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A bit super like introduction to C^*-algebras by probability theory

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Abstract

We begin to study the operator probability theory to some extent.

C*-algebra, probability theory, free probability theory, von Neumann algebra
46L30, 46L51, 46L54, 46L05, 46L60, 60B11, 60B20, 60C05, 60E05, 60F05

1 Introduction

We would like to love people and mathematics for peace and stability to the world. Yes, we can do, as possible, to some extent.

Following [30] we as beginners, outsiders, fools, or not would like to study the free probability theory for operator algebras.

This is a sort of mathematical surfing to make it clear against a mathematical water wall such a lecture notes book. It means a mathematical understanding for some unkind notions such as independence, freeness, and more, together with illustrative helpful examples at the basic level. We made some considerable effort to do this somewhat completely (Yatta mine).

This is nothing but a review.

We use the standard notation by our taste.

The contents are as follows.

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1.1 The first outlooking at the background

The commutation relation of (linear) operators X, Y, and I the identity operator on a Hilbert space in quantum mechanics is given as

$$XY - YX = [X,Y] = \alpha I, \quad \alpha \in \mathbb{C}.$$

In this case, the commutator $[X,Y]$ commutes with X and Y.

Let F_n denote the free group of n generators with respective n inverses. Then elements of F_n are viewed as words generated by $2n$ characters, with cancellation as $gg^{-1} = 1$. where such words may have multiplicity free as g^2. Even usual words such as Good or Look do have multiplicity, but they are not so many? We may consider only multiplicity non-free words such as God or so.

Let H be a Hilbert space. We consider the infinite (or l^2-)direct sum (or product) of n-fold tensor product spaces $\otimes^n H = H \otimes \cdots \otimes H$ of H. Namely, let

$$T(H) = F(H) = \oplus_{n=0}^{\infty} \otimes^n H, \quad \otimes^0 H = \mathbb{C} = \mathbb{C}1$$

named as the Fock (tensor sum) space of states of (elementary) particles with $\otimes^0 H$ vacuum state (cf. [3]).

For any $h \in H$, the left creation (or tensor multiplication) operator L_h on $T(H)$ a Hilbert space is defined as

$$L_h \xi = h \otimes \xi = (h \otimes \xi_n)$$

for $\xi = (\xi_n) \in T(H)$, $\xi_n \in \otimes^n H$.

The operator L_h creates new states of particles by tensor with shift.

The operator L_h and its adjoint operator L_h^* generate the extended Cuntz (?) C^*-algebra.

$\star$ Note that (cf. [14])

$$\|h \otimes \xi_n\|^2 = \langle h \otimes \xi_n, h \otimes \xi_n \rangle = \langle h, h \rangle \langle \xi_n, \xi_n \rangle = \|h\|^2 \|\xi_n\|^2.$$

As well, $\|\xi\|^2 = \sum_{n=0}^{\infty} \|\xi_n\|^2 < \infty$ for $\xi \in T(H)$. Therefore,

$$\|L_h \xi\|^2 = \sum_{n=0}^{\infty} \|h \otimes \xi_n\|^2 = \|h\|^2 \|\xi\|^2.$$

Thus, $\|L_h\| \leq \|h\|$ by definition of the operator supremum norm. Namely, L_h is a bounded operator. Conversely,

$$\|L_h(1, 0, 0, \cdots)\| = \|(0, h, 0, \cdots)\| = \|h\|.$$

Therefore, $\|L_h\| = \|h\|$. In particular, L_h is an isometry for h with norm 1.

We also compute that with $T_h = L_h$,

$$\begin{aligned} \langle \xi, T_h^* T_h \eta \rangle = \langle T_h \xi, T_h \eta \rangle &= \sum_{n=0}^{\infty} \langle h \otimes \xi_n, h \otimes \eta_n \rangle \\ &= \|h\|^2 \sum_{n=0}^{\infty} \langle \xi_n, \eta_n \rangle = \|h\|^2 \langle \xi, \eta \rangle. \end{aligned}$$

Therefore, if $\|h\| = 1$, then $T_h^* T_h$ is the identity operator 1 on $T(H)$. As well,

$$T_h^*(T_h \eta) = T_h^*(0, h \otimes \eta_0, h \otimes \eta_1, \cdots) = (\eta_0, \eta_1, \cdots) = \eta.$$

That's it! Moreover,

$$\langle \xi, T_h^*(\eta_0', 0, \cdots) \rangle = \langle T_h \xi, (\eta_0, 0, \cdots) \rangle = 0,$$

so that $T_h^*(\eta_0', 0, \cdots) = 0$.

The operator T_h^* is called the annihilation operator. It annihilates vacuum state of particles.

Annihilation in physics means that when particles collide (or bump) with anti-particles, their mass become energy to disappear.

The C^*-algebra generated by T_h and T_h^* for $h \in H$ with norm 1 is certainly called the Toeplitz algebra.

Suppose now that vectors $h, k \in H$ are orthogonal. Namely, the inner product $\langle h, k \rangle = 0$. Then

$$\begin{aligned} \langle T_h \xi, T_k \eta \rangle &= \sum_{n=0}^{\infty} \langle h \otimes \xi_n, k \otimes \eta_n \rangle \\ &= \sum_{n=0}^{\infty} \langle h, k \rangle \langle \xi_n, \eta_n \rangle = 0. \end{aligned}$$

Therefore, the operators T_h and T_k have ranges orthogonal. In this case, but the direct sum of their ranges is not $F(H)$, so that the C^*-algebra generated by T_h and T_k with their adjoints on $F(H)$ may be called as the Cuntz like (or similar) algebra denoted by $O_2^{\sim}$ by us.

Similarly, for mutually orthogonal n vectors $h_j \in H$, $j = 1, \cdots, n$, the operators T_{h_j} have ranges mutually orthogonal, but the direct sum of their ranges is not $F(H)$, so that the C^*-algebra generated by T_{h_j}, $j = 1, \cdots, n$ may be the Cuntz like (or similar) algebra $O_n^{\sim}$.

However, if we can identify $F(H)$ with the direct sum of those ranges, then our like $O_n^{\sim}$ can be identified with Cuntz O_n.

Moreover, for a sequence of mutually orthogonal vectors $h_n \in H$, $n \in \mathbb{N}$, the operators T_{h_n} have ranges mutually orthogonal, so that the unital C^*-algebra generated by T_{h_n}, $n \in \mathbb{N}$ is just isomorphic to the (universal) Cuntz algebra O_∞. □

Random variables in probability as measurable (real or complex valued) functions on a measurable space are replaced with (some) operators on a Hilbert space (of square summable or integrable measurable functions) in free (or semi-free) or noncommutative probability, as which are also observables in quantum mechanics.

A noncommutative probability (function) space is defined to be a unital (operator) algebra $\mathfrak{A}$ over $\mathbb{C}$ endowed with a (positive) liner functional $\varphi : \mathfrak{A} \to \mathbb{C}$ as probability (?) integral (!) such that $\varphi(1) = 1$ (so that $\varphi(a^* a) \geq 0$ for any $a \in \mathfrak{A}$).

Table 1: A comparison of Classical and Quantum mechanics bases

Classical	Quantum
A function f on a space X	An operator T on a Hilbert space H
Bounded or unbounded	The same as the left
Real valued or Hermite	Self-adjoint or real spectrum
Circle valued	Unitary or circle (part) spectrum
Commutative algebra by f	Operator algebra $\mathfrak{A}$ by T
(L) Integral $\int_X f(x)dx$	Functional value $\varphi(T)$

Elements of $\mathfrak{A}$ are called noncommutative random variables.

The distribution of a family of elements a_j of $\mathfrak{A}$ for $j \in J$ a set is provided by the information of moments $\varphi(a_{j_1} \cdots a_{j_n})$ for $j_1, \cdots, j_n \in J$.

