REPRESENTATION THEORY OF THE α-DETERMINANT AND ZONAL SPHERICAL FUNCTIONS

KAZUFUMI KIMOTO

Abstract. We investigate the structure of the cyclic module $\boldsymbol{V}_{n,l}(\alpha) = \mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^l$ by embedding it to the tensor product space $(\mathbb{C}^n)^{\otimes nl}$ and utilizing the Schur-Weyl duality. We show that the entries of the transition matrices $F^{\lambda}_{n,l}(\alpha)$ are given by a variation of the spherical Fourier transformation of a certain class function on $\mathfrak{S}_{nl}$ with respect to the subgroup $\mathfrak{S}_l^n$ (Theorem 1.4). This result also provides another proof of Theorem ??. Further, we calculate the polynomial $F^{(2l-s,s)}_{2,l}(\alpha)$ by using an explicit formula of the values of zonal spherical functions for the Gelfand pair $(\mathfrak{S}_{2l}, \mathfrak{S}_l \times \mathfrak{S}_l)$ due to Bannai and Ito (Theorem 2.1).

1. Irreducible decomposition of $\boldsymbol{V}_{n,l}(\alpha)$ and transition matrices

Let us fix $n, l \in \mathbb{N}$. Consider the standard tableau $\mathbb{T}$ with shape (l^n) such that the (i,j)-entry of $\mathbb{T}$ is $(i-1)l + j$. For instance, if $n = 3$ and $l = 2$, then

$$\mathbb{T} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}.$$

We denote by $K = R(\mathbb{T})$ and $H = C(\mathbb{T})$ the row group and column group of the standard tableau $\mathbb{T}$ respectively. Namely,

$$K = \left\{ g \in \mathfrak{S}_{nl} \,\middle|\, \lceil g(x)/l \rceil = \lceil x/l \rceil,\ x \in [nl] \right\}, \tag{1.1}$$

$$H = \left\{ g \in \mathfrak{S}_{nl} \,\middle|\, g(x) \equiv x \pmod{l},\ x \in [nl] \right\}. \tag{1.2}$$

We put

$$e = \frac{1}{|K|} \sum_{k \in K} k \in \mathbb{C}[\mathfrak{S}_{nl}]. \tag{1.3}$$

This is clearly an idempotent element in $\mathbb{C}[\mathfrak{S}_{nl}]$. Let φ be a class function on H. We put

$$\Phi = \sum_{h \in H} \varphi(h) h \in \mathbb{C}[\mathfrak{S}_{nl}].$$

Consider the tensor product space $V = (\mathbb{C}^n)^{\otimes nl}$. We introduce a $(\mathcal{U}(\mathfrak{gl}_n), \mathbb{C}[\mathfrak{S}_{nl}])$-module structure of V by

$$E_{ij} \cdot \boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_{nl}} = \sum_{s=1}^{nl} \delta_{i_s,j}\, \boldsymbol{e}_{i_1} \otimes \cdots \otimes \overset{s\text{-th}}{\boldsymbol{e}_i} \otimes \cdots \otimes \boldsymbol{e}_{i_{nl}},$$

$$\boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_{nl}} \cdot \sigma = \boldsymbol{e}_{i_{\sigma(1)}} \otimes \cdots \otimes \boldsymbol{e}_{i_{\sigma(nl)}} \qquad (\sigma \in \mathfrak{S}_{nl}),$$

where $\{\boldsymbol{e}_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{C}^n$. The main concern of this subsection is to describe the irreducible decomposition of the left $\mathcal{U}(\mathfrak{gl}_n)$-module $V \cdot e\Phi e$.

