Multiple finite Riemann zeta functions

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Abstract

We introduce a new zeta function which we call a *multiple finite Riemann zeta function*. We utilize some q-series identity for the proof of the Euler product of the zeta function. We further study multi-variable and multi-parameter versions of the multiple finite Riemann zeta functions, and their *infinite* counterparts in connection with symmetric polynomials and *powerful numbers*.
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1 Introduction

The divisor function $\sigma_k(N) := \sum_{d|N} d^k$ is a basic multiplicative function and plays an important role from the beginning of the modern arithmetic study. In particular, $\sigma_k(N)$ appears in the Fourier coefficients of the (holomorphic) Eisenstein series $E_{k+1}(\tau)$. It is, however, not so common to regard $\sigma_k(N)$ as a sort of zeta function. In the present paper, we treat $\sigma_k(N)$ as a function of a complex variable $k = -s$. It is clear that when $N = 0$ (or $N \to \infty$) we have $\sigma_{-s}(N) \to \zeta(s)$, the Riemann zeta function. There are at least two interpretations of $Z_N^1(s) := \sum_{n|N} n^{-s} = \sigma_{-s}(N)$ as a zeta function in number theory:

- Fourier coefficients of real analytic Eisenstein series,
- Igusa zeta functions.

Concerning the first, we refer to Bump et al [BCKV] where the so-called "local Riemann hypothesis" is studied. In case of the real analytic Eisenstein series $E(s,\tau)$ for the modular group $SL(2,\mathbb{Z})$, the N-th Fourier coefficient is essentially given by $c_N(s,\tau) := Z_N^1(2s-1)K_{s-\frac{1}{2}}(2\pi N \operatorname{Im}(\tau))e^{2\pi i \operatorname{Re}(\tau)}$. Hence it satisfies the local Riemann hypothesis; if $c_N(s,\tau) = 0$ then $\operatorname{Re}(s) = \frac{1}{2}$. For the second, the interpretation is coming from the (global) Igusa zeta function $\zeta^{\mathrm{Igusa}}(s,R)$ of a ring R defined as

$$\zeta^{\mathrm{Igusa}}(s,R) := \sum_{m=1}^{\infty} {}^{\#}\mathrm{Hom}_{\mathrm{ring}}(R, \mathbb{Z}/(m))m^{-s}.$$

Then, in fact, it is easy to see that $Z_N^1(s) = \zeta^{\text{Igusa}}(s, \mathbb{Z}/(N))$.

The purpose of the present paper is initially to study the function defined by the series

$$Z_N^m(s) := \sum_{n_1|n_2|\cdots|n_m|N} (n_1 n_2 \cdots n_m)^{-s}. \tag{1.1}$$

We call $Z_N^m(s)$ the multiple finite Riemann zeta function of type N. We then study several basic properties of $Z_N^m(s)$ such as an Euler product, a functional equation, and an analogue of the Riemann hypothesis in an elementary way by the help of some q-series identity. We also study the limit case $Z_\infty^m(s) := \sum_{n_1|n_2|\cdots|n_m} (n_1 n_2 \cdots n_m)^{-s}$.

Moreover, we generalize this zeta function in two directions. The first one is to increase the number of variables. We prove that the Euler product of a multi-variable version of $Z_N^m(s)$ can be expressed in terms of the complete symmetric polynomials with a remarkable specialization of variables (Theorem 3.2). The second is to add parameters indexed by a set of positive integers. For general parameters, it seems difficult to calculate an Euler product explicitly using symmetric functions with some meaningful specialization of variables. If we restrict ourselves to the one variable case, however, under a special but non-trivial specialization of parameters, we show that the corresponding multiple zeta functions can be written as a product of $\zeta(cs)$ with several constants c's determined by the given parameters and the Dirichlet series associated with generalized powerful numbers (see Section 4). Moreover, we determine the condition whether the Dirichlet series associated with such generalized powerful numbers can be extended as a meromorphic function to the entire plane $\mathbb{C}$ or not (see Theorem 4.8 and its corollary). As a consequence, the most of such Dirichlet series are shown to have the imaginary axis as a natural boundary. The result is a generalization of the one in [IS]. In the last section, we give two remarks on the number of isomorphism classes of abelian groups and an analogous notion of Eisenstein series.

Throughout the paper, we denote the sets of all integers, positive integers, non-negative integers, real numbers, and complex numbers by $\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{R}$, and $\mathbb{C}$, respectively.

2 Multiple finite Riemann zeta function

We prove the fundamental properties of the multiple finite Riemann zeta function $Z_N^m(s)$ defined by (1.1) and discuss some related Dirichlet series. First we have the

Theorem 2.1. *Let N be a positive integer.*

1. *Euler product* :

$$Z_N^m(s) = \prod_{p\,:\,\text{prime}} \prod_{k=1}^{m} \frac{1 - p^{-s(\operatorname{ord}_p N + k)}}{1 - p^{-sk}}, \tag{2.1}$$

where $\operatorname{ord}_p N$ *denotes the order of p-factor in the prime decomposition of N.*

2. *Functional equation* : $Z_N^m(-s) = N^{ms} Z_N^m(s)$.

3. *Analogue of the Riemann hypothesis* : *All zeros of* $Z_N^m(s)$ *lie on the imaginary axis* $\operatorname{Re} s = 0$. *More precisely, the zeros of* $Z_N^m(s)$ *are of the form* $s = \frac{2\pi i n}{(\operatorname{ord}_p N + k)\log p}$ *for* $k = 1, \ldots, m$, $p|N$ *and* $n \in \mathbb{Z} \setminus \{0\}$. *Consequently, the order* $\operatorname{Mult}^m(n,p,k)$ *of the zero at* $s = \frac{2\pi i n}{(\operatorname{ord}_p N + k)\log p}$ *is given by* $\operatorname{Mult}^m(n,p,k) = \#\{(l,j)\ ;\ 1 \le l \le m,\ j \in \mathbb{Z} \setminus \{0\},\ (\operatorname{ord}_p N + k)j = (\operatorname{ord}_p N + l)n\}$.

