1. Irreducible decomposition of $V_{n,l}(\alpha)$ and transition matrices

Let us fix $n, l \in \mathbb{N}$. Consider the standard tableau $T$ with shape $(l^n)$ such that the $(i,j)$-entry of $T$ is $(i-1)l + j$. For instance, if $n = 3$ and $l = 2$, then

$$T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$  

We denote by $K = R(T)$ and $H = C(T)$ the row group and column group of the standard tableau $T$ respectively. Namely,

\begin{align*}
K &= \{ g \in \mathfrak{S}_{nl} \mid [g(x)/l] = [x/l], \ x \in [nl] \}, \\
H &= \{ g \in \mathfrak{S}_{nl} \mid g(x) \equiv x \pmod{l}, \ x \in [nl] \}.
\end{align*}

We put

$$e = \frac{1}{|K|} \sum_{k \in K} k \in \mathbb{C}[\mathfrak{S}_{nl}].$$

This is clearly an idempotent element in $\mathbb{C}[\mathfrak{S}_{nl}]$. Let $\varphi$ be a class function on $H$. We put

$$\Phi = \sum_{h \in H} \varphi(h) h \in \mathbb{C}[\mathfrak{S}_{nl}].$$

Consider the tensor product space $V = (\mathbb{C}^n)^{\otimes nl}$. We introduce a $\mathcal{U}(\mathfrak{g}_n) \cdot \mathbb{C}[\mathfrak{S}_{nl}]$-module structure of $V$ by

$$E_{ij} \cdot e_{i_1} \otimes \cdots \otimes e_{i_{nl}} = \sum_{s=1}^{nl} \delta_{i,i_j} e_{i_1} \otimes \cdots \otimes e_i^{s-1} \otimes \cdots \otimes e_{i_{nl}},$$

$$e_{i_1} \otimes \cdots \otimes e_{i_{nl}} \cdot \sigma = e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(n)}} \quad (\sigma \in \mathfrak{S}_{nl}),$$

where $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{C}^n$. The main concern of this subsection is to describe the irreducible decomposition of the left $\mathcal{U}(\mathfrak{g}_{nl})$-module $V \cdot e\Phi e$.

We first show that $V_{n,l}(\alpha)$ is isomorphic to $V \cdot e\Phi e$ for a special choice of $\varphi$. Consider the group isomorphism $\theta : H \to \mathfrak{S}_{nl}$ defined by

$$\theta(h) = (\theta(h)_1, \ldots, \theta(h)_l); \ \theta(h)_i(x) = y \iff h((x-1)l + i) = (y-1)l + i.$$  

\begin{abstract}
We investigate the structure of the cyclic module $V_{n,l}(\alpha) = \mathcal{U}(\mathfrak{g}_n) \cdot \det(\alpha)(X)^l$ by embedding it to the tensor product space $(\mathbb{C}^n)^{\otimes nl}$ and utilizing the Schur-Weyl duality. We show that the entries of the transition matrices $F^\lambda_{n,l}(\alpha)$ are given by a variation of the spherical Fourier transformation of a certain class function on $\mathfrak{S}_{nl}$ with respect to the subgroup $\mathfrak{S}_n^l$ (Theorem 1.4). This result also provides another proof of Theorem 77. Further, we calculate the polynomial $F^2_{2,1}(\alpha)$ by using an explicit formula of the values of zonal spherical functions for the Gelfand pair $(\mathfrak{S}_{2l}, \mathfrak{S}_l \times \mathfrak{S}_l)$ due to Bannai and Ito (Theorem 2.1).

\end{abstract}

\textbf{Keywords and phrases.} Alpha-determinant, cyclic modules, Jacobi polynomials, singly confluent Heun ODE, permanent, Kostka numbers, irreducible decomposition, spherical Fourier transformation, zonal spherical functions, Gelfand pair.

\textbf{2000 Mathematics Subject Classification.} Primary 22E47, 33C45; Secondary 43A90, 13A50.