Let $\mathfrak{P} = P(\{X_j\}_{j\in J})$ be the algebra over $\mathbb{C}$ of polynomials of mutually uncommuting inderminates X_j. There is a homomorphism χ from $\mathfrak{P}$ to $\mathfrak{A}$ by sending X_j to a_j. Thus, there is the linear map $\varphi \circ \chi$ from $\mathfrak{P}$ to $\mathbb{C}$. Namely,

$$(\varphi \circ \chi)(X_{j_1} \cdots X_{j_n}) = \varphi(a_{j_1} \cdots a_{j_n}) \in \mathbb{C}.$$

Table 2: Moments and more for functions and operators

Type	Classical (Lebesgue)	Quantum
Mean or expectation	$\mu = \int_X xdx = E[x]$	$\mu = \varphi(T)$
n-th moments	$\mu_n = \int_X x^n dx = E[x^n]$	$\mu_n = \varphi(T^n)$
Variance	$\sigma^2 = \int_X \|x-\mu\|^2 dx$	$\varphi((T^* - \overline{\mu}1)(T - \mu 1))$
Standard deviation	$\sigma \geq 0$	The similar

Let $\mathfrak{A}$ be a (unital) C^*-algebra and $a = a^* \in A$ an Hermitian element and φ a state on $\mathfrak{A}$, that is, a positive linear functional on A with supremum norm 1 on the unit ball of $\mathfrak{A}$. We say that $(\mathfrak{A}, \varphi)$ as above is a C^*-probability space.

There corresponds to a compactly supported probability measure μ_a for the element a on $\mathbb{R}$ such that

$$\varphi(a^n) = \int_{\mathbb{R}} t^n d\mu_a(t)$$

which extends by linearity to polynomials by the element a. In particular, in the unit case, with the support $\mathrm{supp}(\mu_a) \subset \mathbb{R}$ compact (or bounded and closed),

$$1 = \varphi(1) = \int_{\mathbb{R}} 1 d\mu_a(t) = \mu_a(\mathrm{supp}(\mu_a))$$

Therefore, for Hermitian random variables, we obtain corresponding probability (P) measures as in the classical P theory. That is determined completely by the moments collected.

$\star$ The spectrum theory (cf. [17]) implies that the C^*-algebra generated by a normal element a (i.e. $aa^* = a^*a$) and 1 is isomorphic to the C^*-algebra $C(\mathrm{sp}(a))$ of all continuous complex-valued functions on the spectrum $\mathrm{sp}(a)$ of a. If $a = a^*$, then $\mathrm{sp}(a)$ is contained in $\mathbb{R}$ (cf. [6]). The spectrum for any element is contained in $\mathbb{C}$ and always closed. For any bounded operator b, its spectrum is bounded by the operator norm of b. Thus, if an operator is bounded, then its spectrum is compact. For a bounded self-adjoint operator a, the measure μ_a or the corresponding functional φ can be given by the normalized Lebesgue measure restricted to $\mathrm{sp}(a)$. Namely, $\mu_a = \frac{1}{\mu(\mathrm{sp}(a))}\mu$ with μ L measure, and $\varphi(a) = \int_{\mathrm{sp}(a)} t d\mu_a(t)$. $\square$

$\star$ The dual (Banach) space $\mathfrak{A}^*$ of bounded linear functionals on a C^*-algebra $\mathfrak{A}$ is certainly known to somewhat extent. The space of states on $\mathfrak{A}$ is a closed subspace of $\mathfrak{A}^*$. In particular, it is known as the Riesz theorem that the dual (Banach) space $C(I)^*$ of the C^*-algebra $C(I)$ of all continuous functions on a closed interval $I = [a, b]$ is identified with the space $V_0^b(I)$ of bounded variation functions $v(t)$ on I such that $v(a) = 0$, and that the Stieltjes integral of $f \in C(I)$ by v defines the corresponding functional as

$$\varphi(f) = \int_a^b f(t)dv(t), \quad \varphi \in C(I)^*$$

with norm of φ equal to the total variation of v on I as the supremum of finite sums of variations of v with respect to finite partitions of I (cf. [10]). $\square$

The independence distinguishes free and the other quantum or noncommutative, or classical probabilities.

In quantum (or operator) mechanics, the independence is said to be classical independence, modeled on tensor products of algebras.

Two (unital) subalgebras $\mathfrak{B}$ and $\mathfrak{C}$ of $(\mathfrak{A}, \varphi)$ are said to be classically independent if they commute, namely $[\mathfrak{B}, \mathfrak{C}] = 0$, and if

$$\varphi(bc) = \varphi(b)\varphi(c) \quad b \in \mathfrak{B}, c \in \mathfrak{C}.$$

Note that the last condition amounts to that independent random variables (in classical mechanics) factorize under expectation (E).

$\star$ Let $(\mathfrak{B}, \varphi)$ and $(\mathfrak{C}, \psi)$ be C^*-probability spaces with $\mathfrak{B}, \mathfrak{C}$ unital. Then they are identified with respective tensor factors of the tensor product $\mathfrak{B} \otimes \mathfrak{C}$ with a C^*-norm and be classically independent in $(\mathfrak{B} \otimes \mathfrak{C}, \varphi \otimes \psi)$. Indeed, $[\mathfrak{B} \otimes 1, 1 \otimes \mathfrak{C}] = 0$ and

$$(\varphi \otimes \psi)(b \otimes c) = (\varphi \otimes \psi)(b \otimes 1)(\varphi \otimes \psi)(1 \otimes c) = \varphi(b)\psi(c) \quad b \in \mathfrak{B}, c \in \mathfrak{C}$$

by the definition of $\varphi \otimes \psi$. $\square$

⋆ Let $X \subset \mathbb{R}^n$ be a (L) measurable set with (L) measure $\mu(X) = 1$. For two random variables $f(x)$ and $g(y)$ for $(x, y) \in X \times X$, which are integrable on X with respect to μ, we have

$$\begin{aligned} E[f(x)g(y)] &= \int_{X\times X} f(x)g(y)d\mu(x)d\mu(y) \\ &= \int_X f(x)d\mu(x) \int_X g(y)d\mu(y) \\ &= \int_{X\times X} f(x)d\mu(x)d\mu(y) \int_{X\times X} g(y)d\mu(x)d\mu(y) \\ &= E[f(x)]E[g(y)] \end{aligned}$$

by the Fubini theorem. Namely, $f(x)$ and $g(y)$ on $X \times X$ are independent in the usual or classical sense. □

We have the free independence in the free probability theory. A family of unital subalgebras $\mathfrak{A}_j$, $j \in J$ in a C^*-probability space $(\mathfrak{A}, \varphi)$ is said to be freely independent or free if

$$\varphi(a_1 a_2 \cdots a_k) = 0, \quad a_j \in \mathfrak{A}_{i_j}, 1 \leq j \leq k$$

such that $i_1 \neq i_2 \neq \cdots \neq i_k$ in J and $\varphi(a_j) = 0$, $1 \leq j \leq k$. The sets of some variables in $(\mathfrak{A}, \varphi)$ are said to be free if the algebras generated by each of the variables are free.

⋆ Let $f(x) = x$ for $x \in X = [-1, 1] = Y$ the interval. In this case, we have

$$\int_{X^2} (f(x) \otimes 1)\frac{1}{2}dx\frac{1}{2}dy = \int_{-1}^1 x\frac{1}{2}dx \int_{-1}^1 \frac{1}{2}dy = [\frac{x^2}{4}]_{x=-1}^1 = 0.$$

But

$$\begin{aligned} &\int_{X^2} (x \otimes 1)(1 \otimes y)(x \otimes 1)(1 \otimes y)\frac{1}{2}dx\frac{1}{2}dy \\ &= \int_{-1}^1 x^2\frac{1}{2}dx \int_{-1}^1 y^2\frac{1}{2}dy = 2[\frac{x^3}{6}]_{x=-1}^1 = \frac{2}{3} \neq 0. \end{aligned}$$

Therefore, the two variables $x \otimes 1$ and $1 \otimes y$ are not free with respect to $\varphi = \int_{X^2} \frac{1}{2}dx\frac{1}{2}dy$, but they are classically independent. So what are free variables? □

We denote by $vN(F_n)$ the von Neumann algebra by the left regular representation of the free group F_n of n generators.

The problem of Murray and von Neumann is whether the von Neumann algebras $vN(F_n)$ and $vN(F_m)$ are non-isomorphic for $n \neq m$.

This seems to be still unsolved since it is raised some long time about 90 years ago.