We first show that $\boldsymbol{V}_{n,l}(\alpha)$ is isomorphic to $V \cdot e\Phi e$ for a special choice of φ. Consider the group isomorphism $\theta : H \to \mathfrak{S}_n^l$ defined by

$$\theta(h) = (\theta(h)_1, \ldots, \theta(h)_l); \quad \theta(h)_i(x) = y \iff h((x-1)l + i) = (y-1)l + i.$$

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We also define an element $D(X;\varphi) \in \mathcal{A}(\mathrm{Mat}_n)$ by

$$D(X;\varphi) = \sum_{h\in H} \varphi(h) \prod_{q=1}^{n}\prod_{p=1}^{l} x_{\theta(h)_p(q),q} = \sum_{h\in H} \varphi(h) \prod_{q=1}^{n}\prod_{p=1}^{l} x_{q,\theta(h)_p^{-1}(q)}$$
$$= \sum_{\sigma_1,\dots,\sigma_l\in\mathfrak{S}_n} \varphi(\theta^{-1}(\sigma_1,\dots,\sigma_l)) \prod_{q=1}^{n}\prod_{p=1}^{l} x_{\sigma_p(q),q}.$$

We note that $D(X;\alpha^{\nu(\cdot)}) = \det^{(\alpha)}(X)^l$ since $\nu\theta^{-1}(\sigma_1,\dots,\sigma_l) = \nu\sigma_1+\dots+\nu\sigma_l$ for $(\sigma_1,\dots,\sigma_l)\in\mathfrak{S}_l^n$.

Take a class function δ_H on H defined by

$$\delta_H(h) = \begin{cases} 1 & h=1 \\ 0 & h\neq 1. \end{cases}$$

We see that $D(X;\delta_H) = (x_{11}x_{22}\dots x_{nn})^l$. We need the following lemma (The assertion (1) is just a rewrite of Lemma **??**, and (2) is immediate to verify).

Lemma 1.1. (1) *It holds that*

$$\mathcal{U}(\mathfrak{gl}_n)\cdot \boldsymbol{e}_1^{\otimes l}\otimes\dots\otimes\boldsymbol{e}_n^{\otimes l} = V\cdot e = \mathcal{S}^l(\mathbb{C}^n)^{\otimes n},$$
$$\mathcal{U}(\mathfrak{gl}_n)\cdot D(X;\delta_H) = \bigoplus_{\substack{i_{pq}\in\{1,2,\dots,n\}\\ (1\le p\le l,\ 1\le q\le n)}} \mathbb{C}\cdot\prod_{q=1}^{n}\prod_{p=1}^{l} x_{i_{pq}q} \cong \mathcal{S}^l(\mathbb{C}^n)^{\otimes n}.$$

(2) *The map*

$$\mathcal{T}:\mathcal{U}(\mathfrak{gl}_n)\cdot D(X;\delta_H) \ni \prod_{q=1}^{n}\prod_{p=1}^{l} x_{i_{pq}q}$$
$$\longmapsto (\boldsymbol{e}_{i_{11}}\otimes\dots\otimes\boldsymbol{e}_{i_{l1}})\otimes\dots\otimes(\boldsymbol{e}_{i_{1n}}\otimes\dots\otimes\boldsymbol{e}_{i_{ln}})\cdot e\in V\cdot e$$

is a bijective $\mathcal{U}(\mathfrak{gl}_n)$-intertwiner. □

We see that

$$\begin{aligned}
&\mathcal{T}\left(D(X;\varphi)\right)\\
&= \sum_{h\in H}\varphi(h)\mathcal{T}\left(\prod_{q=1}^{n}\prod_{p=1}^{l} x_{\theta(h)_p(q),q}\right)\\
&= \sum_{h\in H}\varphi(h)(\boldsymbol{e}_{\theta(h)_1(1)}\otimes\dots\otimes\boldsymbol{e}_{\theta(h)_l(1)})\otimes\dots\otimes(\boldsymbol{e}_{\theta(h)_1(n)}\otimes\dots\otimes\boldsymbol{e}_{\theta(h)_l(n)})\cdot e\\
&= \boldsymbol{e}_1^{\otimes l}\otimes\dots\otimes\boldsymbol{e}_n^{\otimes l}\cdot\sum_{h\in H}\varphi(h)h\cdot e = \boldsymbol{e}_1^{\otimes l}\otimes\dots\otimes\boldsymbol{e}_n^{\otimes l}\cdot e\Phi e
\end{aligned}$$

by (2) in Lemma 1.1. Using (1) in Lemma 1.1, we have the

Lemma 1.2. *It holds that*

$$\mathcal{U}(\mathfrak{gl}_n)\cdot D(X;\varphi)\cong V\cdot e\Phi e$$

as a left $\mathcal{U}(\mathfrak{gl}_n)$-module. In particular, $V\cdot e\Phi e\cong \boldsymbol{V}_{n,l}(\alpha)$ if $\varphi(h)=\alpha^{\nu(h)}$. □