4. *Special value* : *When* n *is a positive integer one has* $Z_N^m(-n) \in \mathbb{Z}$.

For the proof of the theorem the following lemma is crucial.

Lemma 2.2. *Let* m *be a positive integer. Then*

1. *For any integers* $l \ge 0$ *it holds that*

$$\sum_{d=0}^{l} \begin{bmatrix} m-1+d \\ m-1 \end{bmatrix}_q q^d = \begin{bmatrix} m+l \\ m \end{bmatrix}_q, \tag{2.2}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *is the* q*-binomial coefficient defined by* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k}(1-q^{n+1-j})/(1-q^j)$.

2. *For* $|x|<1$, $|q|<1$ *it holds that*

$$\sum_{d=0}^{\infty} \begin{bmatrix} m+d \\ m \end{bmatrix}_q x^d = \prod_{k=0}^{m} \frac{1}{1-q^k x}. \tag{2.3}$$

Proof. We prove the formula (2.2) by induction on l. When $l=0$, the formula (2.2) clearly holds. Suppose that it holds for l. Then we see that

$$\sum_{d=0}^{l+1} \begin{bmatrix} m-1+d \\ m-1 \end{bmatrix}_q q^d = \begin{bmatrix} m+l \\ m \end{bmatrix}_q + \begin{bmatrix} m+l \\ m-1 \end{bmatrix}_q q^{l+1} = \begin{bmatrix} m+l+1 \\ m \end{bmatrix}_q,$$

whence (2.2) is also true for $l+1$.

The second formula (2.3) can be proved in the same manner by induction, but on m. (It is also obtained from the so-called q-binomial theorem. See, e.g. [AAR]). □

Proof of Theorem 2.1. The functional equation is easily seen from the definition. Actually, we have

$$\begin{aligned} Z_N^m(-s) &= \sum_{n_1|n_2|\cdots|n_m|N} (n_1 n_2 \cdots n_m)^s \\ &= N^{ms} \sum_{n_1|n_2|\cdots|n_m|N} \left(\frac{n_1}{N}\frac{n_2}{N}\cdots\frac{n_m}{N}\right)^s = N^{ms} Z_N^m(s). \end{aligned}$$

To prove (2.1), we show that $Z^m_N(s)$ is multiplicative with respect to N. Suppose N and M are co-prime. Then

$$\begin{aligned} Z^m_{NM}(s) &= \sum_{n_1|n_2|\cdots|n_m|NM} (n_1n_2\cdots n_m)^{-s} \\ &= \sum_{c_1|c_2|\cdots|c_m|N} (c_1c_2\cdots c_m)^{-s} \sum_{d_1|d_2|\cdots|d_m|M} (d_1d_2\cdots d_m)^{-s} = Z^m_N(s)Z^m_M(s), \end{aligned}$$

because every divisor n of NM is uniquely written as $n = cd$ where $c|N$ and $d|M$. By means of this fact, to get the Euler product expression (2.1) of $Z^m_N(s)$ it suffices to calculate the case where N is a power of prime p. In this case, one proves the expression

$$Z^m_{p^l}(s) = \prod_{k=1}^{m} \frac{1-p^{-(l+k)s}}{1-p^{-sk}} \tag{2.4}$$

by induction on m as follows: It is clear for $m = 1$. Suppose (2.4) is true for $m-1$. Then

$$\begin{aligned} Z^m_{p^l}(s) &= \sum_{0\le j_1\le\cdots\le j_m\le l} p^{-(j_1+\cdots+j_m)s} = \sum_{j_m=0}^{l} Z^{m-1}_{p^{j_m}}(s)p^{-j_m s} \\ &= \sum_{d=0}^{l}\prod_{k=1}^{m-1} \frac{1-p^{-(d+k)s}}{1-p^{-sk}} p^{-ds} = \sum_{d=0}^{l} \begin{bmatrix} m-1+d \\ d \end{bmatrix}_{p^{-s}} p^{-ds}. \end{aligned}$$

Therefore the assertion follows immediately from the formula (2.2) in Lemma 2.2. This proves (2.4), whence the Euler product for $Z^m_N(s)$ follows.

Using the Euler product (2.1), we observe that the meromorphic function $Z^m_N(s)$ may have zeros at each $s = \frac{2\pi i n}{(\mathrm{ord}_p N+k)\log p}$ for $k = 1,\ldots,m$, $p|N$ and $n \in \mathbb{Z}$. Note however, since

$$Z^m_N(0) = \prod_{p\,:\,\mathrm{prime}} \binom{\mathrm{ord}_p N + m}{m} \neq 0, \tag{2.5}$$

$s = 0$ is not a zero of $Z^m_N(s)$. Suppose now $\frac{2\pi i n}{(\mathrm{ord}_p N+k)\log p} = \frac{2\pi i m}{(\mathrm{ord}_q N+l)\log q}$ holds for some $1 \le k,l \le m$, $p,q|N$ and $n,j \in \mathbb{Z}\setminus\{0\}$. Then it becomes $p^{(\mathrm{ord}_p N+k)j} = q^{(\mathrm{ord}_q N+l)n}$. This immediately shows that $p = q$ and $(\mathrm{ord}_p N + k)j = (\mathrm{ord}_p N + l)n$. Hence the order of zero at $s = \frac{2\pi i n}{(\mathrm{ord}_p N+k)\log p}$ is given by $\mathrm{Mult}^m(n,p,k)$. Obviously, one has $\mathrm{Mult}^1(n,p,1) = 1$.

