REPRESENTATION THEORY OF THE α-DETERMINANT AND ZONAL SPHERICAL FUNCTIONS

KAZUFUMI KIMOTO

Abstract. Consider the cyclic span of a power of a certain polynomial called *α-determinant*, which is a common generalization of the determinant and permanent, under the action of the universal enveloping algebra of the general linear algebra. We show that the multiplicity of each irreducible component in this cyclic module is given by the rank of a certain associated matrix called *transition matrices*, whose entries are polynomials in the parameter α. We also give several explicit examples of such matrices. In particular, in the case where the size of the matrix for the α-determinant is two, the polynomials in the transition matrices are essentially given by *Jacobi polynomials*.

1. Introduction

Let us consider the representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n=\mathfrak{gl}_n(\mathbb{C})$ on the polynomial algebra $\mathcal{A}(\mathrm{Mat}_n)$ of n^2 variables x_{ij} $(1\le i,j\le n)$ defined by

$$E_{pq}\cdot f(X)=\sum_{r=1}^{n} x_{pr}\frac{\partial f(X)}{\partial x_{qr}}$$

for $f(X)\in\mathcal{A}(\mathrm{Mat}_n)$ $(X=(x_{ij})_{1\le i,j\le n})$, where $\{E_{pq}\}_{1\le p,q\le n}$ is the standard basis of $\mathfrak{gl}_n$. It is well known that the cyclic modules generated by the determinant and permanent are both irreducible. Actually, if we denote by $\mathcal{M}_n^\lambda$ the irreducible representation of $\mathcal{U}(\mathfrak{gl}_n)$ with highest weight λ (which we will represent as a partition), then

$$\mathcal{U}(\mathfrak{gl}_n)\cdot\det(X)=\mathcal{M}_n^{(1^n)},\qquad \mathcal{U}(\mathfrak{gl}_n)\cdot\mathrm{per}(X)=\mathcal{M}_n^{(n)}.$$

Namely, $\det(X)$ generates the skew-symmetric tensor of the natural representation $\mathbb{C}^n$, and $\mathrm{per}(X)$ the symmetric tensor of $\mathbb{C}^n$.

As a common generalization of the determinant and permanent, the *α-determinant* is defined by

$$\det{}^{(\alpha)}(X)=\sum_{\sigma\in\mathfrak{S}_n}\alpha^{\nu(\sigma)}x_{\sigma(1)1}x_{\sigma(2)2}\cdots x_{\sigma(n)n},$$

where $\nu(\sigma)$ is given by n minus the number of cycles in the disjoint cycle decomposition of σ. In fact, $\det(X)=\det^{(-1)}(X)$ and $\mathrm{per}(X)=\det^{(1)}(X)$. It was Vere-Jones [9] who introduce the α-determinant first (but his definition is a little bit different from ours and he also called his one "α-permanent"). One of his motivation to introduce the α-determinant is an application to the probability theory. For further information, we refer to [9], [8] and the references within.

Regarding that the α-determinant *interpolates* the skew-symmetric tensor and symmetric tensor representations, Matsumoto and Wakayama studied the cyclic $\mathcal{U}(\mathfrak{gl}_n)$-module generated by the α-determinant and determine the irreducible decomposition of it. Precisely, they proved that

$$\boldsymbol{V}_{n,1}(\alpha):=\mathcal{U}(\mathfrak{gl}_n)\cdot\det{}^{(\alpha)}(X)=\bigoplus_{\lambda\vdash n}\left(\mathcal{M}_n^\lambda\right)^{\oplus m_n^\lambda(\alpha)},$$

where the multiplicity $m_n^\lambda(\alpha)$ of the irreducible component $\mathcal{M}_n^\lambda$ is given by

$$m_n^\lambda(\alpha)=\begin{cases}0 & f_\lambda(\alpha)=0,\\ f^\lambda & \text{otherwise,}\end{cases}$$

and

$$f^{\lambda}(\alpha)=\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}(1+(j-i)\alpha)$$

is the (modified) content polynomial for a partition λ ($\ell(\lambda)$ is the length of λ). The point is that the irreducible decomposition of the module $\boldsymbol{V}_{n,1}(\alpha)$ is controlled by simple polynomials $\{f_\lambda(\alpha)\}_{\lambda\vdash n}$, whose roots are reciprocal of non-zero integers, and the multiplicities are "all-or-nothing" (i.e. the possible values of $m_n^\lambda(\alpha)$ is either 0 or f^λ for each λ).

