gamma \frac{dXdY}{Q_T(X,Y)} \tag{4.3}$$

for a certain 2-cycle Γ in $\mathbb{C}^2 \setminus Q_T$. By Lemma 3.1, the integral $I_\Gamma(T)$ becomes

$$I_S(T) := \frac{1}{2}\int_S \frac{dxdy}{C_T(x,y)} \tag{4.4}$$

for some $S \in H_2(\mathbb{C}^2 \setminus C_T, \mathbb{Z})$. By the same 'topological reduction' discussion as in [1], it follows that

$$I_S(T) = \frac{\pi i}{2}\int_\gamma \frac{dx}{y} \tag{4.5}$$

for a certain 1-cycle $\gamma \in H_1(\mathbb{C}^2 \setminus C_T, \mathbb{Z})$. Thus the difference $\widetilde{\mathcal{W}_2}(T) - \mathcal{W}_2(T)$ is a constant multiple of a period of the holomorphic 1-form $\frac{\pi i dx}{2y}$. Hence $\mathcal{W}_2(T)$ (and $\widetilde{\mathcal{W}_2}(T)$) and the period $I_S(T)$ of $\frac{\pi i dx}{2y}$ satisfy the same differential equation (see also Remark 4.2).

Let us determine the differential equation for the integral

$$I_\gamma(T) := \pi i \int_\gamma \frac{dx}{y}. \tag{4.6}$$

It is well known that this kind of integral $I_\gamma(T)$ satisfies a second order Fuchs-type differential equation. The local exponents of $I_\gamma(T)$ at the singularities $0, 1, -1, \infty$ are respectively given by $0, 0, 0, 1$. Therefore, the differential equation for $I_\gamma(T)$ is of the form

$$(T^3 - T)F''(T) + (3T^2 - 1)F'(T) + (T + A)F(T) = 0 \tag{4.7}$$

for a certain constant A. Now we take γ as a closed path in the x-plane surrounding just two of the singularities $\frac{1 \pm 6T + T^2}{12}$ of the 1-form $\frac{dx}{2y}$, which go to $\frac{1}{12}$ as $T \to 0$, so that this integral $I_\gamma(T)$ gives the *holomorphic* solution of this differential equation (4.7) around the origin $T = 0$. Calculating the integral $I_\gamma(T)$ directly using the residue theorem, we notice that the Taylor expansion of $I_\gamma(T)$ for sufficient small T is given by

$$I_\gamma(T) = 1 + \frac{1}{4}T^2 + \frac{9}{64}T^4 + \cdots. \tag{4.8}$$

From this expansion (4.8), it follows that $A = 0$.

Hence, put the discussions above together, we now conclude the

Theorem 4.1. *The differential equation $\mathcal{L}(\mathcal{W}_2) = 0$ is the Picard-Fuchs equation associated with the family*

$$C_T : y^2 = x^3 - \frac{T^4 + 14T^2 + 1}{48}x + \frac{T^6 - 33T^4 - 33T^2 + 1}{864} \tag{4.9}$$

of elliptic curves equipped with rational 4-torsion. $\square$

Remark 4.2. Let $\widetilde{\mathcal{L}}(I_\gamma) = 0$ be the second order linear differential equation for the period $I_\gamma(T)$ of the holomorphic 1-form $\frac{dx}{2y}$ associated to the elliptic curves $\{C_T\}$. From the equation $\widetilde{\mathcal{W}_2}(T) - \mathcal{W}_2(T) = \pi i I_\gamma(T)$, we have $\widetilde{\mathcal{L}}(\widetilde{\mathcal{W}_2}) = \widetilde{\mathcal{L}}(\mathcal{W}_2)$. Hence we see that the monodromy group of $\widetilde{\mathcal{L}}(\mathcal{W}_2)$ is trivial so that it is a rational function. In our case, we know a priori that the function $\mathcal{W}_2(T)$ itself satisfy the second order linear differential equation of the form $\mathcal{L}(\mathcal{W}_2) = 0$, and it follows that the operators $\mathcal{L}$ and $\mathcal{L}'$ are identical, up to a constant multiple. ■