By the Schur-Weyl duality, we have

$$V\cong\bigoplus_{\lambda\vdash nl}\mathcal{M}_n^{\lambda}\boxtimes\mathcal{S}^{\lambda}.$$

Here $\mathcal{S}^\lambda$ denotes the irreducible unitary right $\mathfrak{S}_{nl}$-module corresponding to λ. We see that

$$\dim\left(\mathcal{S}^\lambda\cdot e\right) = \left\langle \mathrm{ind}_K^G\,\mathbf{1}_K,\,\mathcal{S}^\lambda\right\rangle_{\mathfrak{S}_{nl}} = K_{\lambda(l^n)},$$

where $\mathbf{1}_K$ is the trivial representation of K and $\langle\pi,\,\rho\rangle_{\mathfrak{S}_{nl}}$ is the intertwining number of given representations π and ρ of $\mathfrak{S}_{nl}$. Since $K_{\lambda(l^n)}=0$ unless $\ell(\lambda)\le n$, it follows the

Theorem 1.3. *It holds that*

$$V\cdot e\Phi e\cong\bigoplus_{\substack{\lambda\vdash nl\\ \ell(\lambda)\le n}}\mathcal{M}_n^\lambda\boxtimes\left(\mathcal{S}^\lambda\cdot e\Phi e\right).$$

In particular, as a left $\mathcal{U}(\mathfrak{gl}_n)$-module, the multiplicity of $\mathcal{M}_n^\lambda$ in $V\cdot e\Phi e$ is given by

$$\dim\left(\mathcal{S}^\lambda\cdot e\Phi e\right)=\mathrm{rk}_{\mathrm{End}(\mathcal{S}^\lambda\cdot e)}(e\Phi e).$$

□

Let $\lambda\vdash nl$ be a partition such that $\ell(\lambda)\le n$ and put $d=K_{\lambda(l^n)}$. We fix an orthonormal basis $\{\boldsymbol{e}_1^\lambda,\dots,\boldsymbol{e}_{f^\lambda}^\lambda\}$ of $\mathcal{S}^\lambda$ such that the first d vectors $\boldsymbol{e}_1^\lambda,\dots,\boldsymbol{e}_d^\lambda$ form a subspace $(\mathcal{S}^\lambda)^K$ consisting of K-invariant vectors and left $f^\lambda-d$ vectors form the orthocomplement of $(\mathcal{S}^\lambda)^K$ with respect to the $\mathfrak{S}_{nl}$-invariant inner product. The matrix coefficient of $\mathcal{S}^\lambda$ relative to this basis is

$$\psi_{ij}^\lambda(g)=\left\langle\boldsymbol{e}_i^\lambda\cdot g,\,\boldsymbol{e}_j^\lambda\right\rangle_{\mathcal{S}^\lambda}\quad(g\in\mathfrak{S}_{nl},\ 1\le i,j\le f^\lambda).\tag{1.4}$$

We notice that this function is K-biinvariant. We see that the multiplicity of $\mathcal{M}_n^\lambda$ in $V\cdot e\Phi e$ is given by the rank of the matrix

$$\left(\sum_{h\in H}\varphi(h)\psi_{ij}^\lambda(h)\right)_{1\le i,j\le d}.$$

As a particular case, we obtain the

Theorem 1.4. *The multiplicity of the irreducible representation $\mathcal{M}_n^\lambda$ in the cyclic module $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)^l$ is equal to the rank of*

$$F_{n,l}^\lambda(\alpha)=\left(\sum_{h\in H}\alpha^{\nu(h)}\psi_{ij}^\lambda(h)\right)_{1\le i,j\le d},\tag{1.5}$$

where $\{\psi_{ij}^\lambda\}_{i,j}$ denotes a basis of the λ-component of the space $C(K\backslash\mathfrak{S}_{nl}/K)$ of K-biinvariant functions on $\mathfrak{S}_{nl}$ given by (1.4). □