In this article, we consider the generalization

$$\boldsymbol{V}_{n,l}(\alpha):=\mathcal{U}(\mathfrak{gl}_n)\cdot\det{}^{(\alpha)}(X)^l.$$

We will see that the multiplicity of each irreducible representation $\mathcal{M}_n^\lambda$ in $\boldsymbol{V}_{n,l}(\alpha)$ is given by the rank of a certain matrix denoted by $F_{n,l}^\lambda(\alpha)$ (Theorem 2.4). In contrast to the case where $l=1$, the multiplicities would take an intermediate value between 0 and the size of the matrix $F_{n,l}^\lambda(\alpha)$ (see Example 2.9), and it seems quite difficult so far to determine the exact values of the multiplicities for a given value of α in an explicit way.

However, we can give a sufficient condition for the matrix $F_{n,l}^\lambda(\alpha)$ to be *scalar* (Proposition 2.5), in which case the multiplicity is controlled by a single polynomial $f_{n,l}^\lambda(\alpha)=\operatorname{tr}F_{n,l}^\lambda(\alpha)$ as in the case of $l=1$. One of the most interesting cases of such a scalar situation is the case where $n=2$; We will see that $f_{n,l}^\lambda(\alpha)$ is written in terms of the *Jacobi polynomials*. As an appendix, we also give several concrete examples of such polynomials.

This article is written based on the talk given at the workshop "Harmonic Analysis on Homogeneous Spaces and Quantization" (February 18–22, 2008) in Fukuoka as well as our recent article [3], which is a joint work with Sho Matsumoto and Masato Wakayama. We will not give proofs of the statements, which one can find in [3].

2. Irreducible decomposition of $\boldsymbol{V}_{n,l}(\alpha)$ and transition matrices

Fix positive integers n,l. Take a standard tableau $\mathbb{T}$ with shape (l^n) such that the (i,j)-entry of $\mathbb{T}$ is $(i-1)l+j$, and denote by $K=\mathrm{Row}(\mathbb{T})$ and $H=\mathrm{Col}(\mathbb{T})$ the row group and column group of $\mathbb{T}$ respectively. The following two elements

$$e:=\frac{1}{|K|}\sum_{k\in K}k,\quad \Phi:=\sum_{h\in H}\varphi(h)h \tag{2.1}$$

in the group algebra $\mathbb{C}[\mathfrak{S}_{nl}]$, where φ is a class function on H, play a key role. We will work on the tensor product space $V=(\mathbb{C}^n)^{\otimes nl}$, which is a $(\mathcal{U}(\mathfrak{gl}_n),\mathbb{C}[\mathfrak{S}_{nl}])$-module by setting

$$\begin{aligned}E_{ij}\cdot\boldsymbol{e}_{i_1}\otimes\cdots\otimes\boldsymbol{e}_{i_{nl}}&=\sum_{s=1}^{nl}\delta_{i_s,j}\,\boldsymbol{e}_{i_1}\otimes\cdots\otimes\overset{s\text{-th}}{\boldsymbol{e}_i}\otimes\cdots\otimes\boldsymbol{e}_{i_{nl}},\\ \boldsymbol{e}_{i_1}\otimes\cdots\otimes\boldsymbol{e}_{i_{nl}}\cdot\sigma&=\boldsymbol{e}_{i_{\sigma(1)}}\otimes\cdots\otimes\boldsymbol{e}_{i_{\sigma(nl)}}\qquad(\sigma\in\mathfrak{S}_{nl}),\end{aligned} \tag{2.2}$$

where $\{\boldsymbol{e}_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{C}^n$. Using the group isomorphism