Date: October 30, 2008.
We also define an element $D(X; \varphi) \in \mathcal{A}(\text{Mat}_n)$ by

$$D(X; \varphi) = \sum_{h \in H} \varphi(h) \prod_{q=1}^n \prod_{p=1}^l x_{\theta(h)p(q),q} = \sum_{h \in H} \varphi(h) \prod_{q=1}^n \prod_{p=1}^l x_{q,\theta(h)_{p}^{-1}(q)}$$

$$= \sum_{\sigma_1, \ldots, \sigma_l \in \mathfrak{S}_n} \varphi(\theta^{-1}(\sigma_1, \ldots, \sigma_l)) \prod_{q=1}^n \prod_{p=1}^l x_{\sigma_p(q),q}.$$  

We note that $D(X; \alpha^{(\cdot)}) = \text{det}^{(\alpha)}(X)^{l}$ since $\nu \theta^{-1}(\sigma_1, \ldots, \sigma_l) = \nu \sigma_1 + \cdots + \nu \sigma_l$ for $(\sigma_1, \ldots, \sigma_l) \in \mathfrak{S}_n$.

Take a class function $\delta_H$ on $H$ defined by

$$\delta_H(h) = \begin{cases} 1 & h = 1 \\ 0 & h \neq 1. \end{cases}$$

We see that $D(X; \delta_H) = (x_{11}x_{22} \cdots x_{nn})^l$. We need the following lemma (The assertion (1) is just a rewrite of Lemma ??, and (2) is immediate to verify).

**Lemma 1.1.**

1. It holds that

$$\mathcal{U}(\mathfrak{gl}_n) \cdot e_{i_1}^{\otimes l} \otimes \cdots \otimes e_{i_n}^{\otimes l} = V \cdot e = \mathcal{S}^l(\mathbb{C}^n)^{\otimes n},$$

$$\mathcal{U}(\mathfrak{gl}_n) \cdot D(X; \delta_H) = \bigoplus_{l_{pq} \in \{1, \ldots, n\} \atop (1 \leq p \leq l, 1 \leq q \leq n)} \mathbb{C} \cdot \prod_{q=1}^n \prod_{p=1}^l x_{i_{pq}q} \cong \mathcal{S}^l(\mathbb{C}^n)^{\otimes n}.$$

2. The map

$$\mathcal{T} : \mathcal{U}(\mathfrak{gl}_n) \cdot D(X; \delta_H) \ni \prod_{q=1}^n \prod_{p=1}^l x_{i_{pq}q} \longrightarrow (e_{i_1} \otimes \cdots \otimes e_{i_1}) \otimes \cdots \otimes (e_{i_n} \otimes \cdots \otimes e_{i_n}) \cdot e \in V \cdot e$$

is a bijective $\mathcal{U}(\mathfrak{gl}_n)$-intertwiner.  \hfill \Box

We see that

$$\mathcal{T} (D(X; \varphi))$$

$$= \sum_{h \in H} \varphi(h) \mathcal{T} \left( \prod_{q=1}^n \prod_{p=1}^l x_{\theta(h)p(q),q} \right)$$

$$= \sum_{h \in H} \varphi(h) (e_{\theta(h)_{i_1}} \otimes \cdots \otimes e_{\theta(h)_{i_1}}) \otimes \cdots \otimes (e_{\theta(h)_{i_n}} \otimes \cdots \otimes e_{\theta(h)_{i_n}}) \cdot e$$

$$= e_{i_1}^{\otimes l} \otimes \cdots \otimes e_{i_n}^{\otimes l} \cdot \sum_{h \in H} \varphi(h) h \cdot e = e_{i_1}^{\otimes l} \otimes \cdots \otimes e_{i_n}^{\otimes l} \cdot e^\Phi,$$

by (2) in Lemma 1.1. Using (1) in Lemma 1.1, we have the

**Lemma 1.2.** It holds that

$$\mathcal{U}(\mathfrak{gl}_n) \cdot D(X; \varphi) \cong V \cdot e^\Phi$$

as a left $\mathcal{U}(\mathfrak{gl}_n)$-module. In particular, $V \cdot e^\Phi \cong V_{n,l}(\alpha)$ if $\varphi(h) = \alpha^{(h)}$.  \hfill \Box

