Selberg zeta functions
of infinite symmetric groups

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Abstract

The main purpose of this paper is to seek a reasonable formulation of Selberg zeta functions of infinite symmetric groups and calculate actual candidates of them. In order to achieve this attempt, we introduce a (Selberg-type) zeta function attached to a finite group action $G \curvearrowright X$. As candidates of Selberg zeta functions of $\mathfrak{S}_\infty$, we calculate a (normalized) limit of the zeta functions of finite symmetric group actions. We also show that this zeta function is a generating function of certain quantities called moments of the action, which determine the multiple transitivity of group actions.

Key words: Selberg-type zeta functions, the symmetric group $\mathfrak{S}_\infty$ of infinite degree, finite group actions.

1 Introduction

In the beginning, we propose the following naive problem, which is our main concern here: What is a (Selberg-type) zeta function attached to (a homogeneous space of) a large group such as an infinite-dimensional group? In this paper we study the case of the symmetric group $\mathfrak{S}_\infty$ of infinite degree. The main theme of this paper is to seek a reasonable formulation of a Selberg zeta function of $\mathfrak{S}_\infty$.

One of the motivations to study infinite-dimensional objects is an expectation that via the infinite-dimensional framework, we may obtain ‘transcendental’ results which are significantly difficult to prove in the finite dimensional framework, as well as we can give better and clear proofs to a known result. A typical example is a classical results on the expansion formulas of the $\eta$-function including the famous pentagonal number theorem due to Euler are obtained in a unified way by calculating the denominator formula for affine Kac-Moody Lie algebras [8].

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On the other hand, it is one of the most important problems, which is interesting of its own right, to formulate Selberg-type zeta functions of infinite-dimensional groups — for instance, the inductive limit groups $GL_\infty$, $Sp_\infty$, $O_\infty$ of finite dimensional groups, the group $\text{Diff}(S^1)$ of all diffeomorphisms on $S^1$, etc. — in an appropriate sense, because it is also related to a formulation problem of trace formulas of infinite-dimensional spaces. The existence and/or a realization of a certain duality often shed light on the situation and make us possible to perform a deeper analysis. A typical example is seen in the particular roles and various applications of the Selberg trace formula on a locally symmetric space, which is clarifying the relation between the information of the Laplacian on that space (a spectral information) and that of primitive closed geodesics (a geometric information). For instance, trace formulas allow us to prove various kinds of distribution theorems like the Prime Geodesic Theorem, which concerns the length distribution of primitive closed geodesics; we can evaluate the spectral side of trace formula by using the Weyl law on the spectrum of Laplacian, and the geometric side is estimated from this evaluation of the spectral side. Such clear descriptions of dualities are ideal, and hence they provide a guiding principle of drawing a similar picture of certain dualities for other situations.

Our particular interest here is poured on a formulation of Selberg-type zeta functions of the symmetric group $S_1$ of infinite degree, the inductive limit group of the finite symmetric groups $S_n$. We mention that we deal with the case of full symmetric group, i.e. the group consisting of all permutations on $\mathbb{N}$, from a spectral point of view in [3, 4].

There are lots of actual examples such that an asymptotic or approximating method works effective. Let us show a typical example of such a situation. In [13], which is one of the pioneering works on infinite-dimensional groups, Thoma determined all the characters of factor representations of $S_\infty$ and gave an explicit formula of them. In [14], Vershik and Kerov give another proof of Thoma’s character formula by using their asymptotic formula of irreducible characters of $S_n$, which also clarifies an interpretation of parameters appearing in the character formula.

According to these facts, it is natural to employ an approximating method when we make an attempt to formulate a Selberg-type zeta function of the infinite symmetric group $S_\infty$. Therefore, one reasonable way of obtaining a zeta function of $S_\infty$ is explained as follows. We introduce a zeta function $Z(u; G, X)$ of a finite group action $G \curvearrowright X$ — we denote by $G \curvearrowright X$ the situation that a group $G$ acts on a set $X$ — and define $Z(u; S_\infty)$ by taking a limit of $Z(u; S_n, [n])$ attached to the natural action $S_n \curvearrowright [n]$: Actually we deal with the modified log-derivative $\Xi(u; G, X)$ of a zeta function $Z(u; G, X)$.

Hence we begin by introducing a zeta function of finite group actions appropriately. We will see that our zeta function $Z(u; G, X)$ attached to a finite group action $G \curvearrowright X$ is also regarded as a kind of generating function of certain quantities $m(k; G, X)$ called moments. We show that these moments describe multiple transitivity of the group action. These facts are also crucial for actual calculation of zeta functions.
2 Zeta functions and moments

In this section we introduce zeta functions and moments attached to finite group actions, and study the general properties of these objects.