$$\begin{aligned}&\theta: H\ni h\mapsto\theta(h)=(\theta(h)_1,\dots,\theta(h)_l)\in\mathfrak{S}_n^l,\\ &\theta(h)_i(x)=y\iff h((x-1)l+i)=(y-1)l+i\qquad(1\le x,y\le n,\ 1\le i\le l),\end{aligned} \tag{2.3}$$

define also an element $D(X;\varphi)\in\mathcal{A}(\mathrm{Mat}_n)$ by

$$D(X;\varphi)=\sum_{h\in H}\varphi(h)\prod_{q=1}^n\prod_{p=1}^l x_{\theta(h)_p(q),q}=\sum_{\sigma_1,\dots,\sigma_l\in\mathfrak{S}_n}\varphi(\theta^{-1}(\sigma_1,\dots,\sigma_l))\prod_{q=1}^n\prod_{p=1}^l x_{\sigma_p(q),q}. \tag{2.4}$$

Notice that $D(X;\alpha^{\nu(\cdot)})=\det^{(\alpha)}(X)^l$ since $\nu(\theta^{-1}(\sigma_1,\dots,\sigma_l))=\nu(\sigma_1)+\cdots+\nu(\sigma_l)$ for $(\sigma_1,\dots,\sigma_l)\in\mathfrak{S}_l^n$. If δ_H is a function on H which is one at the identity element and zero otherwise, then $D(X;\delta_H)=(x_{11}x_{22}\dots x_{nn})^l$. The following lemma is fundamental.

Lemma 2.1. (1) $\mathcal{U}(\mathfrak{gl}_n)\cdot\boldsymbol{e}_1^{\otimes l}\otimes\cdots\otimes\boldsymbol{e}_n^{\otimes l}=V\cdot e=\mathcal{S}^l(\mathbb{C}^n)^{\otimes n}$.

(2) *The map*

$$\mathcal{T} : \mathcal{U}(\mathfrak{gl}_n) \cdot D(X; \delta_H) \ni \prod_{q=1}^{n} \prod_{p=1}^{l} x_{i_{pq}q} \longmapsto (\boldsymbol{e}_{i_{11}} \otimes \cdots \otimes \boldsymbol{e}_{i_{l1}}) \otimes \cdots \otimes (\boldsymbol{e}_{i_{1n}} \otimes \cdots \otimes \boldsymbol{e}_{i_{ln}}) \cdot e \in V \cdot e$$

is a bijective $\mathcal{U}(\mathfrak{gl}_n)$-intertwiner.

(3) $D(X;\varphi) \in \mathcal{U}(\mathfrak{gl}_n) \cdot D(X; \delta_H)$ *for any class function φ on H, and it is mapped to $\boldsymbol{e}_1^{\otimes l} \otimes \cdots \otimes \boldsymbol{e}_n^{\otimes l} \cdot e\Phi e$ by $\mathcal{T}$.* □

Using the lemma, we have the

Lemma 2.2. *It holds that*

$$\mathcal{U}(\mathfrak{gl}_n) \cdot D(X;\varphi) \cong V \cdot e\Phi e \tag{2.5}$$

as a left $\mathcal{U}(\mathfrak{gl}_n)$-module. In particular, $V \cdot e\Phi e \cong \boldsymbol{V}_{n,l}(\alpha)$ if $\varphi(h) = \alpha^{\nu(h)}$. □

The Schur-Weyl duality reads

$$V \cong \bigoplus_{\lambda \vdash nl} \mathcal{M}_n^{\lambda} \boxtimes \mathcal{S}^{\lambda}. \tag{2.6}$$

Here $\mathcal{S}^{\lambda}$ denotes the irreducible unitary right $\mathfrak{S}_{nl}$-module corresponding to λ. We see that

$$\dim\left(\mathcal{S}^{\lambda} \cdot e\right) = \left\langle \operatorname{ind}_K^G \mathbf{1}_K,\, \mathcal{S}^{\lambda} \right\rangle_{\mathfrak{S}_{nl}} = K_{\lambda(l^n)}, \tag{2.7}$$

where $\mathbf{1}_K$ is the trivial representation of K and $\langle \pi,\, \rho \rangle_{\mathfrak{S}_{nl}}$ is the intertwining number of given representations π and ρ of $\mathfrak{S}_{nl}$, and $K_{\lambda\mu}$ is the Kostka number. Since $K_{\lambda(l^n)} = 0$ unless $\ell(\lambda) \le n$, it follows the