5 Modular properties

Let $\lambda(\tau)$ be the classical Legendre modular function

$$\lambda(\tau) = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^8 = 16 \frac{\eta(\tau)^8 \eta(4\tau)^{16}}{\eta(2\tau)^{24}} = 16q - 128q^2 + \cdots , \tag{5.1}$$

where $q = e^{\tau \pi i}$ and $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$. This is a modular function for the congruent subgroup

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\,;\, c \equiv 0 \,(\mathrm{mod}\, 4) \right\}. \tag{5.2}$$

Further, it is known that the modular function field for $\Gamma_0(4)$ is equal to $\mathbb{C}(\lambda(\tau))$. Based on the study [14] (see also Remark 5.3), Zagier pointed out the following theorem from the list therein.

Theorem 5.1. *The equality*

$$w_2(\lambda(\tau)) = \frac{\vartheta_0(\tau)^2}{1-\lambda(\tau)} = \frac{\eta(2\tau)^{22}}{\eta(\tau)^{12} \eta(4\tau)^8} \tag{5.3}$$

holds. Here $\vartheta_0(\tau)$ is the elliptic theta function (which is a $\Gamma_0(4)$-modular form of weight $\frac{1}{2}$)

$$\vartheta_0(\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)} = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = 1 - 2q + 2q^4 + \cdots . \tag{5.4}$$

Notice that $w_2(\lambda)$ is not holomorphic. We show that the theorem is deduced by the same mechanism as the discussion in [1].

Proof. Let $\omega(\tau)$ be a $\Gamma_0(4)$-modular form of weight 1 and put $\tilde{\omega}(\tau) = \tau \omega(\tau)$. Then

$$\omega \left(\frac{a\tau + b}{c\tau + d} \right) = c\,\tilde{\omega}(\tau) + d\,\omega(\tau), \qquad \tilde{\omega} \left(\frac{a\tau + b}{c\tau + d} \right) = a\,\tilde{\omega}(\tau) + b\,\omega(\tau) \tag{5.5}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. Hence, both $\omega(\tau)$ and $\tilde{\omega}(\tau)$, considered as functions in $\lambda = \lambda(\tau)$, satisfy the differential equation

$$\begin{vmatrix} \omega & \omega' \\ \tilde{\omega} & \tilde{\omega}' \end{vmatrix} F'' - \begin{vmatrix} \omega & \omega'' \\ \tilde{\omega} & \tilde{\omega}'' \end{vmatrix} F' + \begin{vmatrix} \omega' & \omega'' \\ \tilde{\omega}' & \tilde{\omega}'' \end{vmatrix} F = 0 \tag{5.6}$$

where the prime denotes the differential with respect to λ. The coefficients in the differential equation are calculated as

$$\begin{vmatrix} \omega & \omega' \\ \tilde{\omega} & \tilde{\omega}' \end{vmatrix} = \frac{\omega^2}{\frac{d\lambda}{d\tau}}, \quad \begin{vmatrix} \omega & \omega'' \\ \tilde{\omega} & \tilde{\omega}'' \end{vmatrix} = \left(\frac{\omega^2}{\frac{d\lambda}{d\tau}} \right)', \quad \begin{vmatrix} \omega' & \omega'' \\ \tilde{\omega}' & \tilde{\omega}'' \end{vmatrix} = \frac{2(\frac{d\omega}{d\tau})^2 - \omega \frac{d^2\omega}{d\tau^2}}{(\frac{d\lambda}{d\tau})^3}. \tag{5.7}$$

Notice that these determinants are $\Gamma_0(4)$-modular functions so that they are rational functions in λ.