The last claim about the special value of $Z^m_N(s)$ is clear from the definition. □

Remark 2.1. Note that $Z^m_{p^l}(s) = Z^l_{p^m}(s)$ from (2.4). □

We next consider a Dirichlet series defined via the multiple finite Riemann zeta functions. Put

$$\zeta^m(s,t) := \sum_{n=1}^{\infty} Z^m_n(s)n^{-t}.$$

Then, from the formula (2.3), $\zeta^m(s,t)$ has the Euler product

$$\zeta^m(s,t)=\prod_{p\,:\,\mathrm{prime}}\left(\sum_{l=0}^{\infty}Z_{p^l}^m(s)p^{-lt}\right)=\prod_{k=0}^{m}\zeta(sk+t). \tag{2.6}$$

We give here a few examples:

Example 2.1. It is well-known that $\zeta^1(-k,t)=\sum_{n=1}^{\infty}\sigma_k(n)n^{-t}=\zeta(t)\zeta(t-k)$ for $\mathrm{Re}\,t>k+1$. Also,
by (2.5), for $\mathrm{Re}\,t>1$, it follows that $\zeta^m(0,t)=\zeta(t)^{m+1}$. □

Example 2.2. Let l be a positive integer. For $\mathrm{Re}\,t>1-l$, we have $\zeta^m(l,t+l)=\prod_{k=1}^{m+1}\zeta(t+lk)$. As $m\to\infty$, we have $\lim_{m\to\infty}\zeta^m(l,t+l)=\prod_{k=1}^{\infty}\zeta(t+lk)$, which is the higher Riemann zeta function $\zeta_{l\infty}(s)$ studied in [KMW] (see also [KW]). Note that $\zeta_{l\infty}(s)$ possesses a functional equation. □

3 Multivariable version

We study a generalization of $Z_N^m(s)$. For $\gamma=(\gamma_1,\dots,\gamma_m)\in\mathbb{Z}_{>0}^m$ and $N\in\mathbb{Z}_{>0}$, define

$$Z_N^{\gamma}(t_1,\dots,t_m):=\sum_{n_m^{\gamma_m}|\cdots|n_1^{\gamma_1}|N}n_1^{-\gamma_1t_1}\cdots n_m^{-\gamma_mt_m}. \tag{3.1}$$

This is multiplicative with respect to N. Notice that $Z_N^m(s)=Z_N^{\gamma}(s,\dots,s)$ when $\gamma=(1^m)=(\overbrace{1,\dots,1}^{m})$. We can prove the following lemma in a similar way to Theorem 2.1.

Lemma 3.1. *For $l\geq 0$, define a function $G_l^{\gamma}(q_1,q_2,\dots,q_m)$ by*

$$G_l^{\gamma}(q_1,\dots,q_m)=\sum_{\substack{l\geq\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0\\ \gamma_j|\lambda_j}}q_1^{\lambda_1}\cdots q_m^{\lambda_m}, \tag{3.2}$$

where the sum is taken over all partitions $\lambda=(\lambda_1,\lambda_2,\dots)$ of length $\leq m$ such that $\lambda_1\leq l$ and $\gamma_j|\lambda_j$ for $1\leq j\leq m$. Then the Euler product of $Z_N^{\gamma}(t_1,\dots,t_m)$ is given as

$$Z_N^{\gamma}(t_1,\dots,t_m)=\prod_{p\,:\,\mathrm{prime}}G_{\mathrm{ord}_p N}^{\gamma}(p^{-t_1},\dots,p^{-t_m}).\quad\square$$

Remark 3.1. Let $f(n_1,\dots,n_m)$ be a multiplicative function with respect to each n_j. We define a multiple zeta function by $Z_N^{\gamma}(f)=\sum_{n_m^{\gamma_m}|\cdots|n_1^{\gamma_1}|N}f(n_1,\dots,n_m)$. Then one can show that $Z_N^{\gamma}(f)$ is multiplicative, whence has the Euler product $Z_N^{\gamma}(f)=\prod_{p\,:\,\mathrm{prime}}Z_{p^{\mathrm{ord}_p N}}^{\gamma}(f)$. □

We first look at the simplest case $\gamma=(1^m)$. We abbreviate respectively $Z_N^{(1^m)}(t_1,\dots,t_m)$ and $G_l^{(1^m)}(q_1,\dots,q_m)$ to $Z_N^m(t_1,\dots,t_m)$ and $G_l^m(q_1,\dots,q_m)$.

Theorem 3.2. *Let* $h_j(x_1,\dots,x_m)=\sum_{\substack{i_1+\cdots+i_m=j,\\ i_k\in\mathbb{Z}_{\geq 0}}} x_1^{i_1}\cdots x_m^{i_m}$ *be the* j*-th complete symmetric polynomial. Then we have*

$$G_l^m(q_1,\dots,q_m)=\sum_{j=0}^{l} h_j(q_1,q_1q_2,\dots,q_1q_2\cdots q_m). \tag{3.3}$$

In particular, we have $G_\infty^m(q_1,\dots,q_m):=\lim_{l\to\infty}G_l^m(q_1,\dots,q_m)=\prod_{k=1}^m(1-q_1q_2\cdots q_k)^{-1}$.

Proof. Since

$$\begin{aligned}h_j(q_1,q_1q_2,\dots,q_1q_2\cdots q_m)&=\sum_{i_1+i_2+\cdots+i_m=j} q_1^{i_1+i_2+\cdots+i_m}q_2^{i_2+\cdots+i_m}\cdots q_m^{i_m}\\ &=\sum_{j=\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0} q_1^{\lambda_1}q_2^{\lambda_2}\cdots q_m^{\lambda_m},\end{aligned}$$

the first formula (3.3) is clear from the definition (3.2). The second formula follows from (3.3) together with the fact $\sum_{j=0}^\infty h_j(x_1,\dots,x_m)z^j=\prod_{k=1}^m(1-x_kz)^{-1}$ (see [M]). □

As a corollary of the theorem, we obtain the Euler product of $Z_N^m(t_1,\dots,t_m)$.