2.1 Preliminary observation of Selberg zeta function

We recall the definition of the original Selberg zeta function $Z_M(s)$ for a locally symmetric space $M$. It is defined by the following Euler product form (See e.g. [2]):

$$Z_M(s) := \prod_{m=1}^{\infty} \prod_{\gamma} (1 - N(\gamma)^{-s+m}).$$

Here the inner product is taken over all primitive closed geodesics $\gamma$ on $M$ (or all primitive hyperbolic conjugacy classes of the fundamental group $\Gamma_M$ of $M$) where $N(\gamma)$ is the norm of $\gamma$. Thus we regard that $Z_M(s)$ is a zeta function attached to the space $M$, or attached to the fundamental group $\Gamma_M$.

We want to define an analogous function of a Selberg zeta function attached to a finite group action $G \curvearrowright X$. The original Selberg zeta function case suggests us that we first need to introduce a corresponding object of a ‘(closed) curve’ on $X$ and a notion of ‘homotopy’ among such curves. Since $X$ is discrete, it is natural to think that a curve on $X$ is a map from $\mathbb{Z}$ to $X$, or an infinite sequence $(x_n)_{n \in \mathbb{Z}}$ on $X$. Then, a periodic sequence on $X$ is suitable for a corresponding object of a closed curve. At this moment the existence of the action of $G$ on $X$ is not considered; here we reflect the existence of the action $G \curvearrowright X$ on the definition of a homotopy on $X$, which is defined as an equivalence relation $\sim$ among the infinite sequences in $X$. Based on these interpretations we introduce a zeta function $Z(u; G, X)$ attached to a finite group action $G \curvearrowright X$ as a zeta function of a certain discrete dynamical system induced from the action $G \curvearrowright X$ as we explain in the following subsection.

2.2 Zeta functions of finite group actions

Let $G$ be a finite group acting on a finite set $X$. Denote by $X^\mathbb{Z}$ the set consisting of all sequences $x = (x_n)_{n \in \mathbb{Z}}$ on $X$ indexed by $\mathbb{Z}$, the set of all integers. The shift operator $\Delta$ on $X^\mathbb{Z}$ is defined by

$$\Delta x = (x_{n-1})_{n \in \mathbb{Z}} \quad (x = (x_n)_{n \in \mathbb{Z}}),$$

that is, $\Delta$ shifts the indices of sequences. The pair $(\Delta, X^\mathbb{Z})$ forms a discrete dynamical system which is called a full shift.

For any couple of sequences $x, y \in X^\mathbb{Z}$, we say $x$ and $y$ are $(G$-)homotope if and only if there exists a certain element $g \in G$ such that $x = gy$ ($x_j = gy_j$ for all $j \in \mathbb{Z}$), and we write $x \sim y$. This relation $\sim$ clearly defines an equivalence relation on $X^\mathbb{Z}$. It is also easy
to see that the relation ~ and the shift operator ∆ is compatible, that is, \( x \sim y \) always implies \( \Delta x \sim \Delta y \). Hence we can define the action of the shift operator \( \Delta \) on the quotient set \( \mathcal{O} := X^Z / \sim \). In order to specify the acting set of the operator \( \Delta \), we denote by \( \Delta\mathcal{O} \) the shift operator acting on the quotient set \( \mathcal{O} \). It is obvious that \( (\Delta\mathcal{O}, \mathcal{O}) \) defines a discrete dynamical system.

The **zeta function** \( Z(u; G, X) \) of a given action \( G \curvearrowright X \) is defined as a zeta function of the dynamical system \( (\Delta\mathcal{O}, \mathcal{O}) \). Namely, \( Z(u; G, X) \) is defined to be a generating function of the number \( \text{fix}(\Delta^k) \) of \( k \)-periodic points of \( \Delta\mathcal{O} \) as follows:

\[
Z(u; G, X) := \exp \left( \sum_{k=1}^{\infty} \frac{\text{fix}(\Delta^k) u^k}{k} \right),
\]

which converges absolutely when \( |u| < 1/\#X \).