By the Schur-Weyl duality, we have

$$V \cong \bigoplus_{\lambda \vdash n l} M_\lambda^n \boxtimes \mathcal{S}^\lambda.$$

Here $\mathcal{S}^\lambda$ denotes the irreducible unitary right $\mathfrak{S}_n$-module corresponding to $\lambda$. We see that

$$\dim (\mathcal{S}^\lambda \cdot e) = \binom{\text{ind}_K^G 1_{K^*}}{\mathcal{S}^\lambda} \equiv K_{\lambda(\tau)}.$$

where \(1_K\) is the trivial representation of \(K\) and \(\langle \pi, \rho \rangle_{\mathfrak{S}_n}\) is the intertwining number of given representations \(\pi\) and \(\rho\) of \(\mathfrak{S}_n\). Since \(K_{\lambda|\mathfrak{r}} = 0\) unless \(\ell(\lambda) \leq n\), it follows the

**Theorem 1.3.** It holds that

\[
V \cdot e\Phi \cong \bigoplus_{\lambda|\mathfrak{r}, \ell(\lambda) \leq n} \mathcal{M}_n^\lambda \otimes (S^\lambda \cdot e\Phi).
\]

In particular, as a left \(U(\mathfrak{g}_n)\)-module, the multiplicity of \(\mathcal{M}_n^\lambda\) in \(V \cdot e\Phi\) is given by

\[
dim (S^\lambda \cdot e\Phi) = \operatorname{rk}_{\mathbb{C}}(S^\lambda \cdot e\Phi).
\]

Let \(\lambda \vdash n\) be a partition such that \(\ell(\lambda) \leq n\) and put \(d = K_{\lambda|\mathfrak{r}}\). We fix an orthonormal basis \(\{e_1^\lambda, \ldots, e_{d_\lambda}^\lambda\}\) of \(S^\lambda\) such that the first \(d\) vectors \(e_1^\lambda, \ldots, e_{d_\lambda}^\lambda\) form a subspace \((S^\lambda)^K\) consisting of \(K\)-invariant vectors and left \(f^\lambda - d\) vectors form the orthocomplement of \((S^\lambda)^K\) with respect to the \(\mathfrak{S}_n\)-invariant inner product. The matrix coefficient of \(S^\lambda\) relative to this basis is

\[
(1.4) \quad \psi_{ij}^\lambda(g) = \langle e_i^\lambda \cdot g, e_j^\lambda \rangle_{S^\lambda}, \quad (g \in \mathfrak{S}_n, \ 1 \leq i, j \leq f^\lambda).
\]

We notice that this function is \(K\)-biinvariant. We see that the multiplicity of \(\mathcal{M}_n^\lambda\) in \(V \cdot e\Phi\) is given by the rank of the matrix

\[
\left( \sum_{h \in H} \varphi(h) \psi_{ij}^\lambda(h) \right)_{1 \leq i, j \leq d}.
\]

As a particular case, we obtain the

**Theorem 1.4.** The multiplicity of the irreducible representation \(\mathcal{M}_n^\lambda\) in the cyclic module \(U(\mathfrak{g}_n) \cdot \det^{(\alpha)}(X)^l\) is equal to the rank of

\[
(1.5) \quad F_{n,l}^\lambda(\alpha) = \left( \sum_{h \in H} \alpha^{\nu(h)} \psi_{ij}^\lambda(h) \right)_{1 \leq i, j \leq d},
\]

where \(\{\psi_{ij}^\lambda\}_{i,j}\) denotes a basis of the \(\lambda\)-component of the space \(C(K \backslash \mathfrak{S}_n / K)\) of \(K\)-biinvariant functions on \(\mathfrak{S}_n\) given by (1.4).