Remark 1.5.
(1) We have $F_{n,l}^\lambda(0)=I$ by the definition of the basis $\{\psi_{ij}^\lambda\}_{i,j}$ in (1.4).
(2) Since $\alpha^{\nu(g^{-1})}=\alpha^{\nu(g)}$ and $\psi_{ij}^\lambda(g^{-1})=\overline{\psi_{ji}^\lambda(g)}$ for any $g\in\mathfrak{S}_{nl}$, the transition matrices satisfy $F_{n,l}^\lambda(\alpha)^*=F_{n,l}^\lambda(\overline{\alpha})$.
(3) In Examples 1.6 and 1.8 below, the transition matrices are given by *diagonal matrices*. We expect that any transition matrix $F_{n,l}^\lambda(\alpha)$ is *diagonalizable* in $\mathrm{Mat}_{K_{\lambda(l^n)}}(\mathbb{C}[\alpha])$.

Example 1.6. If $l=1$, then $H=G=\mathfrak{S}_n$ and $K=\{1\}$. Therefore, for any $\lambda\vdash n$, we have

$$F_{n,1}^\lambda(\varphi)=\frac{n!}{f^\lambda}\left\langle\varphi,\,\chi^\lambda\right\rangle_{\mathfrak{S}_n}I\tag{1.6}$$

by the orthogonality of the matrix coefficients. Here χ^λ denotes the irreducible character of $\mathfrak{S}_n$ corresponding to λ. In particular, if $\varphi=\alpha^{\nu(\cdot)}$, then

$$F_{n,1}^\lambda(\alpha)=f_\lambda(\alpha)I\tag{1.7}$$

since the Fourier expansion of $\alpha^{\nu(\cdot)}$ (as a class function on $\mathfrak{S}_n$) is

$$\alpha^{\nu(\cdot)}=\sum_{\lambda\vdash n}\frac{f^\lambda}{n!}f_\lambda(\alpha)\chi^\lambda,\tag{1.8}$$

which is obtained by specializing the Frobenius character formula for $\mathfrak{S}_n$ (see, e.g. [10]).

Example 1.7. Let us calculate $F_{n,l}^{(nl)}(\alpha)$ by using Theorem 1.4. Since $\mathcal{S}^{(nl)}$ is the trivial representation, it follows that $(\mathcal{S}^{(nl)})^K = \mathcal{S}^{(nl)}$ and

$$F_{n,l}^{(nl)}(\alpha) = \sum_{h\in H} \alpha^{\nu(h)} \langle \boldsymbol{e}\cdot h,\, \boldsymbol{e}\rangle = \sum_{\sigma_1,\dots,\sigma_l\in\mathfrak{S}_n} \alpha^{\nu(\sigma_1)}\dots\alpha^{\nu(\sigma_l)} = \left((1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)\right)^l,$$

where $\boldsymbol{e}$ denotes a unit vector in $\mathcal{S}^{(nl)}$.

Example 1.8. Let us calculate $F_{n,l}^{(nl-1,1)}(\alpha)$ by using Theorem 1.4. As is well known, the irreducible (right) $\mathfrak{S}_{nl}$-module $\mathcal{S}^{(nl-1,1)}$ can be realized in $\mathbb{C}^{nl}$ as follows:

$$\mathcal{S}^{(nl-1,1)} = \left\{ (x_j)_{j=1}^{nl} \in \mathbb{C}^{nl} \;\middle|\; \sum_{j=1}^{nl} x_j = 0 \right\}.$$

This is a unitary representation with respect to the ordinary hermitian inner product $\langle\cdot,\,\cdot\rangle$ on $\mathbb{C}^{nl}$. It is immediate to see that

$$\left(\mathcal{S}^{(nl-1,1)}\right)^K = \left\{ (x_j)_{j=1}^{nl} \in \mathcal{S}^{(nl-1,1)} \;\middle|\; x_{pl+1} = x_{pl+2} = \dots = x_{(p+1)l} \quad (0\le p<n) \right\}.$$