Corollary 3.3. *Let* m *and* N *be positive integers. Then we have*

$$Z_N^m(t_1,\dots,t_m)=\prod_{p\,:\,\mathrm{prime}}\left(\sum_{j=0}^{\mathrm{ord}_p N} h_j(p^{-t_1},p^{-t_1-t_2},\dots,p^{-t_1-t_2-\cdots-t_m})\right).$$

Further, for $\mathrm{Re}\,t_j>1$ $(1\leq j\leq m)$*, we have*

$$Z_\infty^m(t_1,\dots,t_m):=\sum_{n_m|\cdots|n_1} n_1^{-t_1}\cdots n_m^{-t_m}=\prod_{k=1}^m\zeta(t_1+t_2+\cdots+t_k). \quad \square$$

Since

$$\begin{aligned}G_l^\gamma(q_1,\dots,q_m)&=\sum_{0\leq n\leq l/\gamma_1} q^{\gamma_1 n}\sum_{\substack{\gamma_1 n\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0,\\ \gamma_j|\lambda_j\ (2\leq j\leq m)}} q_2^{\lambda_2}\cdots q_m^{\lambda_m}\\ &=\sum_{n=0}^{\lfloor l/\gamma_1\rfloor} q_1^{\gamma_1 n}G_{\gamma_1 n}^{(\gamma_2,\dots,\gamma_m)}(q_2,\dots,q_m),\end{aligned}$$

the recurrence equation among $G_l^\gamma(q_1,\dots,q_m)$'s can be obtained as follows.

Lemma 3.4. *For* $\gamma=(\gamma_1,\dots,\gamma_m)\in\mathbb{Z}_{>0}^m$*, we have*

$$G_l^\gamma(q_1,\dots,q_m)=\sum_{n=0}^{\lfloor l/\gamma_1\rfloor} q_1^{\gamma_1 n}G_{\gamma_1 n}^{(\gamma_2,\dots,\gamma_m)}(q_2,\dots,q_m).$$

Here $\lfloor x\rfloor$ *is the largest integer not exceeding* x. □

The following lemma shows that to study $G_l^\gamma(q_1,\dots,q_m)$ it is enough to study the case where $\gamma_1,\dots,\gamma_m$ are relatively prime. The proof is straightforward.

Lemma 3.5. *For $\gamma=(dc_1,dc_2,\dots dc_m)$, we have*

$$G_l^\gamma(q_1,\dots,q_m)=G_{\lfloor l/d\rfloor}^{(c_1,\dots,c_m)}(q_1^d,\dots,q_m^d).$$

□

Let us calculate several $G^\gamma(q_1,\dots,q_m)$ with special parameters γ.

Example 3.1. Let $\gamma=(c,1)$. Putting $d=\lfloor l/c\rfloor$, we see by Lemma 3.4 that

$$G_l^{(c,1)}(q_1,q_2)=\frac{(1-q_1^{c(d+1)})(1-(q_1q_2)^c)-q_2(1-q_1^c)(1-(q_1q_2)^{c(d+1)})}{(1-q_2)(1-q_1^c)(1-(q_1q_2)^c)}. \tag{3.4}$$

Therefore we have

$$Z_\infty^{(c,1)}(t_1,t_2)=\zeta(t_2)\zeta(ct_1)\zeta(c(t_1+t_2))\prod_{p\,:\,\mathrm{prime}}(1-p^{-t_2}+p^{-ct_1-t_2}-p^{-ct_1-ct_2})$$

for $\mathrm{Re}\,t_j>1$ $(j=1,2)$. Thus, we may have various possibility of Euler products of the form $\prod_{p\,:\,\mathrm{prime}}H(p^{-s},p^{-t})$, where $H(S,T)\in 1+S\cdot\mathbb{C}[S,T]+T\cdot\mathbb{C}[S,T]$ is arising from $Z_\infty^{(c,1)}(t_1,t_2)$. □

Example 3.2. We calculate $G_\infty^{(cd,c,1)}(q,q,q)$. If we set $q_1=q_2=q$ in (3.4), then we have

$$G_l^{(c,1)}(q,q)=\frac{(1-q^{c(d+1)})(1-q+q^c-q^{cd+c+1})}{(1-q)(1-q^{2c})}.$$

It follows from Lemma 3.4 that

$$G_\infty^{(cd,c,1)}(q,q,q)=\frac{1}{(1-q)(1-q^{2c})}\left\{\frac{1-q+q^c}{1-q^{cd}}-\frac{q^c(1+q^c)}{1-q^{2cd}}+\frac{q^{2c+1}}{1-q^{3cd}}\right\}.$$

In particular, putting $d=1$, we obtain

$$\begin{aligned}Z_\infty^{(c,c,1)}(s)&=\sum_{n_3|n_2^c|n_1^c}(n_1^cn_2^cn_3)^{-s}\\&=\zeta(s)\zeta(2cs)\zeta(3cs)\prod_{p\,:\,\mathrm{prime}}(1-p^{-s}+p^{-cs}-p^{-(c+1)s}+p^{-2cs}).\end{aligned}$$ □

4 Multiple zeta functions and powerful numbers

In this section, we study $Z_\infty^\gamma(s)$ for $\gamma = (k, k, \ldots, k, 1)$ in connection with a certain generalized notion of *powerful numbers*.

Let us first recall the definition of powerful numbers. A positive number n is called a k-powerful number if $\mathrm{ord}_p n \geq k$ for any prime number p unless $\mathrm{ord}_p n = 0$ (see, e.g. [IS], [I]). Extending this, we arrive at a new notion, *l-step k-powerful numbers*; a positive integer n is said to be an l-step k-powerful number if n satisfies the condition that $\mathrm{ord}_p n = 0, k, 2k, \ldots, (l-1)k$ or $\mathrm{ord}_p n \geq lk$ for any prime number p. Clearly, if n is an l-step k-powerful number, then n is again a j-step k-powerful number for each j $(1 \leq j \leq l)$. In particular, 1-step k-powerful numbers are nothing but the usual k-powerful numbers. Note also that every natural number is an l-step 1-powerful number for any l; this agrees with the claim for $k = 1$ in Theorem 4.8 below.