Now we take $\omega(\tau) = \frac{\vartheta_0(\tau)^2}{1-\lambda(\tau)}$. Using well-known formulas among the values of elliptic functions and theta functions, we see that the equations

$$\begin{vmatrix}\omega & \omega' \\ \tilde{\omega} & \tilde{\omega}'\end{vmatrix} = \frac{1}{\pi i}\frac{1}{\lambda(1-\lambda)^2}, \quad \begin{vmatrix}\omega & \omega'' \\ \tilde{\omega} & \tilde{\omega}''\end{vmatrix} = -\frac{1}{\pi i}\frac{(1-3\lambda)(1-\lambda)}{\lambda^2(1-\lambda)^4}, \quad \begin{vmatrix}\omega' & \omega'' \\ \tilde{\omega}' & \tilde{\omega}''\end{vmatrix} = \frac{1}{\pi i}\frac{\lambda-\frac{3}{4}}{\lambda^2(1-\lambda)^4} \tag{5.8}$$

holds (we postpone the calculation; see Lemma 5.2 below). Thus we find that ω satisfies

$$\lambda(1-\lambda)^2\omega''(\lambda) - (1-3\lambda)(1-\lambda)\omega'(\lambda) + (\lambda - \frac{3}{4})\omega(\lambda) = 0. \tag{5.9}$$

Comparing this differential equation with (1.9), we have proved the lemma. □

Lemma 5.2. *The equations of determinants in* (5.8) *hold.*

Proof. We use the convention for elliptic and/or theta functions in [4]. Denote by ω_1, ω_2 the fundamental periods so that $\tau = \frac{\omega_2}{\omega_1}$, and by $\vartheta_j(v,\tau)$ $(j = 0,1,2,3)$ the elliptic theta functions. We put $\vartheta_j = \vartheta_j(0,\tau)$ $(j = 0,2,3)$ and $\vartheta_1' = \frac{\partial\vartheta_1}{\partial v}(0,\tau)$. We also put $e_1 = \wp\left(\frac{\omega_1}{2}\right)$ and $e_2 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$, where $\wp(z) = \wp(z;\omega_1,\omega_2)$ is the Weierstrass $\wp$-function. Then we have

$$e_1 = \frac{4\pi i}{\omega_1^2}\left(\frac{1}{3}\frac{d\log\vartheta_1'}{d\tau} - \frac{d\log\vartheta_2}{d\tau}\right), \qquad e_2 = \frac{4\pi i}{\omega_1^2}\left(\frac{1}{3}\frac{d\log\vartheta_1'}{d\tau} - \frac{d\log\vartheta_3}{d\tau}\right), \tag{5.10}$$

and it follows that

$$\vartheta_0^4 = \frac{\omega_1^2}{\pi^2}(e_1 - e_2) = \frac{4}{\pi i}\left(\frac{d\log\vartheta_2}{d\tau} - \frac{d\log\vartheta_3}{d\tau}\right). \tag{5.11}$$

On the other hand, since $\lambda(\tau) = \left(\frac{\vartheta_2}{\vartheta_3}\right)^4$, we have

$$\frac{d\log\lambda}{d\tau} = 4\left(\frac{d\log\vartheta_2}{d\tau} - \frac{d\log\vartheta_3}{d\tau}\right) = \pi i\vartheta_0^4. \tag{5.12}$$

Thus we obtain

$$\begin{vmatrix}\omega & \omega' \\ \tilde{\omega} & \tilde{\omega}'\end{vmatrix} = \frac{\omega^2}{\frac{d\lambda}{d\tau}} = \frac{\vartheta_0^4}{\lambda(1-\lambda)^2\frac{d\log\lambda}{d\tau}} = \frac{1}{\pi i}\frac{1}{\lambda(1-\lambda)^2}. \tag{5.13}$$