Take an orthonormal basis $\boldsymbol{e}_1,\dots,\boldsymbol{e}_{n-1}$ of $\left(\mathcal{S}^{(nl-1,1)}\right)^K$ by

$$\boldsymbol{e}_j = \frac{1}{\sqrt{nl}}(\overbrace{\omega^j,\dots,\omega^j}^{l}, \overbrace{\omega^{2j},\dots,\omega^{2j}}^{l}, \dots, \overbrace{\omega^{nj},\dots,\omega^{nj}}^{l}) \qquad (1\le j\le n-1),$$

where ω is a primitive n-th root of unity. Then, the (i,j)-entry of the transition matrix $F_{n,l}^{(nl-1,1)}(\alpha)$ is

$$\begin{aligned}\sum_{h\in H} \alpha^{\nu(h)} \langle \boldsymbol{e}_i\cdot h,\, \boldsymbol{e}_j\rangle &= \frac{1}{nl} \sum_{\sigma_1,\dots,\sigma_l\in\mathfrak{S}_n} \sum_{p=1}^{n}\sum_{q=1}^{l} \alpha^{\nu(\sigma_1)}\dots\alpha^{\nu(\sigma_l)}\omega^{\sigma_q(p)i-pj} \\ &= \left(\sum_{\tau\in\mathfrak{S}_n}\alpha^{\nu(\tau)}\right)^{l-1} \left(\frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n}\sum_{p=1}^{n}\alpha^{\nu(\sigma)}\omega^{\sigma(p)i-pj}\right).\end{aligned}$$

The first factor is $\left((1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)\right)^{l-1}$. We show that

$$\frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n}\sum_{p=1}^{n}\alpha^{\nu(\sigma)}\omega^{\sigma(p)i-pj} = (1-\alpha)(1+\alpha)(1+2\alpha)\dots(1+(n-2)\alpha)\delta_{ij} \qquad (i,j=1,2,\dots,n-1).$$

For this purpose, by comparing the coefficients of α^{n-m} in both sides, it is enough to prove

$$\frac{1}{n}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \nu(\sigma)=n-m}}\sum_{p=1}^{n}\omega^{\sigma(p)i-pj} = \left\{ \begin{bmatrix} n-1\\ m-1\end{bmatrix} - \begin{bmatrix} n-1\\ m\end{bmatrix} \right\}\delta_{ij} \qquad (i,j,m=1,2,\dots,n-1),$$

where $\begin{bmatrix} n\\ m\end{bmatrix}$ denotes the Stirling number of the first kind (see, e.g. [5] for the definition). Since

$$|\{\sigma\in\mathfrak{S}_n\,;\,\nu(\sigma)=n-m,\ \sigma(p)=x\}| = \begin{cases} \begin{bmatrix} n-1\\ m-1\end{bmatrix} & x=p,\\ \begin{bmatrix} n-1\\ m\end{bmatrix} & x\ne p\end{cases}$$

for each $p, x \in [n]$, it follows that

$$\frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n}\sum_{p=1}^{n}\alpha^{\nu(\sigma)}\omega^{\sigma(p)i-pj} = \frac{1}{n}\sum_{p=1}^{n}\omega^{-pj}\left\{\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}\omega^{pi} + \sum_{x\neq p}\begin{bmatrix} n-1 \\ m \end{bmatrix}\omega^{xi}\right\}$$
$$= \left\{\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix}\right\}\frac{1}{n}\sum_{p=1}^{n}\omega^{p(i-j)} = \left\{\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix}\right\}\delta_{ij},$$

which is the required conclusion. Here we notice that $\sum_{x\neq p}\omega^{xi} = -\omega^{pi}$ since $1 \le i < n$. Consequently, we obtain

$$F_{n,l}^{(nl-1,1)}(\alpha) = \Big((1-\alpha)\left((1+\alpha)(1+2\alpha)\dots(1+(n-2)\alpha)\right)^{l}(1+(n-1)\alpha)^{l-1}\delta_{ij}\Big)_{1\le i,j\le n-1},$$

so that the multiplicity of $\mathcal{M}_n^{(nl-1,1)}$ in $\boldsymbol{V}_{n,l}(\alpha)$ is zero if $\alpha = -1/k$ $(k = 1, 2, \dots, n-1)$ and $n-1$ otherwise.