As an example of l-step k-powerful numbers, we have the first few of 2-step 2-powerful numbers; 1, 4, 9, 16, 25, 32, 36, 49, 64, 81, 100, 121, 128, 144, 169, 196, 225, 243, ... Note that in general an l-step k-powerful number n has the canonical representation; $n = a_1^k a_2^{2k} \cdots a_l^{(l-1)k} \times m$, where $a_1, \cdots, a_l$ are square-free, m is (lk)-powerful and these satisfy $\gcd(a_1, \cdots, a_l, m) = 1$. Remark also that a k-powerful number m is uniquely expressed as $m = b_1^k b_2^{k+1} \cdots b_k^{2k-1}$ if we stipulate that $b_2, \cdots b_k$ are all square-free. Let us put

$$f_{k,l}(n) := \begin{cases} 1 & \text{if } n \text{ is an } l\text{-step } k\text{-powerful number} \\ 0 & \text{otherwise} \end{cases},$$

for a positive integer n. We define also $F_{k,l}(s) := \sum_{n=1}^\infty f_{k,l}(n) n^{-s}$. The arithmetic function $f_{k,l}(n)$ is multiplicative with respect to n. Note that $F_{1,l}(s) = \zeta(s)$ for any l. We show that $Z_\infty^\gamma(s)$ is represented by the product of the Riemann zeta functions times $F_{k,l}(s)$.

Theorem 4.1. *Let k, l be positive integers, and put $\gamma = (\overbrace{k, k, \ldots, k}^{l}, 1)$. Then we have*

$$(4.1) \qquad Z_\infty^\gamma(s) = \sum_{n_{l+1} | n_l^k | \cdots | n_1^k} (n_1^k \cdots n_l^k n_{l+1})^{-s} = F_{k,l}(s) \prod_{j=2}^{l+1} \zeta(jks) \qquad (\mathrm{Re}\, s > 1).$$

In particular, we have $Z_\infty^m(s) = \prod_{j=1}^m \zeta(js)$.

To prove the theorem, we need the following two lemmas.

Lemma 4.2. *Let $\gamma = (\overbrace{k, k, \ldots, k}^{l}, 1)$. Then we have*

$$G_\infty^\gamma(q) = G_\infty^\gamma(\overbrace{q, q, \ldots, q}^{l+1}) = \prod_{j=1}^{l+1} \frac{1}{1 - q^{jk}} \cdot \frac{1 - q + q^{lk+1} - q^{k(l+1)}}{1 - q}.$$

Proof. By definition, we have

$$\begin{aligned}
G_\infty^{(k,\dots,k,1)}(q) &= \sum_{\substack{\lambda_1\ge\cdots\ge\lambda_l\ge\lambda_{l+1}\ge 0\\ k|\lambda_j\ (1\le j\le l)}} q^{\lambda_1+\cdots+\lambda_l+\lambda_{l+1}} = \sum_{n=0}^{\infty} q^n \sum_{\substack{\lambda_1\ge\cdots\ge\lambda_l\ge n\\ k|\lambda_j\ (1\le j\le l)}} q^{\lambda_1+\cdots+\lambda_l} \\
&= \sum_{\substack{\lambda_1\ge\cdots\ge\lambda_l\ge 0\\ k|\lambda_j\ (1\le j\le l)}} q^{\lambda_1+\cdots+\lambda_l} + \sum_{a=0}^{\infty}\sum_{b=1}^{k} q^{ak+b} \sum_{\substack{\lambda_1\ge\cdots\ge\lambda_l\ge ak+b\\ k|\lambda_j\ (1\le j\le l)}} q^{\lambda_1+\cdots+\lambda_l} \\
&= \sum_{\mu_1\ge\cdots\ge\mu_l\ge 0} q^{k(\mu_1+\cdots+\mu_l)} + \sum_{a=0}^{\infty}\sum_{b=1}^{k} q^{ak+b} \sum_{\mu_1\ge\cdots\ge\mu_l\ge a+1} q^{k(\mu_1+\cdots+\mu_l)} \\
&= G_\infty^l(q^k) + \sum_{a=0}^{\infty} q^{ak} \sum_{b=1}^{k} q^b \sum_{\nu_1\ge\cdots\ge\nu_l\ge 0} q^{k\{(\nu_1+a+1)+\cdots+(\nu_l+a+1)\}} \\
&= G_\infty^l(q^k) + q^{lk}\left(\sum_{a=0}^{\infty} q^{k(l+1)a}\right)\left(\sum_{b=1}^{k} q^b\right)\left(\sum_{\nu_1\ge\cdots\ge\nu_l\ge 0} q^{k(\nu_1+\cdots+\nu_l)}\right).
\end{aligned}$$

Since $G_\infty^l(q) = \prod_{j=1}^{l}(1-q^j)^{-1}$, we have

$$\begin{aligned}
G_\infty^{(k,\dots,k,1)}(q) &= \prod_{j=1}^{l} \frac{1}{1-q^{jk}} \left(1 + \frac{q^{kl}}{1-q^{k(l+1)}} \frac{q(1-q^k)}{1-q}\right) \\
&= \prod_{j=1}^{l+1} \frac{1}{1-q^{jk}} \frac{1-q+q^{lk+1}-q^{k(l+1)}}{1-q}.
\end{aligned}$$

This proves the assertion. □

The following lemma is easily obtained.