⋆ A C^*-algebra can be represented isometrically as an operator (sup) norm closed, involutive (or $*$-) subalgebra of the C^*-algebra $B(H)$ of all bounded operators on a Hilbert space H. A von Neuamnn algebra is defined to be an operator strongly (or weakly) closed, $*$-subalgebra of $\mathbb{B}(H)$ for some Hilbert H.

Note that for an operator $T \in B(H)$ and vectors $\xi, \eta \in H$, we have the Cauchy-Schwarz inequality and the vector to operator norm estimate

$$|\langle T\xi, \eta\rangle| \leq \|T\xi\|\|\eta\| \leq \|T\|\|\xi\|\|\eta\|,$$

which implies that norm convergence for operators implies strong convergence and strong convergence implies weak convergence.

Let V be a strongly closed subset of $B(H)$. If a sequence of operators of V converges to an operator T in $B(H)$ in norm, then it converges strongly to T, so that T belongs to V. Thus V is norm closed. It then follows that a von Neumann algebra is a C^*-algebra. □

⋆ We denote by $l^2(F_n)$ the Hilbert space of square summable complex-valued functions on the free (or any discrete) group F_n as a space. The left regular (unitary) representation λ of F_n on $l^2(F_n)$ is defined to be that

$$\lambda_g f(x) = f(g^{-1}x), \quad g, x \in F_n, f \in l^2(F_n).$$

Note that for $g_1, g_2 \in F_n$, we have

$$\lambda_{g_1 g_2} f(x) = f(g_2^{-1} g_1^{-1} x) = (\lambda_{g_2} f)(g_1^{-1} x) = \lambda_{g_1}(\lambda_{g_2} f)(x).$$

Also, we compute

$$\begin{aligned} \langle \lambda_g^* f, h\rangle = \langle f, \lambda_g h\rangle &= \sum_{x \in F_n} f(x)\overline{h(g^{-1}x)} \quad (s = g^{-1}x) \\ &= \sum_{s \in F_n} f(gs)\overline{h(s)} = \langle \lambda_{g^{-1}} f, h\rangle. \end{aligned}$$

Hence, $\lambda_g^* = \lambda_{g^{-1}} = \lambda_g^{-1}$, that is unitary. The free group von Neumann algebra $vN(F_n)$ is defined to be the von Neumann algebra generated by the unitary operators λ_g, $g \in F_n$ on $l^2(F_n)$. What is difficulty? It's F_n, which is non-amenable. □

⋆ By the way, the free groups F_n and F_m are non-isomorphic for $n \neq m$. This hard question has been solved by (Pimsner-Voiculescu), Cuntz (and Blackadar) by using the K-theory of the full (or reduced) group C^*-algebras $C^*(F_n)$ of the free groups F_n. Indeed, the K_0-group of $C^*(F_n)$ is $\mathbb{Z}$, but the K_1 is $\mathbb{Z}^n$, which implies the non-isomorphism (cf. [2]). □

⋆ The operator weak or strong closures for some sets of operators are larger than the operator norm closure in general. For instance, the commutative C^*-algebra $C(I)$ of continuous functions on the interval $I = [0, 1] \subset \mathbb{R}$ is strictly contained in the abelian von Neumann algebra $L^\infty(I)$ of essentially bounded measurable complex-valued functions on I. □

⋆ The C^*-algebras corresponding to groups or spaces do remember topological space invariants to somewhat extents. On the other hand, the von Neumann algebras to them do only Borel space invariants in some sense. Therefore, the von Neumann problem is probably solved to be the unique isomorphism class by this sense only. □

The von Neumann tracial state of $vN(F_n)$ is defined to be that

$$\tau(T) = \langle T\chi_e, \chi_e\rangle, \quad T \in vN(F_n)$$

where e is the identity element of F_n, and χ_e is the characteristic function on F_n at $\{e\}$ as support. The set $\{\chi_g \mid g \in F_n\}$ is the canonical basis of the Hilbert space $l^2(F_n)$.

⋆ For $T, S \in vN(F_n)$ and $\alpha, \beta \in \mathbb{C}$, we have

$$\tau(\alpha T + \beta S) = \langle(\alpha T + \beta S)\chi_e, \chi_e\rangle = \alpha\tau(T) + \beta\tau(S).$$

As well, $\tau(T^*T) = \|T\chi_e\|^2 \geq 0$. Thus, τ is a positive functional on $vN(F_n)$. Moreover, $\tau(1) = \|\chi_e\|^2 = 1$. Hence, τ is a state on $vN(F_n)$. □

⋆ We compute that for $\lambda_g, \lambda_h \in vN(F_n)$ for $g, h \in F_n$ with $gh \neq e$,

$$\langle \lambda_g\lambda_h\chi_e, \chi_e\rangle = \sum_{x\in F_n} \lambda_g(\lambda_h\chi_e)(x)\overline{\chi_e(x)} = \sum_{x\in F_n} \chi_e(h^{-1}g^{-1}x)\overline{\chi_e(x)} = 0,$$

and similarly, $\langle\lambda_h\lambda_g\chi_e, \chi_e\rangle = 0$, so that $\tau(\lambda_g\lambda_h) = \tau(\lambda_h\lambda_g) = 0$. Moreover, $\tau(\lambda_g) = 0$ if $g \neq e$. On the other hand, $\tau(\lambda_e) = 1$ with $\lambda_e = 1 \in vN(F_n)$. It then follows that τ is a tracial state on $vN(F_n)$. Namely, $\tau(TS) = \tau(ST)$ for any $T, S \in vN(F_n)$. Indeed, suppose that nets of elements generated by λ_g for $g \in F_n$ with the trace τ zero converge weakly to some element T of $vN(F_n)$. Then it follows that $\tau(T) = 0$, so that $\tau(TS) = 0 = \tau(ST)$ for $TS \neq 1 \neq ST$. □

⋆ It also follows in particular the following. Let a, b be the generators of F_2. Then the (unital) (involutive) subalgebra generated by λ_a and the subalgebra generated by λ_b in $vN(F_2)$ are freely independent with respect to the von Neumann trace τ. □

⋆ Let $T_1 = \sum_{g\in F_n} \alpha_g\lambda_g \in vN(F_n)$ and $T_2 = \sum_{h\in F_m} \beta_h\lambda_h \in vN(F_m)$ with $\alpha_g, \beta_h \in \mathbb{C}$ and with λ_g and λ_h distinguished. We have $F_{n+m} \cong F_n * F_m$ the free product group of F_n and F_m and also $C^*(F_{n+m}) \cong C^*(F_n) * C^*(F_m)$ the (unital) free product of $C^*(F_n)$ and $C^*(F_m)$ (and as well the reduced version and the von Neumann version). The (multiplication) operators T_1 and T_2 both extend to those on the $vN(F_{n+m})$ with 1 on $vN(F_m)$ and $vN(F_n)$ respectively. Let τ be the von Neumann trace on $vN(F_{n+m})$. Then

$$\tau(T_1 + T_2) = \alpha_e + \beta_e = \tau(T_1) + \tau(T_2).$$

On the other hand, we compute that

$$\tau((T_1 + T_2)^2) = \alpha_e^2 + 2\alpha_e\beta_e + \beta_e^2 = \tau(T_1^2) + 2\tau(T_1T_2) + \tau(T_2^2).$$

Moreover,

$$\begin{aligned}\tau((T_1 + T_2)^3) &= \alpha_e^3 + 3\alpha_e^2\beta_e + 3\alpha_e\beta_e^2 + \beta_e^3 \\ &= \tau(T_1^3) + 3\tau(T_1^2T_2) + 3\tau(T_1T_2^2) + \tau(T_2^3).\end{aligned}$$

It then certainly follows inductively that the n-moments of T_1+T_2 depend on the n-moments of T_1 and T_2 as well as the n-moments of $T_1^{n-k}T_2^k$ for $1 \le k \le n-1$, but not on the (first) moments of T_1 and T_2. Note also that we do have

$$\tau(T_1^{n-k}T_2^k) = \tau(T_1^{n-k})\tau(T_2^k).$$

By the way, we have

$$\begin{aligned} T_1T_2 &= \sum_{g\in F_n} \alpha_g \lambda_g \sum_{h\in F_m} \beta_h \lambda_h \\ &= \sum_{g\in F_n} \sum_{h\in F_m} \alpha_g \beta_h \lambda_g \lambda_h \quad (k = gh \in F_{n+m}) \\ &= \sum_{k\in F_{n+m}, k=gh, g\in F_n, h\in F_m} \alpha_g \beta_h \lambda_k. \end{aligned}$$