The trace of the transition matrix $F_{n,l}^{\lambda}(\alpha)$ is

$$f_{n,l}^{\lambda}(\alpha) = \operatorname{tr} F_{n,l}^{\lambda}(\alpha) = \sum_{h\in H}\alpha^{\nu(h)}\omega^{\lambda}(h), \tag{1.9}$$

where ω^{λ} is the *zonal spherical function* for λ with respect to K defined by

$$\omega^{\lambda}(g) = \frac{1}{|K|}\sum_{k\in K}\chi^{\lambda}(kg) \quad (g \in \mathfrak{S}_{nl}).$$

This polynomial is regarded as a generalization of the modified content polynomial since $f_{n,1}^{\lambda}(\alpha) = f^{\lambda}f_{\lambda}(\alpha)$ as we see above. It is much easier to handle these polynomials than the transition matrices. If we could prove that a transition matrix $F_{n,l}^{\lambda}(\alpha)$ is a scalar matrix, then we would have $F_{n,l}^{\lambda}(\alpha) = d^{-1}f_{n,l}^{\lambda}(\alpha)I$ $(d = \dim(\mathcal{S}^{\lambda})^K)$ and hence we see that the multiplicity of $\mathcal{M}_n^{\lambda}$ in $\boldsymbol{V}_{n,l}(\alpha)$ is completely controlled by the single polynomial $f_{n,l}^{\lambda}(\alpha)$. In this sense, it is desirable to obtain a characterization of the irreducible representations whose corresponding transition matrices are scalar as well as to get an explicit expression for the polynomials $f_{n,l}^{\lambda}(\alpha)$. Here we give a sufficient condition for $\lambda \vdash nl$ such that $F_{n,l}^{\lambda}(\alpha)$ is a scalar matrix.

Proposition 1.9. (1) *Denote by $N_H(K)$ the normalizer of K in H. The transition matrix $F_{n,l}^{\lambda}(\alpha)$ is scalar if $(\mathcal{S}^{\lambda})^K$ is irreducible as a $N_H(K)$-module.*
(2) *If λ is of hook-type (i.e. $\lambda = (nl-r, 1^r)$ for some $r < n$), then $F_{n,l}^{\lambda}(\alpha)$ is scalar.*

Proof. Notice that $N_H(K) \cong \mathfrak{S}_n$. Consider a linear map $T \in \operatorname{End}((\mathcal{S}^{\lambda})^K)$ given by

$$T(\boldsymbol{x}) = \sum_{j=1}^{d}\left(\sum_{h\in H}\alpha^{\nu(h)}\left\langle \boldsymbol{x}\cdot h, \boldsymbol{e}_j^{\lambda}\right\rangle_{\mathcal{S}^{\lambda}}\right)\boldsymbol{e}_j^{\lambda} \qquad (\boldsymbol{x} \in (\mathcal{S}^{\lambda})^K),$$

where $d = \dim(\mathcal{S}^{\lambda})^K$. It is direct to check that T gives an intertwiner of $(\mathcal{S}^{\lambda})^K$ as a $N_H(K)$-module. Hence, by Schur's lemma, T is a scalar map (and $F_{n,l}^{\lambda}(\alpha)$ is a scalar matrix) if $(\mathcal{S}^{\lambda})^K$ is an irreducible $N_H(K)$-module. When $\lambda = (nl-r, 1^r)$ for some $r < n$, it is proved in [2, Proposition 5.3] that $(\mathcal{S}^{(nl-r,1^r)})^K \cong \mathcal{S}^{(n-r,1^r)}$ as $N_H(K)$-modules. Thus we have the proposition. □

Example 1.10. Let us calculate $f_{n,l}^{(nl-1,1)}(\alpha)$. Notice that $\chi^{(nl-1,1)}(g) = \operatorname{fix}_{nl}(g) - 1$ where fix_{nl} denotes the number of fixed points in the natural action $\mathfrak{S}_{nl} \curvearrowright [nl]$. Hence we see that

$$f_{n,l}^{(nl-1,1)}(\alpha) = \sum_{h\in H}\alpha^{\nu(h)}\frac{1}{|K|}\sum_{k\in K}(\operatorname{fix}_{nl}(kh) - 1)$$
$$= \sum_{h\in H}\alpha^{\nu(h)}\frac{1}{|K|}\sum_{k\in K}\sum_{x\in[nl]}\delta_{khx,x} - \sum_{h\in H}\alpha^{\nu(h)}.$$