Theorem 2.3. *The irreducible decomposition*

$$V \cdot e\Phi e \cong \bigoplus_{\substack{\lambda \vdash nl \\ \ell(\lambda) \le n}} \mathcal{M}_n^{\lambda} \otimes \left(\mathcal{S}^{\lambda} \cdot e\Phi e\right) \tag{2.8}$$

holds, so that the multiplicity of $\mathcal{M}_n^{\lambda}$ in $V \cdot e\Phi e$ is given by

$$\dim\left(\mathcal{S}^{\lambda} \cdot e\Phi e\right) = \operatorname{rk}_{\operatorname{End}(\mathcal{S}^{\lambda} \cdot e)}(e\Phi e). \tag{2.9}$$

□

As a special case, we now obtain the

Theorem 2.4. *Let $d = K_{\lambda(l^n)}$. Fix an orthonormal basis $\{\boldsymbol{e}_j^{\lambda}\}_{j=1}^{f^{\lambda}}$ of $\mathcal{S}^{\lambda}$, and denote by $\{\psi_{ij}^{\lambda}\}$ the matrix coefficients relative to this basis. Suppose that the first d vectors $\boldsymbol{e}_1^{\lambda}, \ldots, \boldsymbol{e}_d^{\lambda}$ spans $\left(\mathcal{S}^{\lambda}\right)^K$. Then the multiplicity of the irreducible representation $\mathcal{M}_n^{\lambda}$ in the cyclic module $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^l$ is equal to the rank of the matrix*

$$F_{n,l}^{\lambda}(\alpha) := \left(\sum_{h \in H} \alpha^{\nu(h)} \psi_{ij}^{\lambda}(h) \right)_{1 \le i,j \le d}. \tag{2.10}$$

□

We refer to the matrix $F_{n,l}^{\lambda}(\alpha)$ as a *transition matrix for* λ. The transition matrix itself does *depend on* the choice of the basis $\{\boldsymbol{e}_j^{\lambda}\}_{j=1}^{f^{\lambda}}$ of $\mathcal{S}^{\lambda}$ in the theorem, while its rank *does not*. The trace of the transition matrix $F_{n,l}^{\lambda}(\alpha)$ is

$$f_{n,l}^{\lambda}(\alpha) = \operatorname{tr} F_{n,l}^{\lambda}(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \omega^{\lambda}(h), \tag{2.11}$$

where ω^{λ} is the *zonal spherical function* for λ with respect to K defined by

$$\omega^{\lambda}(g) = \frac{1}{|K|} \sum_{k \in K} \chi^{\lambda}(kg) \quad (g \in \mathfrak{S}_{nl}). \tag{2.12}$$

If the matrix $F^\lambda_{n,l}(\alpha)$ is *scalar*, then $F^\lambda_{n,l}(\alpha) = d^{-1} f^\lambda_{n,l}(\alpha) I_d$ ($d = \dim(\mathcal{S}^\lambda)^K$) and hence the multiplicity of $\mathcal{M}^\lambda_n$ in $\boldsymbol{V}_{n,l}(\alpha)$ is completely controlled by the single polynomial $f^\lambda_{n,l}(\alpha)$ as in the case where $l = 1$. Thus it is desirable to obtain a characterization of the triplets (n, l, λ) such that $F^\lambda_{n,l}(\alpha)$ are scalar. The following is a sufficient condition for λ when n and l are given.

Proposition 2.5. *Denote by $N_H(K)$ the normalizer of K in H (Notice that $N_H(K) \cong \mathfrak{S}_n$). The transition matrix $F^\lambda_{n,l}(\alpha)$ is scalar if $(\mathcal{S}^\lambda)^K$ is irreducible as a $N_H(K)$-module.*

Example 2.6 (Matsumoto-Wakayama case)**.** If $l = 1$, then $K = 1$ and $N_H(K) = H$ so that $(\mathcal{S}^\lambda)^K = \mathcal{S}^\lambda$ is an irreducible $N_H(K)$-module, and hence all the transition matrices $F^\lambda_{n,1}(\alpha)$ are scalar. In fact, we have $F^\lambda_{n,1}(\alpha) = f_\lambda(\alpha) I$ and $f^\lambda_{n,1}(\alpha) = f^\lambda f_\lambda(\alpha)$.