The third determinant $\begin{vmatrix}\omega & \omega'' \\ \tilde{\omega} & \tilde{\omega}''\end{vmatrix}$ is also calculated in the same way, but the calculation is rather complicated. The second determinant $\begin{vmatrix}\omega & \omega'' \\ \tilde{\omega} & \tilde{\omega}''\end{vmatrix} = -\frac{1}{\pi i}\frac{(1-3\lambda)(1-\lambda)}{\lambda^2(1-\lambda)^4}$ is readily obtained by differentiate $\frac{1}{\pi i}\frac{1}{\lambda(1-\lambda)^2}$ by λ. □

Remark 5.3. Consider the differential equation

$$\left((t^3 + at^2 + bt)F'(t)\right)' + (t - \lambda)F(t) = 0 \tag{5.14}$$

with rational parameters a, b, λ, which is due to Beukers. When $a = 11$, $b = -1$ and $\lambda = -3$, this equation (5.14) is exactly the one for the generating function $A(t)$ of the Apéry numbers a_n. In [14], Zagier searches the triplets (a, b, λ) of integers within a certain domain such that (5.14) has an integral solution (i.e. solutions in $\mathbb{Z}[[t]]$), and presents a list of 36 such solutions. It is shown that the twelve solutions of them have parametrizations in terms of modular functions. Zagier noticed and pointed out that our generating function $w_2(t)$ of $J_2(n)$ is #19 in his list. ■

6 Closing remarks

In the final position of the note, we give several remarks for the future study.

6.1

In the discussions above, we see that the numbers $J_2(n)$ arising from the value $\zeta_Q(2)$ acquire the $\Gamma_0(4)$-modularity associated with the family of elliptic curves equipped with rational 4-torsion. However, at this moment, we have no intrinsic explanation from the level of non-commutative harmonic oscillators why such things hold.

6.2

The Apéry-like numbers $J_2(n)$ are arising in the expression (1.6) for $\zeta_Q(2)$ via the generating function

$$g_2(z) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} J_2(n) z^n. \tag{6.1}$$

This function satisfies the differential equation $\mathcal{D}(g_2) = 0$ (see [8]). Here $\mathcal{D}$ is given by

$$\mathcal{D} = 8z^2(1+z)^2 \frac{d^3}{dz^3} + 24z(1+z)(1+2z)\frac{d^2}{dz^2} + 2(4+27z+27z^2)\frac{d}{dz} + 3(1+2z). \tag{6.2}$$

Does the function $g_2(z)$ have a modular form interpretation like $w_2(t)$? If this is true, then $g_2(z)$ should be a modular form of weight 2 from the result in [13]. We also note that $g_2(z)$ has the following explicit expressions by hypergeometric functions [9]:

$$g_2(z) = \frac{1}{\sqrt{1+z^2}}\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{z^2}{1+z^2}\right) = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -z^2\right)^2. \tag{6.3}$$

6.3

The Apéry numbers a_n and b_n are corresponding to $\zeta(2)$, satisfying the same recurrence formula. Their generating functions satisfy the differential equations $L_2(A) = 0$ and $L_2(B) = -5$ for the same operator L_2 but one is homogeneous and the other is inhomogeneous.

Denote by α_n and β_n the Apéry numbers corresponding to $\zeta(3)$. They satisfy the same recurrence formula

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n-1)^3 u_{n-2} = 0 \tag{6.4}$$

with initial conditions $\alpha_0 = 0$, $\alpha_1 = 6$, $\beta_0 = 1$ and $\beta_1 = 5$. Their generating functions $\mathbb{A}(t) = \sum_{n=0}^{\infty} \alpha_n t^n$ and $\mathbb{B}(t) = \sum_{n=0}^{\infty} \beta_n t^n$ satisfy the differential equations $L_3(\mathbb{A}) = 0$ and $L_3(\mathbb{B}) = 5$ for the same operator

$$L_3 = t^2(t^2 - 34t + 1)\frac{d^3}{dt^3} + 3t(2t^2 - 51t + 1)\frac{d^2}{dt^2} + (7t^2 - 112t + 1)\frac{d}{dt} + (t-5), \tag{6.5}$$

but one is homogeneous and the other is inhomogeneous. Thus the situation is exactly the same for the Apéry numbers corresponding to $\zeta(2)$. It was proved in [2] that the differential equation $L_3(\mathbb{A}) = 0$ is the Picard-Fuchs equation associated to a certain family of $K3$ surfaces.