**Example 2.1** (Zeta function of the full shift). Suppose that \( G = \{1\} \). Then \( x \sim y \) is equivalent to the equality \( x = y \). Since every finite sequence \( (x_1, x_2, \ldots, x_k) \) of \( k \)-elements in \( X \) can be the first \( k \)-entries of a \( k \)-periodic element in \( X^Z \), the zeta function of the full shift \( X^Z \) is given by

\[
Z(u; G, X) = \exp \left( \sum_{k=1}^{\infty} \frac{N^k u^k}{k} \right) = (1 - Nu)^{-1},
\]

where we put \( N = \#X \). Here we use the following elementary formula

\[
\sum_{k=1}^{\infty} \frac{1}{k} t^k = \log \frac{1}{1 - t}.
\]

This zeta function \( Z(u; G, X) \) of an action \( G \curvearrowright X \) has the following expression as a generating function of certain quantities concerning the action \( G \curvearrowright X \).

**Theorem 2.1.** We have

(2.1) \( Z(u; G, X) = \exp \left( \sum_{k=1}^{\infty} m(k; G, X) \frac{u^k}{k} \right) \).

Here we put

\[
m(k; G, X) := \frac{1}{\#G} \sum_{g \in G} \text{fix}(g; G, X)^k,
\]

where \( \text{fix}(g; G, X) \) is the number of fixed points of \( g \in G \) on \( X \).
We call the quantity $m(k; G, X)$ the $k$-th **moment** of the action $G \curvearrowright X$. It is convenient to put $m(0; G, X) := 1$. We recall the following fundamental fact concerning the first moment $m(1; G, X)$ which is known as the Frobenius formula.

**Proposition 2.2.** Suppose that a finite group action $G \curvearrowright X$ is given. Then the number of $G$-orbits in $X$ is given by the expectation of $\text{fix}(G, X)$ as follows:

$$m(1; G, X) = \frac{1}{\# G} \sum_{g \in G} \text{fix}(g; G, X) = \# \text{Orb}_G(X).$$

Here $\text{Orb}_G(X)$ denotes the set of $G$-orbits in the $G$-set $X$. In particular, the action $G \curvearrowright X$ is transitive if and only if $m(1; G, X) = 1$. □

**Proof of Theorem 2.1.** First we notice that $\text{fix}(\Delta^k_G) = \# \text{Orb}_G(X^k)$. By the Frobenius formula (2.2) we have

$$\# \text{Orb}_G(X^k) = \frac{1}{\# G} \sum_{g \in G} \text{fix}(g; G, X^k) = m(1; G, X^k).$$

Since it is easy to see that

$$\text{fix}(g; G, X^k) = \text{fix}(g; G, X)^k \implies m(1; G, X^k) = m(k; G, X),$$

we obtain the desired conclusion. □

By the theorem above we also have another expression of $Z(u; G, X)$:

$$Z(u; G, X) = \exp \left( \sum_{k=1}^{\infty} \frac{1}{\# G} \sum_{g \in G} \text{fix}(g; G, X)^k u^k \right),$$

$$= \exp \left( \frac{1}{\# G} \sum_{g \in G} \log \frac{1}{1 - \text{fix}(g; G, X)u} \right),$$

$$= \prod_{g \in G} (1 - \text{fix}(g; G, X)u)^{-\frac{1}{\# G}}.$$  

### 2.3 Moments and multiple transitivity

The normalized Haar measure $\mu$ on $G$ defines a probability measure on $G$. Hence $(G, \mu)$ becomes a discrete probability space. The function $\text{fix}(\cdot; G, X)$ is regarded as a random variable on $G$, and the quantity $m(k; G, X)$ introduced above is the $k$-th moment.

$$\mathbb{E}[\text{fix}(G, X)^k] = \int_G \text{fix}(g; G, X)^k d\mu(g) = \frac{1}{\# G} \sum_{g \in G} \text{fix}(g; G, X)^k.$$
of the random variable $\text{fix}(\cdot; G, X)$. This is why we call the quantity $m(k; G, X)$ the $k$-th moment of the action $G \actson X$. If we put

$$P(\text{fix}(G, X) = m) := \frac{\# \{ g \in G \mid \text{fix}(g; G, X) = m \}}{\# G},$$

then it is immediate by definition that

$$m(k; G, X) = \sum_{m=0}^{\# X} m^k P(\text{fix}(G, X) = m).$$

It is convenient to introduce the $k$-th factorial moment $m(k; G, X)$ defined by

$$m(k; G, X) := \frac{1}{\# G} \sum_{g \in G} \text{fix}(g; G, X)^k,$$

where $a^k$ is the descending product $a^k := a(a - 1) \cdots (a - k + 1)$. This is the first moment of the $G$-set

$$X^k := \{ (x_1, \ldots, x_k) \mid i \neq j \Rightarrow x_i \neq x_j \}.$$

Actually, it is obvious to see that $X^k$ is a $G$-invariant subset of $X^k$ and $\# X^k = (\# X)^k$. Therefore it follows that $m(k; G, X)$ vanishes if and only if $k > \# X$.