Namely, the product of the operators T_1, T_2 is given by a sort of convolution. □

⋆ Let $M_n(\mathbb{C})$ be the C^*-algebra of all $n\times n$ matrices over complex numbers. The trace on $M_n(\mathbb{C})$ is defined to be $\mathrm{tr}((a_{ij})) = \sum_{j=1}^n a_{jj} \in \mathbb{C}$ for $A = (a_{ij}) \in M_n(\mathbb{C})$ as a linear functional. The tracial state on $M_n(\mathbb{C})$ is then given by $\varphi = \frac{1}{n}\mathrm{tr}$. Note that $\varphi(1) = 1$ with $1 \in M_n(\mathbb{C})$ the identity matrix and that

$$\mathrm{tr}(A^*A) = \sum_{k=1}^n \sum_{j=1}^n \overline{a_{jk}} a_{jk} = \sum_{k=1}^n \sum_{j=1}^n |a_{jk}|^2 \ge 0.$$

As well,

$$\mathrm{tr}(AB) = \sum_{k=1}^n \sum_{j=1}^n a_{kj} b_{jk} = \sum_{j=1}^n \sum_{k=1}^n b_{jk} a_{kj} = \mathrm{tr}(BA).$$

□

⋆ Let X be a compact Hausdorff space with μ a probability measure. Let $C(X, M_n(\mathbb{C})) \cong C(X) \otimes M_n(\mathbb{C})$ be the C^*-algebra of continuous $M_n(\mathbb{C})$-valued functions on X. Define a tracial state φ on $C(X, M_n(\mathbb{C}))$ by

$$\varphi(f) = \int_X \frac{1}{n}\mathrm{tr}(f(x))d\mu(x), \quad f = f(x) \in C(X, M_n(\mathbb{C})).$$

In particular, $\varphi(1) = \int_X 1 d\mu(x) = \mu(X) = 1$. If $f = f^*$, then $\varphi(f) \in \mathbb{R}$. The functions f may be viewed as random matrices. □

The von Neumann algebra $vN(F_n)$ can be viewed as being asymptotically generated by random matrices.

There is an isomorphism between $vN(F_\infty)$ and $p(vN(F_\infty))p$ for a projection p with trace rational or not.

There are the Dykema-Radulescu interpolated free group factors $DR(F_r)$ for real $r > 1$ generalizing $vN(F_n)$. In particular, we have $DR(F_{r+s}) \sim DR(F_r) * DR(F_s)$.

2 The second outlooking at the basics

2.1 Operator probability spaces

Let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a positive unital linear functional on a unital C^*-algebra $\mathfrak{A}$ over $\mathbb{C}$. Then $\varphi(a^*) = \overline{\varphi(a)}$ for $a \in \mathfrak{A}$.

Proof. Define a positive sesquilinear form (or an inner product) ρ on $\mathfrak{A}^2$ by $\rho(a, b) = \varphi(b^* a)$ for $(a, b) \in \mathfrak{A}^2$. In particular, $\rho(a, a) \geq 0$. Then we have

$$\rho(a, b) = \overline{\rho(b, a)} = \overline{\varphi(a^* b)}.$$

This is one of the properties of an inner product.

Indeed, the ρ defines a norm on $\mathfrak{A}$ by $\sqrt{\rho(a, a)}$. Then the ρ can be written by the norm as a linear combination. It then follows that that property holds. □

Let $\mathfrak{A}$ be a von Neumann algebra. A normal state φ on $\mathfrak{A}$ is ultra-weakly continuous.

Proof. We may refer to [21]. Assume that $\mathfrak{A}$ is represented on a Hilbert space H. We denote by $\mathfrak{A}_*$ the Banach space of all σ-weakly continuous linear functionals on $\mathfrak{A}$. Each element of $\mathfrak{A}_*$ is said to be normal. The space $\mathfrak{A}_*$ is named as the pre-dual of $\mathfrak{A}$.

The σ-weak topology on $\mathfrak{A}$ is the $\sigma(\mathfrak{A}, \mathfrak{A}_*)$-topology.

Convergence by this topology is given by $|\varphi(x)|$ for $x \in \mathfrak{A}$ and $\varphi \in \mathfrak{A}_*$.

Each φ is given as a limit of sums of vector states on $\mathfrak{A}$. Namely,

$$\varphi(x) = \sum_{j=1}^{\infty} \alpha_j \langle x\xi_j, \eta_j \rangle, \quad \alpha_j \in \mathbb{C}, \xi_j, \eta_j \in H$$

where $\sum_{j=1}^{\infty} |\alpha_j| < \infty$, $\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty$, and $\sum_{j=1}^{\infty} \|\eta_j\|^2 < \infty$. □

A linear functional φ on a C^*-algebra $\mathfrak{A}$ is said to be faithful if $\varphi(a^* a) = 0$, then $a = 0 \in \mathfrak{A}$.

Example 2.1.1. Let (X, μ) be a probability space with $\mu(X) = 1$. We denote by $L^\infty(X)$ the von Neumann algebra of all essentially bounded measurable complex-valued functions on X. Define $\varphi(f) = \int_X f(x) d\mu(x) \in \mathbb{C}$ for $f \in L^\infty(X)$. This is a faithful normal tracial state on $L^\infty(X)$. Normal?

The $L^\infty(X)$ is represented on the Hilbert space $L^2(X)$ by multiplication operators M_f. Note that

$$\|M_f \xi\|_2^2 = \|f\xi\|_2^2 = \int_X |f(x)\xi(x)|^2 d\mu(x) \leq \|f\|_\infty \|\xi\|_2^2.$$

As well, convergence with respect to the inner product

$$\langle M_f \xi, \eta \rangle = \int_X f(x)\xi(x)\overline{\eta(x)} d\mu(x)$$

to $\langle M_g \xi, \eta \rangle$ implies the convergence by φ. Why? We can choose $\xi = 1 = \eta$. That's it!

If $\varphi(f^* f) = 0$, then $\int_X |f|^2 d\mu(x) = 0$. Hence $|f| = 0$ almost everywhere on X. Thus, $f = 0$ up to measure zero sets. Each element of $L^\infty(X)$ is a class of functions up to measure zero sets. $\square$

Example 2.1.2. The tracial state $\varphi = \frac{1}{n}\mathrm{tr}$ on $M_n(\mathbb{C})$ is faithful and normal.

The $M_n(\mathbb{C})$ is represented on $\mathbb{C}^n$ by matrix multiplication. Let $X_k = (x_{ij}(k)) \in M_n(\mathbb{C})$ such that for any $\xi \in \mathbb{C}^n$, there exists $\lim_k X_k \xi = \eta \in \mathbb{C}^n$ for some $\eta \in \mathbb{C}^n$. In particular, for the canonical basis $e_1, \cdots, e_n$ of $\mathbb{C}^n$,

$$\lim_k X_k e_j = \lim_k (x_{ij}(k))_{i=1}^n = (\lim_k x_{ij}(k))_{i=1}^n \in \mathbb{C}^n.$$

Let $X = (\lim_k x_{ij}(k))_{i,j=1}^n \in M_n(\mathbb{C})$. That is the strong limit of the sequence (X_k). Namely, $M_n(\mathbb{C})$ is a von Neumann algebra.

If $\varphi(A^* A) = 0$, then $\mathrm{tr}(A^* A) = 0$. It then follows that $A = (a_{ij}) = (0)$.

If the inner products $\langle X_k e_j, e_i \rangle$ converge to $\langle X e_j, e_i \rangle$ for $i, j = 1, \cdots, n$, then in particular, $x_{jj}(k)$ converge to x_{jj} for $1 \le j \le n$ respectively, so that $\varphi(X_k)$ converge to $\varphi(X)$. $\square$

Example 2.1.3. Let H be a Hilbert space. We have the von Neumann C^*-algebra $\mathbb{B}(H)$ of all bounded linear operators on H with the operator norm.