In our spectral case, the situation seems similar at a glance, but it is different. The Apéry-like numbers $J_2(n)$ are corresponding to the special value $\zeta_Q(2)$, while

$$J_3(n) := 8 \int_0^1 \int_0^1 \int_0^1 \left(\frac{(1-X^4)(1-Y^4Z^4)}{(1-X^2Y^2Z^2)^2} \right)^n \frac{dZ dY dZ}{1 - X^2Y^2Z^2} \tag{6.6}$$

are corresponding to the special value $\zeta_Q(3)$ (see [8]). Let $w_3(t)$ be the generating function for $J_3(n)$ defined by $w_3(t) = \sum_{n=0}^{\infty} J_3(n)t^n$. The generating functions $w_2(t)$ and $w_3(t)$ satisfy the differential equations

$$D(w_2) = 0 \qquad \text{and} \qquad D(w_3) = \frac{1}{2}{}_2F_1\left(1,1;\frac{3}{2};t\right) \tag{6.7}$$

respectively for the same operator

$$D = t(1-t^2)\frac{d^2}{dt^2} - (1-3t)(1-t)\frac{d}{dt} + (t-\frac{3}{4}). \tag{6.8}$$

Notice that the former equation is homogeneous, but the latter is inhomogeneous. This fact shows that the structure of 'pairing' in our theory is different from the case of the Apéry numbers.

6.4

From the expressions (1.5) and (6.6) of the Apéry-like numbers associated to $\zeta_Q(2)$ and $\zeta_Q(3)$, it is quite natural to consider the numbers $J_k(n)$ defined as

$$\begin{aligned} J_k(n) &:= \int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{e^{-(t_1+\cdots+t_k)/2}}{1-e^{-(t_1+\cdots+t_k)}}\left(\frac{(1-e^{-2t_1})(1-e^{-2(t_2+\cdots+t_k)})}{(1-e^{-(t_1+\cdots+t_k)})^2}\right)^n dt_1dt_2\cdots dt_k \\ &= 2^k\int_0^1\int_0^1\cdots\int_0^1 \frac{1}{1-x_1^2\cdots x_k^2}\left(\frac{(1-x_1^4)(1-x_2^4\cdots x_k^4)}{(1-x_1^2\cdots x_k^2)^2}\right)^n dx_1dx_2\cdots dx_k. \end{aligned} \tag{6.9}$$

Note that the numbers $J_k(n)$ are all positive. Their generating functions are

$$\begin{aligned} w_k(z) := \sum_{n=0}^\infty J_k(n)z^n &= 2^k\int_0^1\int_0^1\cdots\int_0^1 \frac{1-x_1^2\cdots x_k^2}{(1-x_1^2\cdots x_k^2)^2-(1-x_1^4)(1-x_2^4\cdots x_k^4)z}dx_1dx_2\cdots dx_k \\ &= 2^k\int_0^1\int_0^1\cdots\int_0^1 \frac{1-x_1^2\cdots x_k^2}{(1-x_1^2\cdots x_k^2)^2-(1-x_1^4)(1-x_2^4\cdots x_k^4)z}dx_1dx_2\cdots dx_k \\ &= \frac{2^{k-1}}{1-z}\Bigg(\int_0^1\int_0^1\cdots\int_0^1 \frac{dx_1dx_2\cdots dx_k}{(1-x_1^2x_2^2\cdots x_k^2)+Z(x_1^2-x_2^2\cdots x_k^2)} \\ &\qquad + \int_0^1\int_0^1\cdots\int_0^1 \frac{dx_1dx_2\cdots dx_k}{(1-x_1^2x_2^2\cdots x_k^2)-Z(x_1^2-x_2^2\cdots x_k^2)}\Bigg) \end{aligned} \tag{6.10}$$

where $Z = \sqrt{\frac{z}{z-1}}$. It is not hard to see the formula $w_k(0) = (2^k-1)\zeta(k)$. Then, we can ask the question; is it true that the function $w_k(z)$ $(k \geq 3)$ satisfies a Picard-Fuchs type differential equation coming from certain family of algebraic varieties?