Lemma 4.3. *We have*

$$\frac{1-q+q^{lk+1}-q^{k(l+1)}}{1-q} = (1-q^k)\left(1+q^k+q^{2k}+\cdots+q^{(l-1)k}+q^{lk}\sum_{j=0}^{\infty} q^j\right).$$

□

Proof of Theorem 4.1. It follows from Lemma 4.2 and Lemma 4.3 that

$$\begin{aligned}
Z_{\infty}^{(k,\ldots,k,1)}(s) &= \prod_{p\,:\,\mathrm{prime}} G_{\infty}^{(k,\ldots,k,1)}(p^{-s}) \\
&= \prod_{p\,:\,\mathrm{prime}} \left(\prod_{j=2}^{\infty} \frac{1}{1-p^{-jks}} \right) \\
&\quad \times \prod_{p\,:\,\mathrm{prime}} (1 + p^{-ks} + p^{-2ks} + \cdots + p^{-(l-1)ks} + p^{-lks} + p^{-(lk+1)s} + \cdots) \\
&= \prod_{j=2}^{l+1} \zeta(jks) \cdot F_{k,l}(s).
\end{aligned}$$

□

Now we determine the condition if the Dirichlet series $F_{k,l}(s)$ can be meromorphically extended to $\mathbb{C}$. We recall the following Estermann's result [E] (see [K] for a generalization). A polynomial $f(T) \in 1 + T \cdot \mathbb{C}[T]$ is said to be *unitary* if and only if there is a unitary matrix M such that $f(T) = \det(1 - MT)$.

Lemma 4.4. *For a polynomial $f(T) \in 1 + T \cdot \mathbb{C}[T]$, put $L(s,f) = \prod_{p\,:\,\mathrm{prime}} f(p^{-s})$. Then*

1. *$f(T)$ is unitary if and only if $L(s,f)$ can be extended as a meromorphic function on $\mathbb{C}$.*
2. *$f(T)$ is not unitary if and only if $L(s,f)$ can be extended as a meromorphic function in $\mathrm{Re}\, s > 0$ with the natural boundary $\mathrm{Re}\, s = 0$; each point on $\mathrm{Re}\, s = 0$ is a limit-point of poles of $L(s,f)$ in $\mathrm{Re}\, s > 0$.* □

Since

$$\begin{aligned}
F_{k,l}(s) &= \prod_{p\,:\,\mathrm{prime}} \left(1 + p^{-ks} + p^{-2ks} + \cdots + p^{-(l-1)ks} + p^{-lks} \sum_{j=0}^{\infty} p^{-js} \right) \\
&= \zeta(s)\zeta(ks) \prod_{p\,:\,\mathrm{prime}} (1 - p^{-s} + p^{-(lk+1)s} - p^{-k(l+1)s})
\end{aligned}$$

by Lemma 4.3, we have only to see whether the polynomial $G_{k,l}(T) := 1 - T + T^{lk+1} - T^{k(l+1)}$ is unitary or not. The polynomial $G_{k,l}(T)$ can be expressed as $G_{k,l}(T) = (1 - T^k) H_{k,l}(T)$ with $H_{k,l}(T) := 1 + (T^k - T) \sum_{j=0}^{l-1} T^{kj}$.

Proposition 4.5. *The polynomial $G_{k,l}(T)$ is unitary if and only if $k = 1, 2$.*

In order to prove this proposition, we need the following two lemmas.

Lemma 4.6. *Let $k \geq 3$. Then the unitary root α (i.e. $|\alpha| = 1$) of the polynomial $G_{k,l}(T)$ must satisfy $\alpha^k = 1$ or $\alpha^{k-2} = 1$.*

Proof. Let $\alpha = e^{2\pi i\theta} \neq 1$ ($\theta \in \mathbb{R}$) be a unitary root of $G_{k,l}(T)$. Since $G_{k,l}(T)/(1-T) = 1 + T^{lk+1}(1-T^{k-1})/(1-T)$, we have $1+\alpha^{lk+1}(1-\alpha^{k-1})/(1-\alpha) = 0$ so that $\left|(1-\alpha^{k-1})/(1-\alpha)\right| = \left|\alpha^{-(lk+1)}\right| = 1$. Hence we see that $\operatorname{Re}\alpha^{k-1} = \operatorname{Re}\alpha$, that is, $\cos 2\pi(k-1)\theta - \cos 2\pi\theta = -2\sin\pi k\theta \sin\pi(k-2)\theta = 0$. Thus we conclude that either $k\theta \in \mathbb{Z}$ or $(k-2)\theta \in \mathbb{Z}$. This proves the lemma. □

Lemma 4.7. *Let $k \geq 3$. Suppose that a complex number α satisfies $\alpha^{k-2} = 1$. Then $G''_{k,l}(\alpha) \neq 0$.*

Proof. Since $G''_{k,l}(T) = (lk+1)lkT^{l(k-2)+2l-1} - (kl+k)(kl+k-1)T^{(k-2)(l+1)+2l}$, if we assume that α satisfies $G''_{k,l}(\alpha) = 0$ and $\alpha^{k-2} = 1$, we have $G''_{k,l}(\alpha) = (lk+1)lk\alpha^{2l-1} - (kl+k)(kl+k-1)\alpha^{2l} = 0$. This shows that $\alpha = \frac{l(lk+1)}{(l+1)(kl+k-1)}$, which contradicts $\alpha^{k-2} = 1$. Hence the assertion follows. □

Proof of Proposition 4.5. Let l be a positive integer. Since the unitarity of $G_{k,l}(T)$ and that of $H_{k,l}(T)$ are equivalent, it suffices to check the unitarity of $H_{k,l}(T)$. It is clear that $H_{k,l}(T)$ is a unitary polynomial when $k = 1, 2$. Actually we have $H_{1,l}(T) = 1$ and $H_{2,l}(T) = 1 - T + T^{2l+1} - T^{2l+2} = (1-T)(1+T^{2l+1})$, which are indeed unitary.