Example 2.7 (hook-type case)**.** If $\lambda = (nl - r, 1^r)$ is of *hook type* ($0 \le r \le n-1$), then $(\mathcal{S}^{(nl-r,1^r)})^K \cong \mathcal{S}^{(n-r,1^r)}$ as $N_H(K)$-modules by [1, Proposition 5.3]. Thus the transition matrix $F^{(n-r,1^r)}_{n,l}(\alpha)$ is scalar. See Appendix for the concrete examples in this case.

Example 2.8 (Gelfand pair case)**.** Suppose that $(\mathfrak{S}_{nl}, K)$ is a *Gelfand pair*, that is, the induced representation $\mathrm{ind}_K^{\mathfrak{S}_{nl}} \mathbf{1}_K$ of the trivial representation $\mathbf{1}_K$ of K to $\mathfrak{S}_{nl}$ is *multiplicity-free* (see, e.g. [6]). Then $(\mathcal{S}^\lambda)^K$ is obviously irreducible as an $N_H(K)$-module since it is one-dimensional. In this case, each transition matrix is just a polynomial (one by one matrix). We give an explicit formula of the transition matrices for the case where $n = 2$ in the next section.

We also give a *non-scalar* example of a transition matrix.

Example 2.9. Taking a suitable orthonormal basis of $\left(\mathcal{S}^{(4,2)}\right)^K$, we have the transition matrix $F^{(4,2)}_{3,2}(\alpha)$ for $\mathcal{M}^{(4,2)}_{3,2}(\alpha)$ in $\boldsymbol{V}_{3,2}(\alpha)$ as

$$F^{(4,2)}_{3,2}(\alpha) = \frac{1}{2}(1+\alpha)^2 \operatorname{diag}(2 - 2\alpha + 3\alpha^2, 1 - \alpha, 1 - \alpha).$$

Hence, the multiplicity $m^{(4,2)}_{3,2}(\alpha)$ of $\mathcal{M}^{(4,2)}_{3,2}(\alpha)$ in $\boldsymbol{V}_{3,2}(\alpha)$ is

$$m^{(6,2)}_{4,2}(\alpha) = \begin{cases} 0 & \alpha = -1, \\ 1 & \alpha = 1, \\ 2 & \alpha = \frac{1 \pm \sqrt{-5}}{3}, \\ 3 & \text{otherwise.} \end{cases}$$

3. Irreducible decomposition of $\boldsymbol{V}_{2,l}(\alpha)$ and Jacobi polynomials

When $n = 2$, as is well known, the pair $(\mathfrak{S}_{2l}, K)$ is a *Gelfand pair*, so that the transition matrices $F^\lambda_{2,l}(\alpha)$ ($\lambda \vdash 2l$, $\ell(\lambda) \le 2$) are scalar (of size one). If we set $g_s = (1, l+1)(2, l+2) \dots (s, l+s) \in \mathfrak{S}_{2n}$, then we have

$$F^\lambda_{2,l}(\alpha) = \operatorname{tr} F^\lambda_{2,l}(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \omega^\lambda(h) = \sum_{s=0}^{l} \binom{l}{s} \omega^\lambda(g_s) \alpha^s. \tag{3.1}$$

Now we write $\lambda = (2l - r, r)$ for some r ($0 \le r \le l$). The value $\omega^{(2l-r,r)}(g_s)$ of the zonal spherical function is calculated by Bannai and Ito [2, p.218] as

$$\omega^{(2l-r,r)}(g_s) = Q_r(s; -l-1, -l-1, l) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \binom{2l-r+1}{j} \binom{l}{j}^{-2} \binom{s}{j}, \tag{3.2}$$

where

$$Q_n(x; \alpha, \beta, N) = \sum_{j=0}^{N} (-1)^j \binom{n}{j} \binom{-n-\alpha-\beta-1}{j} \binom{-\alpha-1}{j}^{-1} \binom{N}{j}^{-1} \binom{x}{j} \tag{3.3}$$

is the Hahn polynomial (see also [6, p.399]).