The relation between moments and factorial moments is described as below. Denote by $[k]$ the set of $k$-points, say $[k] := \{1, 2, \ldots, k\}$. For a disjoint union decomposition

$$[k] = \coprod_{i=1}^{m} A_i$$

of the set $[k]$ into non-empty $m$ subsets $A_1, \ldots, A_m$, the set

$$\left\{ (x_1, \ldots, x_k) \in X^k \mid i, j \in A_l \Rightarrow x_i = x_j \ (l = 1, \ldots, m) \right\}$$

is $G$-equivalent to the $G$-set $X^m$. Consequently we have the

**Proposition 2.3.** The $G$-orbit decomposition of the set $X^k$ is given by

$$X^k \cong \sum_{m=1}^{k} \binom{k}{m} X^m,$$

where we denote by $\binom{k}{m}$ the Stirling number of second kind, that is, the number of the ways of dividing $[k]$ into disjoint union of non-empty $m$ subsets. The moments describe the multiple transitivity of an action. In particular we get

$$m(k; G, X) = \sum_{m=1}^{k} \binom{k}{m} m(m; G, X)$$

by comparing the cardinalities in the decomposition above. \hfill \Box
Now we recall the notion of \textbf{multiple transitivity} of a group action: A group action \( G \acts X \) is said to be \textit{\( m \)-ply transitive} (or simply \( m \)-transitive) if the following condition holds: For any two pairs of collections \( \{ x_1, \ldots, x_m \} \) and \( \{ y_1, \ldots, y_m \} \) consisting of mutually distinct points in \( X \), there exists some \( g \in G \) such that \( gx_j = y_j \) for all \( j \). We remark that if \( G \acts X \) is \( m \)-transitive, then it is obviously \( l \)-transitive for every \( l \leq m \).

\textbf{Theorem 2.4.} Assume that a finite group action \( G \acts X \) is given. Then the following conditions are equivalent for every \( k \leq \#X \).

\begin{enumerate}[(a)]
\item \( G \acts X \) is \( k \)-transitive.
\item \( G \acts X^k \) is transitive.
\item \( m(k; G, X) = \beta_k \).
\item \( m(k; G, X) = 1 \).
\end{enumerate}

Here we denote by \( \beta_k := \sum_{j=0}^{k} \binom{k}{j} \) the \( k \)-th \textbf{Bell number}.

\textbf{Remark 2.1.} By definition, a finite group action \( G \acts X \) cannot be \( k \)-transitive when \( k > \#X \).

\textbf{Proof of Theorem 2.4.} The equivalence (a) \( \iff \) (b) is immediate by the definition of \( k \)-transitiveness. Since \( m(k; G, X) \) is the first moment \( m(1; G, X^k) \), the equivalence (b) \( \iff \) (d) follows from the Frobenius formula (2.2) for \( G \acts X^k \).

We check the equivalence (c) \( \iff \) (d). When \( m(k; G, X) = 1 \), the lower moments \( m(m; G, X) \) \((m \leq k)\) are also equal to 1 because of the equivalence (a) \( \iff \) (b) \( \iff \) (d). Since \( m(k; G, X) \) is a non-negative integer and \( m(k; G, X) = 0 \iff k > \#X \), we have \( m(k; G, X) \geq \beta_k \) by the relation (2.5), and the equality holds if and only if all \( m(j; G, X) \) are equal to 1. Thus we have the equivalence (c) \( \iff \) (d). \( \square \)

Analogous to the usual notion of moment generating functions, we consider the moment generating function for the action \( G \acts X \) defined by

\[ \mathbb{M}(t; G, X) := \sum_{k=0}^{\infty} m(k; G, X) \frac{t^k}{k!} = \sum_{g \in G} e^{\text{fix}(g)t} \frac{1}{\#G}, \]

which is a finite sum of exponential functions. Therefore we introduce a polynomial function \( \mathcal{M}(z; G, X) \) defined by a change of variable as follows:

\[ \mathcal{M}(z; G, X) := \mathbb{M}(\log(1+z); G, X) = \sum_{g \in G} \frac{(1+z)^{\text{fix}(g)}}{\#G}. \]

We call this polynomial the \textbf{moment polynomial} for \( G \acts X \).
Corollary 2.5. A group action $G \curvearrowright X$ is $k$-transitive if and only if $M^{(k)}(0) = 1$ holds.