Suppose $k \geq 3$. If $G_{k,l}(T)$ is unitary, then every root of $H_{k,l}(T)$ satisfies $\alpha^k = 1$ or $\alpha^{k-2} = 1$ by Lemma 4.6. However, if $\alpha^k = 1$ we immediately see that $H_{k,l}(\alpha) = 1 + (1-\alpha)l$. Thus, $H_{k,l}(\alpha)$ can not be 0 because of the unitarity of α. Thus any root of $H_{k,l}(T)$ must satisfy $\alpha^{k-2} = 1$ and $\alpha^k \neq 1$. By Lemma 4.7, the multiplicity of these roots of $H_{k,l}(T)$ is at most 2. Since $H_{k,l}(T)$ is assumed to be unitary and $H_{k,l}(1) \neq 0$, it follows that $2(k-3) \geq \deg H_{k,l}(T) = lk$. This is possible only when $l = 1$. Therefore it is enough to prove that $H_{k,1}(T)$ is not unitary for $k \geq 3$. We put $H_k(T) = H_{k,1}(T) = 1 - T + T^k$ for simplicity. If k is odd, then $H_k(T)$ has a real root in the interval $(-2, -1)$ since $H_k(-1) = 1 > 0$ and $H_k(-2) = 3 - 2^k < 0$. This implies $H_k(T)$ is not unitary.

Thus, only we have to consider is the case where k is even and $k \geq 4$. Suppose that $H_k(T)$ is unitary and let $e^{i\theta}$ $(-\pi < \theta \leq \pi)$ be its unitary root. Then we see that θ satisfies the equations $\cos k\theta = \cos\theta - 1$ and $\sin k\theta = \sin\theta$. Since $1 = \sin^2 k\theta + \cos^2 k\theta = 2 - 2\cos\theta$, we have $\theta = \pm\pi/3$. Further, since $1 = \cos^2\theta + \sin^2\theta = (\cos k\theta + 1)^2 + \sin^2 k\theta = 2\cos k\theta + 2$, we have $\cos(k\pi/3) = -1/2$. Hence we see that either $k \equiv 2$ or $4 \pmod 6$. On the other hand, since $1 = (\cos\theta - \cos k\theta)^2 + (\sin\theta - \sin k\theta)^2 = 2 - 2\cos(k-1)\theta$, we have $\cos((k-1)\pi/3) = 1/2$. It follows that either $k \equiv 0$ or $2 \pmod 6$. Thus we have $k \equiv 2 \pmod 6$.

Now we show that every unitary root of $H_k(T)$ is simple. If we assume that β is a multiple root of $H_k(T)$, it follows that $\beta^k - \beta + 1 = 0$ and $k\beta^{k-1} - 1 = 0$. Then we have $|\beta| = k^{-1/(k-1)}$ and $\beta = k\beta^k$ by the second equation. On the other hand, by the first equation, we obtain $1 = \beta - \beta^k = (k-1)\beta^k$ so that $|\beta| = (k-1)^{-1/k}$. Therefore we have

$k^k = |\beta|^{-k(k-1)} = (k-1)^{k-1}$, but this contradicts to the assumption of the unitarity of $H_k(T)$. This completes the proof of the proposition. □

Finally we obtain the following generalization of the result in [IS] concerning the powerful numbers. The proof follows immeadiately from Lemma 4.4 and Proposition 4.5.

Theorem 4.8. *Let k and l be positive integers. When $k = 1, 2$ we have*

$$F_{1,l}(s) = \zeta(s), \qquad F_{2,l}(s) = \frac{\zeta(2s)\zeta((2l+1)s)}{\zeta(2(2l+1)s)}.$$

When $k \geq 3$, $F_{k,l}(s)$ is meromorphic in $\mathrm{Re}\, s > 0$ *and has a natural boundary* $\mathrm{Re}\, s = 0$. □

Corollary 4.9. *Let $Z_{\infty}^{(k,\ldots,k,1)}(s)$ be as in Theorem 4.1. Then for $k = 1, 2$ we have*

$$Z_{\infty}^{(1,\ldots,1,1)}(s) = \prod_{j=1}^{l+1} \zeta(js), \qquad Z_{\infty}^{(2,\ldots,2,1)}(s) = \frac{\zeta((2l+1)s)}{\zeta(2(2l+1)s)} \prod_{j=1}^{l+1} \zeta(2js).$$

For $k \geq 3$, $Z_{\infty}^{(k,\ldots,k,1)}(s)$ can be meromorphically extended to $\mathrm{Re}\, s > 0$ *with a natural boundary* $\mathrm{Re}\, s = 0$. □

5 Closing remarks

We give two remarks.

• The isomorphism classes of abelian groups A of order n are indexed by the map $\boldsymbol{\lambda}$ from the set of all prime numbers to that of partitions such that $n = \prod_{p\,:\,\mathrm{prime}} p^{|\boldsymbol{\lambda}(p)|}$ and $A \cong \bigoplus_{p\,:\,\mathrm{prime}} \bigoplus_{j=1}^{\ell(\boldsymbol{\lambda}(p))} \mathbb{Z}/p^{\boldsymbol{\lambda}_j(p)}\mathbb{Z}$, where $|\boldsymbol{\lambda}(p)|$ and $\ell(\boldsymbol{\lambda}(p))$ are the size and the length of the partition $\boldsymbol{\lambda}(p) = (\boldsymbol{\lambda}_j(p))_{j\geq 1}$ respectively. The multiple finite Riemann zeta function is expressed also as $Z_N^m(s) = \sum_{n|N^m} g_N^m(n) n^{-s}$. Here $g_N^m(n)$ is the number of isomorphism classes of abelian groups of order n, parametrized by $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda}_1(p) \leq m$ and $\ell(\boldsymbol{\lambda}(p)) \leq \mathrm{ord}_p N$ for all p. It is clear that $g_N^m(n)$ is multiplicative with respect to n and N. Put $g_{\infty}^m(n) := \lim_{N\to\infty} g_N^m(n)$. Then $g_{\infty}^m(n)$ is the number of the isomorphism classes of abelian groups A of order n which is the direct sum of p-groups A_p such that $p^m A_p = 0$ for $p|n$.