**Remark 1.5.**

1. We have \(F_{n,l}^\lambda(0) = I\) by the definition of the basis \(\{\psi_{ij}^\lambda\}_{i,j}\) in (1.4).

2. Since \(\alpha^{\nu(g^{-1})} = \alpha^{\nu(g)}\) and \(\psi_{ij}^\lambda(g^{-1}) = \overline{\psi_{ij}^\lambda(g)}\) for any \(g \in \mathfrak{S}_n\), the transition matrices satisfy \(F_{n,l}^\lambda(\alpha)^* = F_{n,l}^\lambda(\alpha)\).

3. In Examples 1.6 and 1.8 below, the transition matrices are given by diagonal matrices. We expect that any transition matrix \(F_{n,l}^\lambda(\alpha)\) is diagonalizable in \(\operatorname{Mat}_{K_{\lambda|\mathfrak{r}}}(C[\alpha])\).

**Example 1.6.** If \(l = 1\), then \(H = G = \mathfrak{S}_n\) and \(K = \{1\}\). Therefore, for any \(\lambda \vdash n\), we have

\[
(1.6) \quad F_{n,1}^\lambda(\varphi) = \frac{n!}{\chi^\lambda} \langle \varphi, \chi^\lambda \rangle_{\mathfrak{S}_n} I
\]

by the orthogonality of the matrix coefficients. Here \(\chi^\lambda\) denotes the irreducible character of \(\mathfrak{S}_n\) corresponding to \(\lambda\). In particular, if \(\varphi = \alpha^{\nu(\cdot)}\), then

\[
(1.7) \quad F_{n,1}^\lambda(\alpha) = f_{\lambda}(\alpha) I
\]

since the Fourier expansion of \(\alpha^{\nu(\cdot)}\) (as a class function on \(\mathfrak{S}_n\)) is

\[
(1.8) \quad \alpha^{\nu(\cdot)} = \sum_{\lambda \vdash n} \frac{f_{\lambda}}{n!} f_{\lambda}(\alpha) \chi^\lambda,
\]

which is obtained by specializing the Frobenius character formula for \(\mathfrak{S}_n\) (see, e.g. [10]).
Example 1.7. Let us calculate \( F_{n,l}^{(nl)}(\alpha) \) by using Theorem 1.4. Since \( S^{(nl)} \) is the trivial representation, it follows that \( (S^{(nl)})^K = S^{(nl)} \) and
\[
F_{n,l}^{(nl)}(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \langle e \cdot h, e \rangle = \sum_{\sigma_1, \ldots, \sigma_l \in \mathfrak{S}_n} \alpha^{\nu(\sigma_1)} \cdots \alpha^{\nu(\sigma_l)} = ((1 + \alpha)(1 + 2\alpha) \cdots (1 + (n - 1)\alpha))^l,
\]
where \( e \) denotes a unit vector in \( S^{(nl)} \).