Proof. In fact, we have

$$M(t; G, X) = \sum_{k=0}^{\infty} m(k; G, X) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} m(j; G, X) \frac{t^k}{k!} = \sum_{j=0}^{\infty} m(j; G, X) \left( \sum_{k=m}^{\infty} \binom{k}{j} \frac{t^k}{k!} \right) = \sum_{j=0}^{\infty} m(j; G, X) \frac{(e^t - 1)^j}{j!},$$

which means

$$M(z; G, X) = M(\log(1 + z); G, X) = \sum_{j \geq 0} m(j; G, X) \frac{z^j}{j!}.$$

Now the conclusion follows immediately. \hfill \Box

Corollary 2.6. Assume the action $G \curvearrowright X$ is transitive. Let $H = G_x$ be the stabilizer for some $x \in X$, and put $Y := X \setminus \{x\}$. Then we have

$$M(z; H, Y) = \frac{d}{dz} M(z; G, X).$$

Proof. From the expression (2.6) of the moment polynomial, it is sufficient to show the equality $m(l; H, Y) = m(l + 1; G, X)$ for each $l \geq 1$. This equality is also rewritten as $\# H \setminus Y^{(l)} = \# G \setminus X^{(l+1)}$, which follows from the fact that the map

$$H \setminus Y^{(l)} \ni H(p_1, \ldots, p_l) \mapsto (x, p_1, \ldots, p_l) \in G \setminus X^{(l+1)}$$

is well-defined and bijective. \hfill \Box

Example 2.2. Fix a prime power $q$ and let $\mathbb{F}_q$ be a finite field consisting of $q$ elements. The general linear group $GL_2(\mathbb{F}_q)$ acts on the projective line $\mathbb{P}^1(\mathbb{F}_q) \simeq \mathbb{F}_q \cup \{\infty\}$ by

$$g.z := \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q).$$

The moment generating function of this action is calculated as follows.

$$M(z; GL_2(\mathbb{F}_q), \mathbb{P}^1(\mathbb{F}_q)) = 1 + z + \frac{z^2}{2} + \sum_{j=3}^{q+1} \frac{(q - 2)!}{(q - j + 1)!} \frac{z^j}{j!}.$$
It follows that
\[
\mathcal{M}^{(3)}(z) = (1 + z)^{q-2} \implies \mathcal{M}^{(3)}(0) = 1,
\]
\[
\mathcal{M}^{(4)}(z) = (q - 2)(1 + z)^{q-3} \implies \mathcal{M}^{(4)}(0) = q - 2,
\]
which means that the action $GL_2(\mathbb{F}_q) \cong \mathbb{P}^1(\mathbb{F}_q)$ is 3-transitive but not 4-transitive if $q \neq 3$; when $q = 3$, this action is 4-transitive. In fact, $PGL_2(\mathbb{F}_3) \cong \mathbb{P}^1(\mathbb{F}_3) \cong S_4 \lhd [4]$.

By the definition of the moment polynomial it is immediate to see that
\[
(2.10) \quad \mathcal{M}(-1; G, X) = P(\text{fix}(G, X) = 0),
\]
from which we obtain another kind of criterion for transitivity as follows.

**Theorem 2.7.** For a given action $G \cdot X$ of a finite group with $\#X \geq 2$, we have
\[
(2.11) \quad \mathcal{M}(-1; G, X) = 0 \implies G \cdot X \text{ is not transitive.}
\]

**Proof.** Suppose that the action $G \cdot X$ is transitive. Then there is some fixed-point free element $g \in G$, i.e. $\text{fix}(g; G, X) = 0$. In fact, if we assume that $\text{fix}(g; G, X) \geq 1$ for any $g \in G$, then by remarking that $\text{fix}(1; G, X) = \#X \geq 2$ it follows that
\[
1 = m(1; G, X) = \frac{1}{\#G} \sum_{g \in K} \text{fix}(g; G, X) \geq \frac{2}{\#G} + \frac{\#G - 1}{\#G} > 1,
\]
which is the contradiction. Consequently, we have
\[
\mathcal{M}(-1; G, X) = P(\text{fix}(G, X) = 0) \neq 0.
\]
This completes the proof.

### 2.4 Log-derivative of zeta functions

From the expression (2.3) of the zeta function $Z(u; G, X)$, the log-derivative of $Z(u; G, X)$ is given by
\[
(2.12) \quad \frac{Z'(u; G, X)}{Z(u; G, X)} = \frac{1}{\#G} \sum_{g \in G} \frac{\text{fix}(g; G, X)}{1 - \text{fix}(g; G, X)u},
\]
which gives a meromorphic continuation of $Z'(u; G, X)/Z(u; G, X)$ to the whole plane $(u = 0, 1, 1/2, 1/3, \ldots, 1/\#X)$ are possible poles of $Z'(u; G, X)/Z(u; G, X)$ in general.