We now study the asymptotic average for $g_{\infty}^m(n)$ and $Z_n^m(\sigma)$ ($\sigma \in \mathbb{R}$) with respect to n. Thus we need the Tauberian theorem below (see, e.g. [MM]).

Lemma 5.1. *Let $F(t) = \sum_{n=1}^{\infty} a_n n^{-t}$ be a Dirichlet series with non-negative real coefficients which converges absolutely for* $\mathrm{Re}(t) > \beta$. *Suppose that $F(t)$ has a meromorphic continuation to the region* $\mathrm{Re}(t) \geq \beta$ *with a pole of order $\alpha + 1$ at $t = \beta$ for some $\alpha \geq 0$. Put* $c := \frac{1}{\alpha!} \lim_{t\to\beta} (t-\beta)^{\alpha+1} F(t)$. *Then we have $\sum_{n\leq x} a_n = (c + o(1)) x^{\beta} (\log x)^{\alpha}$ as $x \to \infty$.* □

Using this lemma, we have easily the following results: Let m be a positive integer.

1. We have $\sum_{n\le x} g_\infty^m(n) = (\zeta(2)\zeta(3)\cdots\zeta(m)+o(1))x$ as $x\to\infty$. In other words, the asymptotic average of $g_\infty^m(n)$ with respect to n is given by the product $\zeta(2)\zeta(3)\cdots\zeta(m)$.

2. For a fixed $\sigma>0$, when $x\to\infty$,

$$\sum_{n\le x} Z_n^m(\sigma) = (\zeta(\sigma+1)\zeta(2\sigma+1)\cdots\zeta(m\sigma+1)+o(1))x,$$
$$\sum_{n\le x} Z_n^m(-\sigma) = (\zeta(\sigma+1)\zeta(2\sigma+1)\cdots\zeta(m\sigma+1)+o(1))x^{1+m\sigma},$$
$$\sum_{n\le x} Z_n^m(0) = \sum_{n\le x}\prod_{p\,:\,\mathrm{prime}}\binom{\mathrm{ord}_p n+m}{m} = \left(\frac{1}{m!}+o(1)\right)x(\log x)^m.$$

Actually, since $\zeta(s)$ has a single pole at $s=1$ and $\mathrm{Res}_{s=1}\zeta(s)=1$, it follows the first assertion from (2.6) and Lemma 5.1. Next, fix $\sigma\in\mathbb{R}$. By (2.6), we have $\zeta^m(\sigma,t)=\sum_{n=1}^\infty Z_n^m(\sigma)n^{-t}=\prod_{k=0}^m\zeta(t+k\sigma)$. This shows that the abscissa of absolute convergence of the Dirichlet series $\zeta^m(\sigma,t)$ is given by $t=\max\{1,1-m\sigma\}$. Hence the remaining formulas follow similarly.

Remark 5.1. For $g(n):=\lim_{m\to\infty}g_\infty^m(n)$, it is well-known [A] that $\sum_{n\le x}g(n)=\Big(\prod_{k=2}^\infty\zeta(k)\Big)x+O(\sqrt{x})$.

Since $Z_n^1(0)=d(n):=\sum_{d|n}1$, we have $\sum_{n\le x}d(n)\sim x\log x$. It is also well-known [Z] that there exists a constant C such that $\sum_{n\le x}d(n)=x\log x+Cx+O(\sqrt{x})$ in an elementary way. □

• We define a multiple Eisenstein series with parameter s of type m by

$$E_s^m(q)=\sum_{n=1}^\infty Z_n^m(1-s)q^n.$$

We sometimes write $E_s^m(\tau)$ instead of $E_s^m(q)$ when $q=e^{2\pi i\tau}$ with $\tau\in\mathbb{C}$, $\mathrm{Im}(\tau)>0$. It is obvious that $E_k^1(q)$ is (essentially) the usual holomorphic Eisenstein series of weight k. In this remark we make an experimental study of $E_s^m(q)$ when $m=2$. First we have easily the

Lemma 5.2. *We have* $E_{s+1}^2(q)=\sum_{l=1}^\infty\sum_{N=1}^\infty\sigma_s(N)N^sq^{Nl}$. □

Recall now the Fourier expansion of $E_{k+1}(\tau)$ of weight $k+1$ with k being odd;

$$E_{k+1}(\tau)=1+\frac{1}{\zeta(k+1)}\frac{(2\pi i)^{k+1}}{k!}\sum_{n=1}^\infty\sigma_k(n)q^n$$
$$\left(=1+\frac{1}{\zeta(k+1)}\frac{(2\pi i)^{k+1}}{k!}E_{k+1}^1(\tau)\right).$$

Take k-times derivative of $E_{k+1}(\tau)$. Then, if k is odd, by Lemma 5.2 we immediately get

$$E_{k+1}^2(\tau) = \frac{\zeta(k+1)k!}{(2\pi i)^{2k+1}} \sum_{l=1}^{\infty} \left(\frac{d^k}{d\tau^k} E_{k+1} \right) (l\tau).$$

There is also an expression of $E_{k+1}^2(\tau)$ similar to $E_{k+1}(\tau)$, when k is odd.

$$E_{k+1}^2(\tau) = -\frac{(2k)!}{(2\pi i)^{2k+1}} \sum_{l=1}^{\infty} \sum_{\substack{(c,d)=1 \\ c>0}} \sigma_k(cl) l^{-2k-1} (c\tau + d)^{-2k-1}.$$

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