It is easily seen that $khx \neq x$ for any $k \in K$ if $hx \neq x$ ($x \in [nl]$). Thus it follows that

$$\frac{1}{|K|}\sum_{k\in K}\sum_{x\in[nl]}\delta_{khx,x} = \sum_{x\in[nl]}\delta_{hx,x}\frac{1}{|K|}\sum_{k\in K}\delta_{kx,x} = \frac{1}{l}\,\mathrm{fix}_{nl}(h) \qquad (h\in H).$$

Therefore we have

$$\begin{aligned} f_{n,l}^{(nl-1,1)}(\alpha) &= \frac{1}{l}\sum_{h\in H}\alpha^{\nu(h)}\,\mathrm{fix}_{nl}(h) - \sum_{h\in H}\alpha^{\nu(h)} = f_{n,1}^{(n)}(\alpha)^{l-1}f_{n,1}^{(n-1,1)}(\alpha) \\ &= (n-1)(1-\alpha)(1-(n-1)\alpha)^{l-1}\prod_{i=1}^{n-2}(1+i\alpha)^l. \end{aligned}$$

Since the transition matrix $F_{n,l}^{(nl-1,1)}$ is a scalar one and its size is $\dim\mathcal{S}^{(n-1,1)} = n-1$, we get $F_{n,l}^{(nl-1,1)}(\alpha) = (1-\alpha)(1-(n-1)\alpha)^{l-1}\prod_{i=1}^{n-2}(1+i\alpha)^l I_{n-1}$ again.

We will investigate these polynomials $f_{n,l}^{\lambda}(\alpha)$ and their generalizations in [**?**].

2. Irreducible decomposition of $\boldsymbol{V}_{2,l}(\alpha)$ and Jacobi polynomials

In this subsection, as a particular example, we consider the case where $n = 2$ and calculate the transition matrix $F_{2,l}^{\lambda}(\alpha)$ explicitly. Since the pair $(\mathfrak{S}_{2l}, K)$ is a *Gelfand pair* (see, e.g. [10]), it follows that

$$K_{\lambda(l^2)} = \left\langle \mathrm{ind}_K^{\mathfrak{S}_{2l}}\mathbf{1}_K, \mathcal{S}^{\lambda}\right\rangle_{\mathfrak{S}_{2l}} = 1$$

for each $\lambda \vdash 2n$ with $\ell(\lambda) \le 2$. Thus, in this case, the transition matrix is just a polynomial and is given by

$$F_{2,l}^{\lambda}(\alpha) = \mathrm{tr}\,F_{2,l}^{\lambda}(\alpha) = \sum_{h\in H}\alpha^{\nu(h)}\omega^{\lambda}(h) = \sum_{s=0}^{l}\binom{l}{s}\omega^{\lambda}(g_s)\alpha^s. \tag{2.1}$$

Here we put $g_s = (1, l+1)(2, l+2)\dots(s, l+s) \in \mathfrak{S}_{2n}$. Now we write $\lambda = (2l-p, p)$ for some p ($0 \le p \le l$). The value $\omega^{(2l-p,p)}(g_s)$ of the zonal spherical function is calculated by Bannai and Ito [3, p.218] as

$$\omega^{(2l-p,p)}(g_s) = Q_p(s; -l-1, -l-1, l) = \sum_{j=0}^{p}(-1)^j\binom{p}{j}\binom{2l-p+1}{j}\binom{l}{j}^{-2}\binom{s}{j},$$

where

$$\begin{aligned} Q_n(x;\alpha,\beta,N) &= {}_3\tilde{F}_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x \\ \alpha+1, -N\end{matrix}; 1\right) \\ &= \sum_{j=0}^{N}(-1)^j\binom{n}{j}\binom{-n-\alpha-\beta-1}{j}\binom{-\alpha-1}{j}^{-1}\binom{N}{j}^{-1}\binom{x}{j} \end{aligned}$$