Example 1.8. Let us calculate \( F_{n,l}^{(nl-1,1)}(\alpha) \) by using Theorem 1.4. As is well known, the irreducible (right) \( \mathfrak{S}_n \)-module \( S^{(nl-1,1)} \) can be realized in \( \mathbb{C}^{nl} \) as follows:
\[
S^{(nl-1,1)} = \left\{ (x_j)_{j=1}^{nl} \in \mathbb{C}^{nl} \mid \sum_{j=1}^{nl} x_j = 0 \right\}.
\]
This is a unitary representation with respect to the ordinary hermitian inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}^{nl} \). It is immediate to see that
\[
(S^{(nl-1,1)})^K = \left\{ (x_j)_{j=1}^{nl} \in S^{(nl-1,1)} \mid x_{pl+1} = x_{pl+2} = \cdots = x_{(p+1)l} \ (0 \leq p < n) \right\}.
\]
Take an orthonormal \( e_1, \ldots, e_{n-1} \) of \( (S^{(nl-1,1)})^K \) by
\[
e_j = \frac{1}{\sqrt{nl}} (\omega^1, \omega^2, \ldots, \omega^l, \omega^1, \omega^2, \ldots, \omega^l, \omega^1, \omega^2, \ldots, \omega^l) \ (1 \leq j \leq n - 1),
\]
where \( \omega \) is a primitive \( n \)-th root of unity. Then, the \((i, j)\)-entry of the transition matrix \( E_{n,l}^{(nl-1,1)}(\alpha) \) is
\[
\sum_{h \in H} \alpha^{\nu(h)} \langle e_i \cdot h, e_j \rangle = \frac{1}{nl} \sum_{\sigma_1, \ldots, \sigma_l \in \mathfrak{S}_n} \sum_{p=1}^{l} \alpha^{\nu(\sigma_1)} \cdots \alpha^{\nu(\sigma_l)} \sigma_i^p \sigma_j^{(p - p)j} = \left( \sum_{\tau \in \mathfrak{S}_n} \alpha^{\nu(\tau)} \right)^{l-1} \left( \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n \ p=1}^{n} \alpha^{\nu(\sigma)} \sigma_i^{(p - p)j} \right).
\]
The first factor is \((1 + \alpha)(1 + 2\alpha) \cdots (1 + (n - 1)\alpha))^{l-1}. We show that
\[
\frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n}^{n} \alpha^{\nu(\sigma)} \sigma_i^{(p - p)j} = (1 - \alpha)(1 + \alpha)(1 + 2\alpha) \cdots (1 + (n - 2)\alpha) \delta_{ij} \ (i, j = 1, 2, \ldots, n - 1).
\]
For this purpose, by comparing the coefficients of \( \alpha^{n-m} \) in both sides, it is enough to prove
\[
\frac{1}{n} \sum_{\nu(\sigma) = n-m}^{n} \sigma_i^{(p - p)j} = \left\lceil \frac{n-1}{m-1} \right\rceil - \left\lfloor \frac{n-1}{m} \right\rfloor \delta_{ij} \ (i, j, m = 1, 2, \ldots, n - 1),
\]
where \( \left\lceil \frac{n}{m} \right\rceil \) denotes the Stirling number of the first kind (see, e.g. [5] for the definition). Since
\[
|\{ \sigma \in \mathfrak{S}_n ; \nu(\sigma) = n-m, \sigma(p) = x \}| = \begin{cases} \left\lfloor \frac{n-1}{m-1} \right\rfloor & x = p, \\ \left\lfloor \frac{n-1}{m} \right\rfloor & x \neq p \end{cases}
\]
for each \( p, x \in [n] \), it follows that

\[
\frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \sum_{p=1}^{n} \alpha^{\nu(\sigma)}_{\omega^{p}(i-p)j} = \frac{1}{n} \sum_{p=1}^{n} \omega^{-pj} \left\{ \begin{array}{c} n-1 \\ m-1 \end{array} \right\} \omega^m + \sum_{x \neq p} \left\{ \begin{array}{c} n-1 \\ m \end{array} \right\} \omega^x \delta_{ij},
\]

which is the required conclusion. Here we notice that \( \sum_{x \neq p} \omega^x = -\omega^p \) since \( 1 \leq i < n \). Consequently, we obtain

\[
F^{(nl-1,1)}_{n,l}(\alpha) = \left( (1-\alpha)(1+\alpha)(1+2\alpha) \ldots (1+(n-2)\alpha) \right)^{1} (1 + (n-1)\alpha)^{-1} \delta_{ij},
\]

so that the multiplicity of \( \mathcal{M}^{(nl-1,1)}_{n} \) in \( V_{n,l}(\alpha) \) is zero if \( \alpha = -1/k \) \((k = 1, 2, \ldots, n-1)\) and \( n-1 \) otherwise.