We have the C^*-norm condition $\|T^* T\| = \|T\|^2$ for any $T \in \mathbb{B}(H)$.

The operator norm for T is defined to be

$$\|T\| = \sup_{\xi \in H, \xi \ne 0} \frac{\|T\xi\|}{\|\xi\|} = \sup_{\|\xi\| \le 1} \|T\xi\| = \sup_{\|\xi\| = 1} \|T\xi\|.$$

Note that $\|\frac{1}{\|\xi\|}\xi\| = 1$ for $\xi \ne 0$. Also, we obtain that

$$\sup_{\|\xi\|=1} \|T\xi\| \le \sup_{\|\xi\| \le 1} \|T\xi\| = \sup_{0 < \|\xi\| \le 1} \|T\xi\| \le \sup_{\xi \in H, \xi \ne 0} \frac{\|T\xi\|}{\|\xi\|} \le \sup_{\|\xi\|=1} \|T\xi\|.$$

As well, for $\|\xi\| = 1$,

$$\|T\xi\|^2 = \langle T\xi, T\xi \rangle = \langle T^* T\xi, \xi \rangle \le \|T^* T\xi\| \le \|T^* T\|.$$

It then follows that $\|T\|^2 \le \|T^* T\|$.

On the other hand, we have $\|T^* T\| \le \|T^*\| \|T\|$ with $\|T^*\| = \|T\|$.

Indeed, for any $S, T \in \mathbb{B}(H)$, we have that for $\|\xi\| = 1$,

$$\|ST\xi\|^2 = \langle ST\xi, ST\xi \rangle \le \|S^* ST\xi\| \|T\xi\| \le \|S^* ST\| \|T\|.$$

It then follows that $\|ST\|^2 \le \|S^* ST\| \|T\|$. What's this? We next estimate that

$$\begin{aligned} \|ST\xi\| &= \|S(\frac{1}{\|T\xi\|} T\xi)\| \|T\xi\| \quad (\|T\xi\| \ne 0, \|\xi\| = 1) \\ &\le \|S\| \|T\|. \end{aligned}$$

It then follows that $\|ST\| \leq \|S\|\|T\|$. As well, for $\|\xi\| = 1$,

$$\begin{aligned}\|T^*\xi\|^2 &= \langle T^*\xi, T^*\xi\rangle = \langle TT^*\xi, \xi\rangle \\ &\leq \|TT^*\xi\| \leq \|TT^*\| \leq \|T\|\|T^*\|.\end{aligned}$$

It then follows that $\|T^*\|^2 \leq \|T\|\|T^*\|$. Hence $\|T^*\| \leq \|T\|$. Since $T = (T^*)^*$, then $\|T\| \leq \|T^*\|$. Thus, $\|T\| = \|T^*\|$.

A C^*-algebra may be defined to be an involutive Banach (or $*$-)algebra with the C^*-norm condition.

Since H is a Banach space, then $\mathbb{B}(H)$ is a Banach space.

Indeed, for any Cauchy sequence (T_n) of $\mathbb{B}(H)$, and for any $\xi \in H$, we have the Cauchy sequence $(\|T_n\xi\|)$ in H. Thus, the $(T_n\xi)$ converges to some $T\xi \in H$. This is extended to the limit operator $T \in \mathbb{B}(H)$.

Let us have the $\infty \times \infty$ matrix representation such as

$$(T_n) = \begin{pmatrix} t_{11}(n) & t_{12}(n) & \cdots \\ t_{21}(n) & t_{22}(n) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{B}(H)$$

with respect to a (countable or not) basis (e_k) for H. The strong or weak limit T of (T_n) looks like that

$$T = \begin{pmatrix} \lim_n t_{11}(n) & \lim_n t_{12}(n) & \cdots \\ \lim_n t_{21}(n) & \lim_n t_{22}(n) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{B}(H).$$

Indeed, we have

$$\langle T_n e_k, e_l\rangle = t_{lk}(n) \to \lim_n t_{lk}(n) = \langle Te_k, e_l\rangle.$$

Also,

$$\|T_n e_k\|^2 = \sum_{j=1}^{\infty} |t_{jk}(n)|^2 \to \|Te_k\|^2 = \sum_{j=1}^{\infty} |\lim_n t_{jk}(n)|^2 \in \mathbb{R}.$$

The weak or strong limit of bounded operators is always bounded? We can not prove it in general, can do we. But it is only assumed from the first.

For instance, let $X = [0, 2]$ and let $f_n(x) = x^n$ for $x \in X$, $n \in \mathbb{N}$. Then (f_n) is a sequence of bounded functions. But the point-wise (or strong) limit is given by

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & (0 \leq x < 1), \\ 1 & (x = 1), \\ \infty & (1 < x \leq 2). \end{cases}$$

This limit function is not (essetially) bounded.

By the way, the von Neumann double commutant theorem is that for a unital $*$-algebra $\mathfrak{A}$ (of bounded operators) on a Hilbert space H, $\mathfrak{A}$ is a von Neumann algebra if and only if the commutant of the commutant of $\mathfrak{A}$ is the same as $\mathfrak{A}$, i.e., $(\mathfrak{A}')' = \mathfrak{A}$.

In particular,

$$(\mathbb{B}(H)')' = (\mathbb{C}1)' = \mathbb{B}(H).$$

Also, the commutant $\mathfrak{A}'$ is a von Neumann algebra.

Indeed, suppose that a net (T_λ) of $\mathfrak{A}'$ converge strongly to a bounded operator T on H. Then for any $S \in \mathfrak{A}$ $(\subset \mathbb{B}(H))$, we have $T_\lambda S = ST_\lambda$. For any $\xi \in H$, we then have

$$\begin{array}{ccc} \|T_\lambda S\xi\| = \|ST_\lambda \xi\| & \longrightarrow & \|ST\xi\| \\ \downarrow & & \\ \|TS\xi\| & & \end{array}$$

and it then follows (by the uniqueness of the strong or weak limit) that $T \in \mathfrak{A}'$.

Let $\xi \in H$ with $\|\xi\| = 1$. Define a positive linear functional φ on $\mathbb{B}(H)$ by $\varphi(T) = \langle T\xi, \xi\rangle$ with $\varphi(1) = 1$. Thus, φ is a (vector) state on $\mathbb{B}(H)$. This is neither a trace nor faithful if dimension $\dim H \geq 2$. □

Example 2.1.4. Let G be a (finite or not) group with 1_G the unit. The group algebra $\mathbb{C}[G]$ of G is defined to be the (formal) algebra over $\mathbb{C}$ of all finite linear combinations such as $\sum_{j=1}^n \alpha_{g_j} g_j, \sum_{k=1}^m \beta_{h_k} h_k$ with $\alpha_{g_j}, \beta_{h_k} \in \mathbb{C}$, $g_j, h_k \in G$. Elements of G are viewed as a basis for the linear space $\mathbb{C}[G]$ over $\mathbb{C}$. Multiplication on $\mathbb{C}[G]$ is given as

$$\sum_{j=1}^n \alpha_{g_j} g_j \sum_{k=1}^m \beta_{h_k} h_k = \sum_{j=1}^n \sum_{k=1}^m \alpha_{g_j} \beta_{h_k} g_j h_k.$$

This operation on $\mathbb{C}[G]$ extends the group operation of G. The involution on $\mathbb{C}[G]$ is defined by

$$(\sum_{j=1}^n \alpha_{g_j} g_j)^* = \sum_{j=1}^n \overline{\alpha_{g_j}} g_j^{-1}.$$

By definition, for any $f \in \mathbb{C}[G]$, we have $(f^*)^* = f$. Namely, the involution is reflexive (or reflective) or a reflexion (or reflection). Also, for $f_1, f_2 \in \mathbb{C}[G]$ and $\alpha, \beta \in \mathbb{C}$, we have

$$(\alpha f_1 + \beta f_2)^* = \overline{\alpha} f_1^* + \overline{\beta} f_2^*.$$

The involution is conjugate linear. With $f_1 = \sum_{j=1}^n \alpha_{g_j} g_j, f_2 = \sum_{k=1}^m \beta_{h_k} h_k$,

$$\begin{aligned} (f_1 f_2)^* &= (\sum_{j=1}^n \sum_{k=1}^m \alpha_{g_j} \beta_{h_k} g_j h_k)^* \\ &= \sum_{j=1}^n \sum_{k=1}^m \overline{\alpha_{g_j} \beta_{h_k}} (g_j h_k)^{-1} \\ &= \sum_{k=1}^m \overline{\beta_{h_k}} h_k^{-1} \sum_{j=1}^n \overline{\alpha_{g_j}} g_j^{-1} = f_2^* f_1^* \end{aligned}$$

The involution is anti-homomorphism.