Theorem 3.1. *Let l be a positive integer. It holds that*

$$F_{2,l}^{(2l-r,r)}(\alpha)=\sum_{s=0}^{l}\binom{l}{s}Q_r(s;l-1,l-1,l)\alpha^s=\binom{n-l-1}{n}^{-1}(1+\alpha)^{l-r}P_r^{(-l-1,2l-2r+1)}(1+2\alpha) \tag{3.4}$$

for $r=0,1,\dots,l$. Here $P_n^{(a,b)}(x)$ denotes the Jacobi polynomial

$$P_n^{(a,b)}(x)=\binom{n+a}{n}{}_2F_1\left(-n,a+b+n+1,a+1;\frac{1-x}{2}\right). \tag{3.5}$$

Further, all roots of $F_{2,l}^{(2l-r,r)}(\alpha)$ are lying on the unit circle $|z|=1$. □

Thus we obtain the irreducible decomposition

$$\boldsymbol{V}_{2,l}(-1)\cong\mathcal{M}_2^{(l,l)},\qquad \boldsymbol{V}_{2,l}(\alpha)\cong\bigoplus_{\substack{0\le r\le l\\ P_r^{(-l-1,2l-2r+1)}(1+2\alpha)\neq 0}}\mathcal{M}_2^{(2l-r,r)}\quad(\alpha\neq-1) \tag{3.6}$$

of $\boldsymbol{V}_{2,l}(\alpha)$.

4. Remarks on related works

For all *but finite* values of α, $\boldsymbol{V}_{n,l}(\alpha)$ is equivalent to $\mathcal{S}^l(\mathbb{C}^n)^{\otimes n}$ as we see above. It is interesting not only to describe the exceptional *singular values* nicely (as zeros of certain *special polynomials*, for instance), but also to investigate what happens at the singular values.

We study the *quantum* analogue of our problem in [5] from the first point of view. We introduce the α-deformation of the quantum determinant in the quantum matrix algebra, and consider the cyclic span generated by it under the action of the quantum enveloping algebra. What we expect in this direction is to obtain certain special polynomials in α with parameter q as (entries of) transition matrices defined analogously.

From the second point of view, in [4], we study the case where α is a reciprocal of a negative integer. In this case, the $-1/k$-determinants satisfies a "$-1/k$-analogue" of the multiplicativity of the determinant ($k=1,2,\dots,n-1$, n being the size of the matrix). This enables us to construct a certain relative invariant of GL_n, which we call the *wreath determinants*, using the $-1/k$-determinant. It would be interesting to explore wreath analogues of various known determinant formulas. As an example of such ones, we give an analogue of the Cauchy determinant formula (see §6 of [4]).

Appendix A. Examples of traces of the transition matrices for hook-types

Here we give several examples of the trace $f_{n,l}^{\lambda}(\alpha)=\operatorname{tr}F_{n,l}^{\lambda}(\alpha)$ of the transition matrices for the case where λ is of hook-type calculated by MAPLE. Remark that we can calculate $f_{n,l}^{\lambda}(\alpha)$ explicitly for $\lambda=(nl),(nl-1,1)$ as follows:

$$f_{n,l}^{(nl)}(\alpha)=f_{n,1}^{(n)}(\alpha)^l=\prod_{j=1}^{n-1}(1+j\alpha)^l, \tag{A.1}$$

$$f_{n,l}^{(nl-1,1)}n,l=f_{n,1}^{(n)}(\alpha)^{l-1}f_{n,1}^{(n-1,1)}(\alpha)=(n-1)(1-\alpha)(1-(n-1)\alpha)^{l-1}\prod_{j=1}^{n-2}(1+j\alpha)^l. \tag{A.2}$$