is the Hahn polynomial (see also [10, p.399]), and ${}_{n+1}\tilde{F}_n\left(\begin{smallmatrix}a_1,\dots,a_p \\ b_1,\dots,b_{q-1},-N\end{smallmatrix}; x\right)$ is the hypergeometric polynomial

$${}_p\tilde{F}_q\left(\begin{matrix}a_1,\dots,a_p \\ b_1,\dots,b_{q-1},-N\end{matrix}; x\right) = \sum_{j=0}^{N}\frac{(a_1)_j\dots(a_p)_j}{(b_1)_j\dots(b_{q-1})_j(-N)_j}\frac{x^j}{j!}$$

for $p, q, N \in \mathbb{N}$ in general (see [1]). We now re-prove Theorem **??** as follows:

Theorem 2.1. *Let l be a positive integer. It holds that*

$$F_{2,l}^{(2l-p,p)}(\alpha) = \sum_{s=0}^{l}\binom{l}{s}Q_p(s; l-1, l-1, l)\alpha^s = (1+\alpha)^{l-p}G_p^l(\alpha)$$

for $p = 0, 1, \dots, l$.

Proof. Let us put $x = -1/\alpha$. Then we have

$$\begin{aligned}&\sum_{s=0}^{l}\binom{l}{s}Q_p(s;l-1,l-1,l)\alpha^s\\&=\sum_{j=0}^{p}(-1)^j\binom{p}{j}\binom{2l-p+1}{j}\binom{l}{j}^{-1}\alpha^j(1+\alpha)^{l-j}\\&=x^{-l}(x-1)^{l-p}\sum_{j=0}^{p}\binom{p}{j}\binom{2l-p+1}{j}\binom{l}{j}^{-1}(x-1)^{p-j}\end{aligned}$$

and

$$(1+\alpha)^{l-p}G_p^l(\alpha)=x^{-l}(x-1)^{l-p}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\binom{l-p+j}{j}\binom{l}{j}^{-1}(-x)^{p-j}.$$

Here we use the elementary identity

$$\sum_{s=0}^{l}\binom{l}{s}\binom{s}{j}\alpha^s=\binom{l}{j}\alpha^j(1+\alpha)^{l-j}.$$

Hence, to prove the theorem, it is enough to verify

$$\sum_{i=0}^{p}\binom{p}{i}\binom{l-p+i}{i}\binom{l}{i}^{-1}x^{p-i}=\sum_{j=0}^{p}\binom{p}{j}\binom{2l-p+1}{j}\binom{l}{j}^{-1}(x-1)^{p-j}. \tag{2.2}$$

Comparing the coefficients of Taylor expansion of these polynomials at $x = 1$, we notice that the proof is reduced to the equality

$$\sum_{i=0}^{r}\binom{l-i}{l-r}\binom{l-p+i}{l-p}=\binom{2l-p+1}{r} \tag{2.3}$$

for $0 \le r \le p$, which is well known (see, e.g. (5.26) in [5]). Hence we have the conclusion. □

Thus we obtain the irreducible decomposition

$$\boldsymbol{V}_{2,l}(-1)\cong\mathcal{M}_2^{(l,l)},\qquad \boldsymbol{V}_{2,l}(\alpha)\cong\bigoplus_{\substack{0\le p\le l\\ G_p^l(\alpha)\neq 0}}\mathcal{M}_2^{(2l-p,p)}\quad(\alpha\neq -1) \tag{2.4}$$

of $\boldsymbol{V}_{2,l}(\alpha)$ again.

Remark 2.2. (1) The calculation above uses the advantage for the fact that the pair $(\mathfrak{S}_{nl}, \mathfrak{S}_l^n)$ is the Gelfand pair *only when* $n = 2$.
(2) We have used the result in [3, p.218] for the theorem. It is worth mentioning that one may prove conversely the result in [3, p.218] from Theorem **??**.

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Department of Mathematical Sciences, University of the Ryukyus, Nishihara, Okinawa 903-0213, Japan
E-mail address: kimoto@math.u-ryukyu.ac.jp