The trace τ on $\mathbb{C}[G]$ to $\mathbb{C}$ is given as

$$\tau(\sum_{j=1}^{n} \alpha_{g_j} g_j) = \begin{cases} 0 & (g_j \neq 1_G, j = 1, \cdots, n), \\ \alpha_j & (g_j = 1_G). \end{cases}$$

This is certainly a positive linear functional on $\mathbb{C}[G]$ with $\tau(1_G) = 1$, i.e., a state and with traceness. Is this faithful? With $f = \sum_{j=1}^{n} \alpha_{g_j} g_j$, we compute

$$\begin{aligned} \tau(f^* f) &= \tau(\sum_{k=1}^{n} \overline{\alpha_{g_k}} g_k^{-1} \sum_{j=1}^{n} \alpha_{g_j} g_j) \\ &= \tau(\sum_{k=1}^{n} \sum_{j=1}^{n} \overline{\alpha_{g_k}} \alpha_{g_j} g_k^{-1} g_j) \\ &= \tau(\sum_{j=1}^{n} \overline{\alpha_{g_j}} \alpha_{g_j} 1_G) = \sum_{j=1}^{n} |\alpha_{g_j}|^2 = 0 \end{aligned}$$

does imply that $f = 0$. That's it faithfulness! □

Example 2.1.5. Let (X, μ) be a probability space with $X \subset \mathbb{R}^n$ for some integer $n \geq 1$ and Lebesgue like measure $\mu(X) = 1$. We denote by $L^1(X, \mu)$ the Banach $*$-algebra of all integrable measurable functions on (X, μ) up to measure zero sets by convolution. The convolution of $f, g \in L^1(X, \mu)$ is defined to be

$$(f \star g)(x) = \int_X f(x - y) g(y) d\mu(y) \in L^1(X, \mu).$$

Indeed, we have that an integration estimate and the Fubini theorem imply that

$$\begin{aligned} \|f \star g\|_1 &= \int_X |(f \star g)(x)| d\mu(x) \\ &\leq \int_X d\mu(x) \int_X |f(x - y) g(y)| d\mu(y) \\ &= \int_X |g(y)| d\mu(y) \int_X |f(x - y)| d\mu(x) = \|g\|_1 \|f\|_1 < \infty. \end{aligned}$$

Associability for the convolution may be checked by the Fubini. The involution for $f \in L^1(X, \mu)$ is just complex conjugate. Namely, $f^* = \overline{f}$. The integration $\int_X$ on $L^1(X, \mu)$ to $\mathbb{C}$ is a positive linear functional with $\int_X 1 d\mu(x) = 1$. Since $f \star g = g \star f$, then $\int_X$ is a tracial state. This is faithful. As well, we have a picture like that

$$\int_X : L^1(X, \mu) \to \mathbb{C} \cong \mathbb{C}1 \subset L^1(X, \mu)$$

where we identify $\mathbb{C}$ with $\mathbb{C}1$ a (Banach $*$-)subalgebra of $L^1(X, \mu)$. □

More generally, the following notion is considered.

Example 2.1.6. Let $\mathfrak{A}$ be a unital algebra and $\mathfrak{B}$ a unital subalgebra of $\mathfrak{A}$. A conditional expectation E from $\mathfrak{A}$ to $\mathfrak{B}$ is defined to be a linear map $E : \mathfrak{A} \to \mathfrak{B} \subset \mathfrak{A}$ such that

$$E(b) = b \quad b \in \mathfrak{B} \quad \text{and} \quad E(b_1 a b_2) = b_1 E(a) b_2 \quad a \in \mathfrak{A}, b_1, b_2 \in \mathfrak{B}.$$

We may say that the triple $(\mathfrak{A}, E, \mathfrak{B})$ is an operator probability space with operator integration.

For instance, $\int_X \alpha 1 d\mu(x) = \alpha$ for $\alpha \in \mathbb{C} \subset L^1(X, \mu)$. Also,

$$\int_X (\alpha_1 f(x) \alpha_2) d\mu(x) = \alpha_1 \int_X f(x) d\mu(x) \alpha_2.$$

Thus, the triple $(L^1(X, \mu), \int_X, \mathbb{C})$ is an example by functions as operators. □

Example 2.1.7. Let $(\mathfrak{A}, \varphi)$ be an operator probability space with $1 \in \mathfrak{A}$ and $\varphi(1) = 1$. Then $M_2(\mathbb{C})$ is a unital $*$-subalgebra of $M_2(\mathfrak{A})$ the 2×2 matrix algebra over $\mathfrak{A}$. Define a conditional expectation $E : M_2(\mathfrak{A}) \to M_2(\mathbb{C})$ to be

$$E \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix}.$$

We check that

$$E \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \varphi(\alpha_{11} 1) & \varphi(\alpha_{12} 1) \\ \varphi(\alpha_{21} 1) & \varphi(\alpha_{22} 1) \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_2(\mathbb{C}).$$

As well,

$$\begin{aligned}
&E \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \alpha'_{11} & \alpha'_{12} \\ \alpha'_{21} & \alpha'_{22} \end{pmatrix} \right] \\
&= E \left[\begin{pmatrix} \alpha_{11} b_{11} + \alpha_{12} b_{21} & \alpha_{11} b_{12} + \alpha_{12} b_{22} \\ \alpha_{21} b_{11} + \alpha_{22} b_{21} & \alpha_{21} b_{12} + \alpha_{22} b_{22} \end{pmatrix} \begin{pmatrix} \alpha'_{11} & \alpha'_{12} \\ \alpha'_{21} & \alpha'_{22} \end{pmatrix} \right] = \\
&E \begin{pmatrix} (\alpha_{11} b_{11} + \alpha_{12} b_{21}) \alpha'_{11} + (\alpha_{11} b_{12} + \alpha_{12} b_{22}) \alpha'_{21} & (\alpha_{11} b_{11} + \alpha_{12} b_{21}) \alpha'_{12} + (\alpha_{11} b_{12} + \alpha_{12} b_{22}) \alpha'_{22} \\ (\alpha_{21} b_{11} + \alpha_{22} b_{21}) \alpha'_{11} + (\alpha_{21} b_{12} + \alpha_{22} b_{22}) \alpha'_{21} & (\alpha_{21} b_{11} + \alpha_{22} b_{21}) \alpha'_{12} + (\alpha_{21} b_{12} + \alpha_{22} b_{22}) \alpha'_{22} \end{pmatrix} \\
&= \begin{pmatrix} (\alpha_{11} \varphi(b_{11}) + \alpha_{12} \varphi(b_{21})) \alpha'_{11} + (\alpha_{11} \varphi(b_{12}) + \alpha_{12} \varphi(b_{22})) \alpha'_{21} \\ \quad (\alpha_{11} \varphi(b_{11}) + \alpha_{12} \varphi(b_{21})) \alpha'_{12} + (\alpha_{11} \varphi(b_{12}) + \alpha_{12} \varphi(b_{22})) \alpha'_{22} \\ (\alpha_{21} \varphi(b_{11}) + \alpha_{22} \varphi(b_{21})) \alpha'_{11} + (\alpha_{21} \varphi(b_{12}) + \alpha_{22} \varphi(b_{22})) \alpha'_{21} \\ \quad (\alpha_{21} \varphi(b_{11}) + \alpha_{22} \varphi(b_{21})) \alpha'_{12} + (\alpha_{21} \varphi(b_{12}) + \alpha_{22} \varphi(b_{22})) \alpha'_{22} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} E \left[\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right] \begin{pmatrix} \alpha'_{11} & \alpha'_{12} \\ \alpha'_{21} & \alpha'_{22} \end{pmatrix}.
\end{aligned}$$

That's it conditional! □

An expectation from a unital C^*-algebra onto a subalgebra is defined to be a positive unital idempotent map (cf. [4]).