- $(n,l)=(5,2)$:
$$f_{5,2}^{(8,1^2)}(\alpha)=6(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)(1-\alpha)(1+2\alpha-\frac{11}{2}\alpha^2)$$
$$f_{5,2}^{(7,1^3)}(\alpha)=4(1+\alpha)^2(1+2\alpha)(1+3\alpha)(1-\alpha)(1-2\alpha)(1+\alpha-\frac{9}{2}\alpha^2)$$
$$f_{5,2}^{(6,1^4)}(\alpha)=(1+\alpha)(1+2\alpha)^2(1-\alpha)(1-2\alpha)^2(1-6\alpha^2)$$
- $(n,l)=(4,2)$:
$$f_{4,2}^{(6,1^2)}(\alpha)=3(1+\alpha)^2(1+2\alpha)(1-\alpha)(1+\alpha-4\alpha^2)$$
$$f_{4,2}^{(5,1^3)}(\alpha)=(1+\alpha)(1+2\alpha)(1-\alpha)(1-2\alpha)(1-3\alpha^2)$$

- $(n,l)=(3,2)$: $f^{(4,1^2)}_{3,2}(\alpha)=(1+\alpha)(1-\alpha)(1-\frac{5}{2}\alpha^2)$
- $(n,l)=(4,3)$: $f^{(10,1^2)}_{4,3}(\alpha)=3(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)(1-\alpha)(1+\alpha-\frac{10}{3}\alpha^2)$

  $f^{(9,1^3)}_{4,3}(\alpha)=(1+\alpha)^2(1+2\alpha)^2(1-\alpha)(1+\alpha-7\alpha^2-\frac{17}{9}\alpha^3+\frac{94}{9}\alpha^4)$
- $(n,l)=(3,3)$: $f^{(7,1^2)}_{3,3}(\alpha)=(1+\alpha)^2(1+2\alpha)(1-\alpha)(1-2\alpha^2)$
- $(n,l)=(3,4)$: $f^{(10,1^2)}_{3,4}(\alpha)=(1+\alpha)^3(1+2\alpha)^2(1-\alpha)(1-\frac{7}{4}\alpha^2)$

Acknowledgement

The author thanks Itaru Terada for letting him know the work [1], which is used to give Example 2.7, after the talk at the workshop. He also thank Masashi Kosuda for the help in the calculation by MAPLE.

References

1. S. Ariki, J. Matsuzawa and I. Terada, *Representation of Weyl groups on zero weight spaces of $\mathfrak{g}$-modules*, Algebraic and topological theories (Kinosaki, 1984), 546–568, Kinokuniya, Tokyo, 1986. MR1102274
2. E. Bannai and T. Ito, *Algebraic Combinatorics I, Association Schemes*, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984. MR0882540 (87m:05001)
3. K. Kimoto, S. Matsumoto and M. Wakayama, *Alpha-determinant cyclic modules and Jacobi polynomials*, to appear in Trans. Amer. Math. Soc.
4. K. Kimoto and M. Wakayama, *Invariant theory for singular α-determinants*, J. Combin. Theory Ser. A **115** (2008), no. 1, 1–31. MR2378855 (2008m:15019)
5. ———, *Quantum α-determinant cyclic modules of $\mathcal{U}_q(\mathfrak{gl}_n)$*, J. Algebra **313** (2007), 922–956. MR2329577 (2008d:17021)
6. I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd edn.*, Oxford University Press, 1995. MR1354144 (96h:05207)
7. S. Matsumoto and M. Wakayama, *Alpha-determinant cyclic modules of $\mathfrak{gl}_n(\mathbb{C})$*, J. Lie Theory **16** (2006), 393–405. MR2197599 (2007a:17011)
8. T. Shirai and Y. Takahashi, *Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes*, J. Funct. Anal. **205** (2003), 414–463. MR2018415 (2004m:60104)
9. D. Vere-Jones, *A generalization of permanents and determinants*, Linear Algebra Appl. **111** (1988), 119–124. MR0974048 (89j:15014)

Department of Mathematical Sciences, University of the Ryukyus, Nishihara, Okinawa 903-0213, Japan
E-mail address: kimoto@math.u-ryukyu.ac.jp