We remark that the number $J_k(n)$ has the double integral expression

$$J_k(n) = \frac{2^k}{\Gamma(k-1)}\int_0^1\int_0^1 \frac{(-\log X)^{k-2}}{1-X^2Y^2}\left(\frac{(1-X^4)(1-Y^4)}{(1-X^2Y^2)^2}\right)^n dXdY. \tag{6.11}$$

Acknowledgement

The second author would like to thank Lin Weng and Masanobu Kaneko, the organizers of the *Conference on L-functions*, for presenting him the precious occasion to give a talk on the subject. We are also very grateful to Don Zagier, who gave us an accurate suggestion and his storing interest on our Apéry-like numbers, and kindly sent us his unpublished note [14].

References

[1] Beukers, F.: Irrationality of π^2, periods of an elliptic curve and $\Gamma_1(5)$. Diophantine approximations and transcendental numbers (Luminy, 1982), 47–66, Progr. Math., **31**, Birkhäuser, Boston, Mass., 1983.

[2] Beukers, F. and Peters, C. A. M.: A family of $K3$ surfaces and $\zeta(3)$. *J. Reine Angew. Math.* **351** (1984), 42–54.

[3] Deutsch, E. and Sagan, B. E.: Congruences for Catalan and Motzkin numbers and related sequences. *J. Number Theory* **117** (2006), no. 1, 191–215.

[4] Hurwitz, A. and Courant, R.: "Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen." Interscience Publishers, Inc., New York, (1944).

[5] Ichinose, T. and Wakayama, M.: Zeta functions for the spectrum of the non-commutative harmonic oscillators. *Commun. Math. Phys.* **258** (2005), 697–739.

[6] Ichinose, T. and Wakayama, M.: Special values of the spectral zeta function of the non-commutative harmonic oscillator and confluent Heun equations. *Kyushu J. Math.* **59** (2005), 39–100.

[7] Kubert, D. S.: Universal bounds on the torsion of elliptic curves. Proc. London Math. Soc. **33** (1976), 193–237.

[8] Kimoto, K. and Wakayama, M.: Apéry-like numbers arising from special values of spectral zeta functions for non-commutative harmonic oscillators. Kyushu J. Math. **60** (2006) (ArXiv math.NT/0603700).

[9] Ochiai, H.: A special value of the spectral zeta function of the non-commutative harmonic oscillators. To appear in *Ramanujan J.* (ArXiv math.RT/0507111).

[10] Parmeggiani, A. and Wakayama, M.: Oscillator representations and systems of ordinary differential equations. *Proc. Natl. Acad. Sci. USA* **98** (2001), 26–30.

[11] Van der Poorten, A. J.: A proof that Euler missed…Apéry's proof of the irrationality of $\zeta(3)$. *Math. Intelligencer* **1** (1978/1979), 195–203.

[12] Silverman, J. H. and Tate, J.: "Rational points on elliptic curves." Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1992.

[13] Zagier, D.: Modular forms and differential operators. *Proc. Indian Acad. Sci. Math. Sci.* **104** (1994), no 1, 57–75.

[14] Zagier, D.: Integral solutions of Apéry-like recurrence equations. Preprint.

Kazufumi KIMOTO
Department of Mathematical Science, University of the Ryukyus.
Senbaru, Nishihara, Okinawa 903-0231, JAPAN.
`kimoto@math.u-ryukyu.ac.jp`

Masato WAKAYAMA
Faculty of Mathematics, Kyushu University.
Hakozaki, Fukuoka 812-8518, JAPAN.
`wakayama@math.kyushu-u.ac.jp`