Our main concern is an appropriate formulation of a zeta function attached to a large group like an inductive limit group of a certain system of finite groups. If we deal with such an inductive system $\{G_n \cdot X_n\}_{n=1}^\infty$ of finite group actions with $\#X_n \to \infty$ and take a limit
\[ \lim_{n \to \infty} \frac{Z'}{Z}(u; G_n, X_n), \] then the origin would be an accumulation point of singularities of the limit function. In order to avoid this matter, we change the variable and put

\[ \Xi(u; G, X) := \frac{1}{u^2} \frac{Z'}{Z} \left( -\frac{1}{u}; G, X \right) = \frac{d}{du} \log \left( -\frac{1}{u}; G, X \right). \]

In the sequel we treat this modified log-derivative of \( Z(u; G, X) \) instead of the zeta function itself.

From the expression (2.12) we immediately have the

**Proposition 2.8** (Special value formula). We have

\[
(2.13) \quad \frac{1}{\# G} \sum_{g \in G} \frac{1}{1 + \text{fix}(g; G \cap X)} = 1 - \Xi(1; G, X). 
\]

\[ \square \]

**Remark 2.2.** For a once punctured torus \( M \), the following identity is known to hold [9]:

\[
(2.14) \quad \sum_{\gamma} \frac{1}{1 + \ell(\gamma)} = \frac{1}{2},
\]

where the sum above is over all prime geodesics \( \gamma \) on \( M \) (or primitive hyperbolic conjugacy classes of the fundamental group \( \Gamma(M) \) of \( M \)), and \( \ell(\gamma) \) denotes the length of \( \gamma \). The relation (2.13) expresses an analogous formula to this identity.

We notice that (2.12) is also rewritten as follows:

\[
\Xi(u; G, X) = \frac{1}{u} - \frac{1}{\# G} \sum_{g \in G} \frac{1}{u + \text{fix}(g; G, X)}
\]

\[
= \frac{1}{u} - \sum_{k=0}^{\infty} \frac{P(\text{fix}(G, X) = k)}{u + k},
\]

which implies the

**Proposition 2.9.** The possible poles of the function \( \Xi(u; G, X) \) are located in \( u = 0, -1, -2, \ldots, -\# X \) which are all simple. The residue of \( \Xi(u; G, X) \) at \( u = -k \) is given by

\[ \text{Res}_{u=-k} \Xi(u; G, X) = \delta_{0,k} - P(\text{fix}(G, X) = k) \]

for \( k = 0, 1, \ldots, \# X \). These exhaust all singularities of \( \Xi(u; G, X) \). \[ \square \]

From (2.1) the log-derivative of the zeta function \( Z(u; G, X) \) is given by

\[
\frac{Z'}{Z}(u; G, X) = \sum_{k=1}^{\infty} m(k; G, X) u^{k-1}. 
\]
By using the relation (2.5) we have
\[
\frac{Z'}{Z}(u; G, X) = \frac{1}{u} \sum_{k=1}^{\infty} \sum_{j=1}^{k} \binom{k}{j} m(j; G, X) u^k
= \frac{1}{u} \sum_{j=1}^{\infty} m(j; G, X) \sum_{k=j}^{\infty} \binom{k}{j} u^k.
\]
Here we recall the well-known identity
\[
\sum_{k=j}^{\infty} \binom{k}{j} z^k = \frac{z^j}{(1 - z)(1 - 2z) \cdots (1 - jz)}
\]
for \(j \geq 0\). Then we have
\[
(2.15) \quad \frac{Z'}{Z}(u; G, X) = \frac{1}{u} \sum_{j=1}^{\infty} \frac{m(j; G, X) u^j}{(1 - u)(1 - 2u) \cdots (1 - ju)},
\]
which is rewritten as follows:
\[
\Xi(u; G, X) = \frac{1}{u^2} (-u) \sum_{j=1}^{\infty} \frac{m(j; G, X) (-u^{-1})^j}{(1 + u^{-1})(1 + 2u^{-1}) \cdots (1 + ju^{-1})}
= -\frac{1}{u} \sum_{j=1}^{\infty} m(j; G, X) \frac{(-1)^j}{(u + 1)_j}.
\]
Since \(m(j; G, X)\) vanishes if \(j > \#X\), the summation in the expression above is actually a finite sum. From this expression we have the

**Proposition 2.10.** The residue of the function \(\Xi(u; G, X)\) at \(u = -k\) is given by
\[
\text{Res}_{u=-k} \Xi(u; G, X) = \delta_{0,k} - \frac{1}{k!} \sum_{j=0}^{\infty} (-1)^j \frac{m(k+j; G, X)}{j!}
\]
for \(k = 0, 1, \ldots, \#X\).