The trace of the transition matrix \( F^{\lambda}_{n,l}(\alpha) \) is

\[
f^{\lambda}_{n,l}(\alpha) = \text{tr} F^{\lambda}_{n,l}(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \omega^{\lambda}(h),
\]

where \( \omega^{\lambda} \) is the zonal spherical function for \( \lambda \) with respect to \( K \) defined by

\[
\omega^{\lambda}(g) = \frac{1}{|K|} \sum_{k \in K} \chi^{\lambda}(kg) \quad (g \in \mathfrak{S}_n).
\]

This polynomial is regarded as a generalization of the modified content polynomial since \( f^{\alpha}_{n,1}(\alpha) = f^{\lambda}_{n,l}(\alpha) \) as we see above. It is much easier to handle these polynomials than the transition matrices. If we could prove that a transition matrix \( F^{\lambda}_{n,l}(\alpha) \) is a scalar matrix, then we would have \( F^{\lambda}_{n,l}(\alpha) = d^{-1} f^{\lambda}_{n,l}(\alpha) I \) \((d = \dim(S^{\lambda}|K))\) and hence we see that the multiplicity of \( \mathcal{M}^{(nl-1,1)}_{n} \) in \( V_{n,l}(\alpha) \) is completely controlled by the single polynomial \( f^{\lambda}_{n,l}(\alpha) \).

In this sense, it is desirable to obtain a characterization of the irreducible representations whose corresponding transition matrices are scalar as well as to get an explicit expression for the polynomials \( f^{\lambda}_{n,l}(\alpha) \). Here we give a sufficient condition for \( \lambda \vdash nl \) such that \( F^{\lambda}_{n,l}(\alpha) \) is a scalar matrix.

**Proposition 1.9.**  \( \text{(1) Denote by } N_H(K) \text{ the normalizer of } K \text{ in } H \). The transition matrix \( F^{\lambda}_{n,l}(\alpha) \) is scalar if \( (S^{\lambda}|K) \) is irreducible as a \( N_H(K) \)-module.

\( \text{(2) If } \lambda \text{ is of hook-type (i.e. } \lambda = (nl-r,1') \text{ for some } r < n, \text{ then } F^{\lambda}_{n,l}(\alpha) \) is scalar. \)

**Proof.** Notice that \( N_H(K) \cong \mathfrak{S}_n \). Consider a linear map \( T \in \text{End}((S^{\lambda}|K)) \) given by

\[
T(x) = \sum_{j=1}^{d} \left( \sum_{h \in H} \alpha^{\nu(h)} \langle x \cdot h, e^\lambda_j \rangle_{S^{\lambda}} \right) e^\lambda_j \quad (x \in (S^{\lambda}|K)),
\]

where \( d = \dim(S^{\lambda}|K) \). It is direct to check that \( T \) gives an intertwiner of \( (S^{\lambda}|K) \) as a \( N_H(K) \)-module. Hence, by Schur’s lemma, \( T \) is a scalar map (and \( F^{\lambda}_{n,l}(\alpha) \) is a scalar matrix) if \( (S^{\lambda}|K) \) is an irreducible \( N_H(K) \)-module.

When \( \lambda = (nl-r,1') \) for some \( r < n \), it is proved in [2, Proposition 5.3] that \( (S^{(nl-r,1')})^{K} \cong S^{(n-r,1')} \) as \( N_H(K) \)-modules. Thus we have the proposition.

**Example 1.10.** Let us calculate \( f^{(nl-1,1)}_{n,l}(\alpha) \). Notice that \( \chi^{(nl-1,1)}(g) = \text{fix}_{nl}(g) - 1 \) where \( \text{fix}_{nl} \) denotes the number of fixed points in the natural action \( \mathfrak{S}_n \cap [nl] \). Hence we see that

\[
f^{(nl-1,1)}_{n,l}(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \frac{1}{|K|} \sum_{k \in K} \left( \text{fix}_{nl}(kh) - 1 \right)
\]