Example 2.1.8. A conditional expectation E from a unital C^*-algebra $\mathfrak{A}$ onto a C^*-subalgebra $\mathfrak{B}$ is an expectation.

We may refer to [22]. Let $\mathfrak{A}$ be a von Neumann algebra and $\mathfrak{B}$ a von Neumann subalgebra. A conditional expectation (co-expe) E from $\mathfrak{A}$ to $\mathfrak{B}$ is defined to be a linear map $E : \mathfrak{A} \to \mathfrak{B}$ which satisfies the following desirable conditions: (1) $E(b) = b$ for $b \in \mathfrak{B}$, (2) $E(a^*) = E(a)^*$ for $a \in \mathfrak{A}$, (3) $E(a^*a)$ is positive in $\mathfrak{B}$, (4) $\|E(a)\| \leq \|a\|$ for $a \in \mathfrak{A}$, (5) $E(a^*a) = 0$ if and only if $a = 0$. (6) $E(b_1ab_2) = b_1E(a)b_2$, (7) $E(a^*a) \geq E(a)^*E(a)$, (8) For the limit a of a monotone increasing sequence of positive elements $a_j \in \mathfrak{A}$, $E(a)$ is the limit of the monotone increasing sequence of $E(a_j) \in \mathfrak{B}$. (9) For any normal state φ of $\mathfrak{A}$, we have $\varphi(E(a)) = \varphi(a)$ for $a \in \mathfrak{B}$.

The co-expe map E on $\mathfrak{A}$ may be denoted as $E_\varphi(\cdot\,|\mathfrak{B})$.

Similarly, we may define a co-expe map from a (unital) C^*-algebra $\mathfrak{A}$ to a (unital) C^*-subalgebra $\mathfrak{B}$, where the (weak or strong) limit in (8) is replaced by the norm limit and normal states in (9) are replaced by just states.

Therefore, by this definition, in particular, a conditional expectation E is always positive. Since $E(1) = 1 \in \mathfrak{B} \subset \mathfrak{A}$, then E is unital. Also, $E(a) \in \mathfrak{B}$ for any $a \in \mathfrak{A}$. Thus, $E(E(a)) = E(a)$. Namely, $E^2 = E$. It says that E is an idempotent. It also follows from the condition (4) that the operator norm $\|E\| \leq 1$. It says that E is contractive. Since $\|E(1)\| = \|1\| = 1$, then we have the norm $\|E\|$ one. □

Example 2.1.9. The condition (2) like that

$$E\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}^*\right] = \begin{pmatrix} \varphi(a^*) & \varphi(c^*) \\ \varphi(b^*) & \varphi(d^*) \end{pmatrix} = \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix}^*$$

is provided if $\varphi(a^*) = \varphi(a)^* = \overline{\varphi(a)}$ for any $a \in \mathfrak{A}$.

A positive linear functional τ on a C^*-algebra $\mathfrak{A}$ (unital or not) has that $\tau(a^*) = \overline{\tau(a)}$ for $a \in \mathfrak{A}$.

Let (u_λ) be an approximate unit for $\mathfrak{A}$ as an increasing net of positive elements of the closed unit ball of $\mathfrak{A}$. Then

$$\tau(a^*) = \lim_\lambda \tau(a^*u_\lambda) = \lim_\lambda \overline{\tau(u_\lambda a)} = \overline{\tau(a)}.$$

Also, as for the condition (3), we have

$$E(M^*M) = E\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = E\begin{pmatrix} a^*a + c^*c & a^*b + c^*d \\ b^*a + d^*c & b^*b + d^*d \end{pmatrix}$$
$$= \begin{pmatrix} \varphi(a^*a) + \varphi(c^*c) & \varphi(a^*b) + \varphi(c^*d) \\ \varphi(b^*a) + \varphi(d^*c) & \varphi(b^*b) + \varphi(d^*d) \end{pmatrix} \in M_2(\mathbb{C}).$$

This matrix is self-adjoint. Is this positive? Let

$$N = \begin{pmatrix} a & z \\ \overline{z} & b \end{pmatrix}, \quad a, b \geq 0, z \in \mathbb{C}.$$

Then the eigen equation for the matrix N is

$$\begin{aligned}0 &= \det(\lambda 1_2 - N) = (\lambda - a)(\lambda - b) - |z|^2 \\ &= \lambda^2 - (a+b)\lambda + ab - |z|^2.\end{aligned}$$

This is solved as

$$\lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4(ab - |z|^2)}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4|z|^2}}{2}.$$

As well, we always have $a + b \geq |a - b|$. This is strictly positive if and only if $ab > 0$. Anyhow, the eigen values λ are not always positive. But for $ab > 0$, if a and b are near and if $|z|$ is small, then N can become positive.

A bounded operator T on a Hilbert space H is positive (definite), i.e., $\langle T\xi, \xi\rangle \geq 0$ for any $\xi \in H$, if and only if $T = S^*S$ for some $S \in \mathbb{B}(H)$, if and only if the spectrum of T is contained in the interval $[0, \infty) \subset \mathbb{R}$.

Consequently, the condition (3) may be removed from the definition from the first.

As for the condition (5), $E(M^*M) = 0 \in M_2(\mathbb{C})$ implies that $\varphi(a^*a) = 0$, $\varphi(c^*c) = 0$, $\varphi(b^*b) = 0$, and $\varphi(d^*d) = 0$. Thus, if φ is faithful, then $a = 0$, $c = 0$, $b = 0$, and $d = 0$, so that $M = 0$. The converse also holds.

As for the condition (7), we compute

$$\begin{aligned}E(M)^*E(M) &= \begin{pmatrix} \overline{\varphi(a)} & \overline{\varphi(c)} \\ \overline{\varphi(b)} & \overline{\varphi(d)} \end{pmatrix} \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix} \\ &= \begin{pmatrix} |\varphi(a)|^2 + |\varphi(c)|^2 & \overline{\varphi(a)}\varphi(b) + \overline{\varphi(c)}\varphi(d) \\ \overline{\varphi(b)}\varphi(a) + \overline{\varphi(d)}\varphi(c) & |\varphi(b)|^2 + |\varphi(d)|^2 \end{pmatrix}.\end{aligned}$$

Thus, we obtain

$$E(M^*M) - E(M)^*E(M) =$$
$$\begin{pmatrix} \varphi(a^*a) + \varphi(c^*c) - |\varphi(a)|^2 - |\varphi(c)|^2 & \varphi(a^*b) + \varphi(c^*d) - \overline{\varphi(a)}\varphi(b) - \overline{\varphi(c)}\varphi(d) \\ \varphi(b^*a) + \varphi(d^*c) - \overline{\varphi(b)}\varphi(a) - \overline{\varphi(d)}\varphi(c) & \varphi(b^*b) + \varphi(d^*d) - |\varphi(b)|^2 - |\varphi(d)|^2 \end{pmatrix}$$

with $\|\varphi\| = 1$ and $\varphi(a^*a) = \varphi(a^*a)\|\varphi\| \geq |\varphi(a)|^2$. This matrix is a type of the matrix N above. Consequently, the condition (7) may be removed from the (special) definition from the first.

As for the condition (4), we have

$$\begin{aligned}&\|E(M)\begin{pmatrix}\alpha \\ \beta\end{pmatrix}\|^2 = \|\begin{pmatrix}\varphi(a)\alpha + \varphi(b)\beta \\ \varphi(c)\alpha + \varphi(d)\beta\end{pmatrix}\|^2 \quad \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \in \mathbb{C}^2, \|\begin{pmatrix}\alpha \\ \beta\end{pmatrix}\| = 1 \\ &= |\varphi(a)\alpha + \varphi(b)\beta|^2 + |\varphi(c)\alpha + \varphi(d)\beta|^2.\end{aligned}$$

On the other hand, if E is positive, then it is bounded so that $\|E(M)\| \leq \|E\|\|M\|$ for any $M \in M_2(\mathfrak{A})$ with $\|E\| = \|E(1_2) = 1_2\| = 1$. That's it that the (4) holds by positiveness for E.