**Proof.** In fact, it follows that
\[
(u + k)\Xi(u; G, X) = \frac{u + k}{u} \left( 1 - \sum_{j=0}^{\infty} m(j; G, X) \frac{(-1)^j}{(u + 1)_j} \right)
= \frac{u + k}{u} - \frac{u + k}{u} \sum_{j=0}^{k-1} m(j; G, X) \frac{(-1)^j}{(u + 1)_j}
- \frac{1}{u} \frac{u + k}{(u + 1)_k} \sum_{j=0}^{\infty} m(k+j; G, X) \frac{(-1)^{k+j}}{(u + k + 1)_j}
\]
\[
\underset{u\to-k}{\Rightarrow} \delta_{0,k} - \frac{1}{k!} \sum_{j=0}^{\infty} (-1)^j \frac{m(k+j; G, X)}{j!}.
\]
as we desired. 

Comparing the residues we have the following formula.

**Corollary 2.11.** We have

\[
P(\text{fix}(G, X) = k) = \frac{1}{k!} \sum_{j \geq 0} (-1)^j \frac{m(k + j; G, X)}{j!}
\]

for any non-negative integer \(k\).

### 3 Symmetric group case

In this section we exclusively deal with the case of defining actions \(\mathfrak{S}_n \curvearrowright [n]\) of symmetric groups \(\mathfrak{S}_n\). For abbreviation we put

\[
\Xi(u; \mathfrak{S}_n, [n]) := \Xi(u; S_n, [n]),
\]

\[
\text{fix}(\sigma; n) := \text{fix}(\sigma; S_n, [n]),
\]

\[
m(k; n) := m(k; S_n, [n]).
\]

The definition of \(\Xi(u; G, X)\) does not work when \(G = \mathfrak{S}_\infty\) and \(X = [\infty] = \mathbb{N}\), but the series \(\{\Xi(u; n)\}_{n=1}^\infty\) does have a limit function as follows.

**Theorem 3.1.** The sequence \(\{\Xi(u; n)\}_{n=1}^\infty\) converges absolutely and uniformly on any compact domain in \(\mathbb{C}\). The limit function

\[
\Xi(u; \infty) := \lim_{n \to \infty} \Xi(u; n)
\]

have the expression

\[
\Xi(u; \infty) = \frac{1 - \, _1F_1(1; u + 1; -1)}{u}.
\]

Here \(\, _1F_1(a, c; z)\) is the confluent hypergeometric function of Kummer’s type

\[
\, _1F_1(a; c; z) := \sum_{n=0}^\infty \frac{(a)_n}{(c)_n} \frac{z^n}{n!},
\]

where \((a)_n := a(a + 1) \cdots (a + n - 1)\) is Pochhammer’s symbol. The poles of \(\Xi(u; \infty)\) are given by \(u = 0, -1, -2, \ldots\), which are simple and their residues are given by

\[
\text{Res}_{u=-m} \Xi(u; \infty) = \delta_{m,0} - \frac{1}{m!} \quad (m = 0, 1, 2, \ldots).
\]

These exhaust all poles of \(\Xi(u; \infty)\).
Proof. Since the action $\mathfrak{S}_n \triangleleft [n]$ is $n$-transitive, we see that

$$m(k; n) = \begin{cases} 1 & 1 \leq k \leq n, \\ 0 & k > n. \end{cases}$$

Hence we have

$$\Xi(u; n) = -\frac{1}{u} \sum_{m=1}^{n} \frac{(-u^{-1})^m}{(1 + u^{-1})(1 + 2u^{-1}) \cdots (1 + mu^{-1})} = -\frac{1}{u} \sum_{m=1}^{n} \frac{(-1)^m}{(u + 1)_m}.$$ 

Thus it follows

$$\Xi(u; \infty) = \lim_{n \to \infty} \Xi(u; n) = -\frac{1}{u} \sum_{m=1}^{\infty} \frac{(-1)^m}{(u + 1)_m} = \frac{1}{u} \mathrm{F}_1(1; u + 1; -1).$$