\[
= \sum_{h \in H} \alpha^{\nu(h)} \frac{1}{|K|} \sum_{k \in K} \sum_{x \in [nl]} \delta_{kx,x} - \sum_{h \in H} \alpha^{\nu(h)}.
\]
It is easily seen that $khx \neq x$ for any $k \in K$ if $hx \neq x$ ($x \in [nl]$). Thus it follows that
\[
\frac{1}{|K|} \sum_{k \in K} \sum_{x \in [nl]} \delta_{khx,x} = \sum_{x \in [nl]} \frac{1}{|K|} \sum_{k \in K} \delta_{kx,x} = \frac{1}{l} \text{fix}_n(h) \quad (h \in H).
\]
Therefore we have
\[
f_{n,l}^{(n-1,1)}(\alpha) = \frac{1}{l} \sum_{h \in H} \alpha^{\nu(h)} \text{fix}_n(h) - \sum_{h \in H} \alpha^{\nu(h)} = f_{n,1}^{(n)}(\alpha)^{l-1} f_{n,1}^{(n-1,1)}(\alpha)
\]
\[
= (n-1)(1-\alpha)(1-(n-1)\alpha)^{l-1} \prod_{i=1}^{n-2}(1+ia)^i.
\]
Since the transition matrix $F_{n,l}^{(n-1,1)}$ is a scalar one and its size is $\dim S^{(n-1,1)} = n-1$, we get $F_{n,l}^{(n-1,1)}(\alpha) = (1-\alpha)(1-(n-1)\alpha)^{l-1} \prod_{i=1}^{n-2}(1+ia)^i I_{n-1}$ again.

We will investigate these polynomials $f_{n,l}^{(\lambda)}(\alpha)$ and their generalizations in [?].

2. Irreducible Decomposition of $V_{2,l}(\alpha)$ and Jacobi Polynomials

In this subsection, as a particular example, we consider the case where $n = 2$ and calculate the transition matrix $F_{2,l}^{(\lambda)}(\alpha)$ explicitly. Since the pair $(\mathfrak{g}_{2l}, K)$ is a Gelfand pair (see, e.g. [10]), it follows that
\[
K_{\lambda(\mathfrak{g}_l)} = \langle \text{ind}_{\mathfrak{g}_{2l}}^K 1_K, S^{\lambda} \rangle_{\mathfrak{g}_{2l}} = 1
\]
for each $\lambda \vdash 2n$ with $\ell(\lambda) \leq 2$. Thus, in this case, the transition matrix is just a polynomial and is given by
\[
F_{2,l}^{(\lambda)}(\alpha) = \text{tr} F_{2,l}^{(\lambda)}(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \omega^\lambda(h) = \sum_{s=0}^{l} \binom{l}{s} \omega^\lambda(g_s) \alpha^s.
\]
Here we put $g_s = (1, l+1)(2, l+2)\ldots(s,l+s) \in \mathfrak{g}_{2l}$. Now we write $\lambda = (2l-p, p)$ for some $p$ ($0 \leq p \leq l$). The value $\omega^{(2l-p,p)}(g_s)$ of the zonal spherical function is calculated by Bannai and Ito [3, p.218] as
\[
\omega^{(2l-p,p)}(g_s) = Q_p(s;l-1,-l-1,l) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \binom{2l-p+1}{j} \binom{l-2}{j} \binom{s}{j},
\]
where
\[
Q_p(x;\alpha, \beta, N) = _3F_2 \left( \begin{array}{c}
-n, n + \alpha + \beta + 1, -x \\
\alpha + 1, -N
\end{array} \mid 1 \right)
\]
\[
= \sum_{j=0}^{N} (-1)^j \binom{n}{j} \binom{-n - \alpha - \beta - 1}{j} \binom{-\alpha - 1}{j} \binom{N}{j} \binom{x}{j}
\]
is the Hahn polynomial (see also [10, p.399]), and $n+1 \tilde{F}_p \left( \begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_{q-1}, -N \end{array} \mid x \right)$ is the hypergeometric polynomial
\[
\tilde{F}_p \left( \begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_{q-1}, -N \end{array} \mid x \right) = \sum_{j=0}^{N} \binom{a_1}{j} \cdots \binom{a_p}{j} \binom{b_1}{j} \cdots \binom{b_{q-1}}{j} \binom{-N}{j} \frac{x^j}{j!}
\]
for $p, q, N \in \mathbb{N}$ in general (see [1]). We now re-prove Theorem ?? as follows:

Theorem 2.1. Let $l$ be a positive integer. It holds that
\[
F_{2,l}^{(2l-p,p)}(\alpha) = \sum_{s=0}^{l} \binom{l}{s} Q_p(s;l-1,-l-1,l) \alpha^s = (1 + \alpha)^{l-p} G_p^{(l)}(\alpha)
\]
for $p = 0, 1, \ldots, l$. 
Proof. Let us put \( x = -1/\alpha \). Then we have

\[
\sum_{s=0}^{l} \left( \begin{array}{l} l \\ s \end{array} \right) Q_p(s; l-1, l-1, l) \alpha^s \\
= \sum_{j=0}^{p} (-1)^j \left( \begin{array}{l} p \\ j \end{array} \right) \left( \begin{array}{l} 2l - p + 1 \\ j \end{array} \right) \alpha^j (1 + \alpha)^{l-j} \\
= x^{-l} (x-1)^{l-p} \sum_{j=0}^{p} \left( \begin{array}{l} p \\ j \end{array} \right) \left( \begin{array}{l} 2l - p + 1 \\ j \end{array} \right) \alpha^{-j} (x-1)^{p-j}
\]

and

\[
(1 + \alpha)^{l-p} G_p^1(\alpha) = x^{-l} (x-1)^{l-p} \sum_{j=0}^{p} (-1)^j \left( \begin{array}{l} p \\ j \end{array} \right) \left( \begin{array}{l} l-p+j \\ j \end{array} \right) \alpha^{j} (x-1)^{p-j}.
\]

Here we use the elementary identity

\[
\sum_{s=0}^{l} \left( \begin{array}{l} l \\ s \end{array} \right) \left( \begin{array}{l} s \\ j \end{array} \right) \alpha^s = \left( \begin{array}{l} l \\ j \end{array} \right) \alpha^j (1 + \alpha)^{l-j}.
\]

Hence, to prove the theorem, it is enough to verify

\[
\sum_{i=0}^{p} \left( \begin{array}{l} p \\ i \end{array} \right) \left( \begin{array}{l} l - p + i \\ i \end{array} \right) \alpha^{-i} x^{p-i} = \sum_{j=0}^{p} \left( \begin{array}{l} p \\ j \end{array} \right) \left( \begin{array}{l} 2l - p + 1 \\ j \end{array} \right) \alpha^{-j} (x-1)^{p-j}.
\]

Comparing the coefficients of Taylor expansion of these polynomials at \( x = 1 \), we notice that the proof is reduced to the equality

\[
\sum_{i=0}^{r} \left( \begin{array}{l} l-i \\ l-r \end{array} \right) \left( \begin{array}{l} l-p+i \\ l-p \end{array} \right) = \left( \begin{array}{l} 2l - p + 1 \\ r \end{array} \right)
\]

for \( 0 \leq r \leq p \), which is well known (see, e.g. (5.26) in [5]). Hence we have the conclusion. \( \square \)

Thus we obtain the irreducible decomposition

\[
V_{2,l}(\alpha) \cong M_{2,l}^{(l,l)}, \quad V_{2,l}(\alpha) \cong \bigoplus_{0 \leq p \leq l} M_{2,l}^{(2l-p,p)} (\alpha \neq -1)
\]

of \( V_{2,l}(\alpha) \) again.

Remark 2.2. (1) The calculation above uses the advantage for the fact that the pair \((\mathfrak{S}_{nl}, \mathfrak{S}_n^l)\) is the Gelfand pair only when \( n = 2 \).

(2) We have used the result in [3, p.218] for the theorem. It is worth mentioning that one may prove conversely the result in [3, p.218] from Theorem ??.

Acknowledgement

The author would thank Professor Itaru Terada for noticing that his work [2] is useful for the discussion in Section 6.2.

References

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