The residue at $u = -m$ $(m = 1, 2, 3, \ldots)$ is calculated as follows. Since

$$(u + m)\Xi(u; \infty)$$

$$= -\frac{(u + m)}{u} \sum_{j=1}^{m-1} \frac{(-1)^j}{(u + 1)_j} - \frac{(-1)^m}{(u)_m} \sum_{j=m}^{\infty} \frac{(-1)^{j-m}}{(u + m + 1) \cdots (u + j)},$$

we have

$$\lim_{u \to -m} (u + m)\Xi(u; \infty) = -\frac{1}{m!} \sum_{j=m}^{\infty} \frac{(-1)^{j-m}}{(j - m)!} = -\frac{1}{m!} e^{-1}$$

as we required. The residue at $u = 0$ is also given by

$$u\Xi(u; \infty) = -\sum_{j=1}^{\infty} \frac{(-1)^j}{(u + 1)_j} u \to 1 - e^{-1}.$$ 

The holomorphy on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$ is clear. 

**Corollary 3.2.** We have

$$\Xi(u; \infty) = \frac{1}{u} - \frac{1}{e} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{u + m}.$$ 

This also implies that the random variable $\text{fix}(\cdot ; n)$ on $\mathfrak{S}_n$ asymptotically follows to the Poisson distribution $P(1)$ with parameter 1.
Proof. This is just a partial fraction decomposition of $\Xi(u; \infty)$. This expression also indicates that

$$\lim_{n \to \infty} P(\text{fix}(n) = m) = \frac{1}{m!} e^{-1}.$$

\[\square\]

**Corollary 3.3.** We have

$$\lim_{n \to \infty} \frac{1}{n!} \sum_{g \in S_n} \frac{1}{1 + \text{fix}(g; n)} = 1 - \frac{1}{e}. \quad (3.3)$$

**Proof.** In fact, the relation (2.13) implies that the left hand side of (3.3) is given by $\lim_{n \to \infty} (1 - \Xi(1; n)) = 1 - \Xi(1; \infty) = 1 - e^{-1}$ as we required. \[\square\]

**Remark 3.1.** Since the action $S_n \actson [n]$ is $k$-transitive if $n \geq k$, we have

$$\beta_k = \sum_{m=0}^{n} m^k P(\text{fix}(n) = m).$$

By Corollary 2.11 we have the following expressions of the $k$-th Bell number $\beta_k$:

$$\beta_k = \sum_{m=0}^{n} m^k \sum_{j=0}^{n-m} \frac{(-1)^j}{j!} \sum_{m=0}^{\infty} \frac{m^k}{m!}.$$

This function $\Xi(u; \infty)$ has a functional equation as follows.

**Theorem 3.4 (Functional equation).** The function $\Xi(u; \infty)$ satisfies

$$\Xi(u + 1; \infty) = -u \Xi(u; \infty) + \frac{1}{u + 1}. \quad (3.4)$$

**Proof.** In fact, we have

$$\Xi(u + 1; \infty) = -\frac{1}{u + 1} \sum_{j=1}^{\infty} \frac{(-1)^j}{(u + 2)_j}$$

$$= -u \sum_{j=1}^{\infty} \frac{(-1)^{j+2}}{(u)_{j+2}}$$

$$= -u \left( \Xi(u; \infty) - \frac{1}{u(u + 1)} \right)$$

$$= -u \Xi(u; \infty) + \frac{1}{u + 1}$$

as we required. \[\square\]
We observe that the function \( \Xi(u; \infty) \) is similar to the usual Gamma function \( \Gamma(z) \). Actually, it is well-known that \( \Gamma(z) \) satisfies the functional equation

\[
\Gamma(z + 1) = z\Gamma(z),
\]

and has simple poles at \( z = -m \) \( (m = 0, 1, 2, \ldots) \) whose residues are given by

\[
\text{Res}_{z=-m} \Gamma(z) = \frac{(-1)^m}{m!} \quad (m = 0, 1, 2, \ldots).
\]

Further, analogous to the integral expression (or definition)

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (\text{Re}(z) > 0)
\]

of \( \Gamma(z) \), we have the following integral expression

\[
\Xi(u; \infty) = \frac{1}{u} + (-1)^{-u} \int_1^2 e^{1-t}t^{u-1}dt
\]

\[= \frac{1}{u} + (-1)^{-u}(\gamma(u, 2) - \gamma(u, 1)) \quad (\text{Re}(u) > 0), \tag{3.5}
\]

for \( \text{Re}(u) > 0 \), which is immediately obtained from the general integral formula of \( \,_{1}F_{1}(a; c; z) \).

Here \( \gamma(z, p) \) denotes the incomplete gamma function defined by

\[
\gamma(z, p) := \int_0^p e^{-t}t^{z-1}dt \quad (\text{Re}(z) > 0).
\]

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References