As for the condition (8), positiveness of E implies boundedness of E so that E is norm continuous.

As for the condition (9), we let ψ be a state on $M_2(\mathfrak{A})$.

By the way, a positive linear functional on a C^*-algebra is bounded. A bounded linear functional on a C^*-algebra is positive!

Anyhow, $M_2(\mathbb{C})^* \cong (\mathbb{C}^4)^* \cong \mathbb{C}^4$. As a possible choice, we can write

$$\psi(M) = \psi(a_{11}, a_{12}, a_{21}, a_{22}) = \sum_{i,j=1}^{2} \beta_{ij}\psi^{\sim}(a_{ij})$$

for some $\beta_{ij} \in \mathbb{C}$ and $\psi^{\sim} \in \mathfrak{A}^*$ with $\psi^{\sim}(1) = 1$. In this case, we also have

$$\psi(E(M)) = \sum_{i,j=1}^{2} \beta_{ij}\varphi(a_{ij}).$$

The condition requires to that $\varphi = \psi^{\sim}$. This seems to be that it is difficult in general. Thus, the condition may be removed from the (C^*-)definition from the first. □

2.2 Operator freeness

The operator freeness is omitted. Because we have outlooked the kindness.

There is another notion named as Boolean independence. As well there are two notions named as monotone independence and anti-monotone independence. We may refer to [13].

There are related concepts such as traffic freeness and matricial freeness. We may refer to [11] and [9] respectively.

The Voiculescu Bifreeness concept is obtained in [28].

Example 2.2.1. Let f, g be independent random variables of $L^\infty(X)$ with X a probability space by a probability measure μ in the classical sense that

$$E(fg) = \int_X f(x)g(x)d\mu(x) = E(f)E(g) = \int_X f(x)d\mu(x) \int_X g(x)d\mu(x).$$

It then follows that for any positive integers k, l, we have $E(f^k g^l) = E(f^k)E(g^l)$. Namely, f and g are independent in tensor. Perhaps, that's the definition.

As checked some above, on $L^\infty(X \times X) \cong L^\infty(X) \otimes L^\infty(X)$ with $\mu \times \mu$ and $\mu \otimes \mu$ respectively, we certainly have

$$E[x^k \otimes y^l] = \int_{X\times X} x^k \otimes y^l d(\mu \otimes \mu)(x, y) =$$
$$\int_{X\times X} x^k \otimes 1 d(\mu \otimes \mu)(x, y) \int_{X\times X} 1 \otimes y^l d(\mu \otimes \mu)(x, y)$$
$$= E[x^k \otimes 1]E[1 \otimes y^l].$$

What we need to have is the following?

$$E(a^k b^l) = \Pi_{i=1}^k E(a)\Pi_{j=1}^l E(b), \quad E(a^k) = \Pi_{i=1}^k E(a), \quad E(b^l) = \Pi_{j=1}^l E(b).$$

This seems to be wrong. That claim says that multiples f^k and g^l are independent from independence of f and g. □

Example 2.2.2. We consider the free group F_n of n generators. We have $F_{n+m} = F_n * F_m$ the free product. The groups F_n and F_m sit freely inside F_{n+m}.

Let $G = F_{n+m}$ of generators x_j for $1 \leq j \leq n+m$ with no relations and with 1 the unit of G. Let G_1 and G_2 be subgroups of G generated by $x_1, \cdots, x_n$ and $x_{n+1}, \cdots, x_{n+m}$ respectively. Then G_1 and G_2 are free in the sense that for $g_j \in G_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_k$ with $g_j \neq 1$, we have $g_1 \cdots g_n \neq 1$. It says that no relations between G_1 and G_2. This may be the definition of freeness of two subgroups of a group.

That notion can be applied to the case of group algebras. Let $A = \mathbb{C}F_{n+m}$, $A_1 = \mathbb{C}F_n$, and $A_2 = \mathbb{C}F_m$ be the group algebras of free groups. For finite sums $a_j = \sum_l \alpha_{g_l} g_l \in A_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_k$ such that all $g_l \neq 1$, we have $a_1 \cdots a_k = \sum \beta_g g \in A$ has no term for $g = 1$.

We would like to formalize free sitting of the free group operator factors LF_n and LF_m inside LF_{n+m}. This seems to be difficult to say so because of topology.

Those group algebras of free groups are also said to be free if we have the trace $\tau(a_j) = 0$ for such indices j, then $\tau(a_1 \cdots a_n) = 0$.

This notion by trace can be extended to the case of the enveloping von Naumann group algebras with strong topology as well as the group C^*-algebras with operator norm topology. □

Example 2.2.3. Let H be a Hilbert space. The full Fock space for H is defined to be l^2 direct sum Hilbert space of tensor product Hilbert spaces as

$$F(H) = \mathbb{C}\omega \oplus [\oplus_{n=1}^{\infty} (\otimes^n H)]$$

where ω is the vaccum vector with norm 1. The C^*- and von Naumann algebra $\mathbb{B}(F(H)) = \mathbb{B}$ of bounded operators on $F(H)$ is viewed as an operator probability space with the vector space $\varphi(T) = \langle T\omega, \omega \rangle$ for $T \in \mathbb{B}$, so $\varphi(1) = \|\omega\|^2 = 1$.

The left creation operator $l(\xi)$ for $\xi \in H$ is defined by $l(\xi)\omega = \xi \in H$ and

$$l(\xi)(\eta_1 \otimes \cdots \eta_k) = \xi \otimes \eta_1 \otimes \cdots \otimes \eta_k \in \otimes^{k+1} H, \quad k \geq 1.$$

The adjoint operator $l(\xi)^*$ is the left annihilation operator.

For an orthonormal system $\{\xi_1, \cdots, \xi_k\}$ of H, the operators $l(\xi_1), \cdots, l(\xi_k)$ with their adjoints are (dual) free in $\mathbb{B}$ with respect to φ in the sense that $*$-algebras A_j generated by $l(\xi_j)$ and $l(\xi_j)^*$, $1 \leq j \leq k$ are free by φ. Namely, if $\varphi(a_j) = 0$ for $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_l$, then $\varphi(a_1 \cdots a_l) = 0$.

Note that we have $l(\xi_j)^* l(\xi_j) = 1$, so $\varphi(1) \neq 0$. Also, $\varphi(l(\xi_j)) = 0$. As well, $l(\xi)^*\omega = 0$, so $\varphi(l(\xi)^*) = 0$.

Similarly, the right creation operators $r(\xi)$ are defined by tensor placing ξ to the right side of tensor vectors of $F(H)$. The play of left and right creation operators on the Fock start with to the Voiculescu bi-freenss. $\square$

$\star$ Let $\xi, \eta \in H$ be orthogonal, so $\langle \xi, \eta \rangle = 0$. Then we have

$$\begin{aligned}
&\langle l(\xi)(\alpha, \oplus_{j=1}^{\infty} x_j), l(\eta)(\beta, \oplus_{j=1}^{\infty} y_j)\rangle \\
&= \langle (0, \alpha\xi, \oplus_{j=1}^{\infty} \xi \otimes x_j), (0, \beta\eta, \oplus_{j=1}^{\infty} \eta \otimes y_j)\rangle \\
&= \alpha\overline{\beta}\langle \xi, \eta\rangle + \sum_{j=1}^{\infty} \langle \xi, \eta\rangle \langle x_j, y_j\rangle = 0.
\end{aligned}$$

It thus follows that the ranges of $l(\xi)$ and $l(\eta)$ are orthogonal. Hence the products $l(\eta)^* l(\xi)$ and $l(\xi)^* l(\eta)$ are zero. But their products $l(\eta)l(\xi)$ and $l(\xi)l(\eta)$ and their adjoints may be not. $\square$

Tensor like classical independence for two elements a, b of an operator probability space $(\mathfrak{A}, \varphi)$ with moments $\varphi(a^n)$ and $\varphi(b^m)$ for n, m natural numbers assumed to be known implies that the mixed moments $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ are obtained so. We now assume that $a, b \in \mathfrak{A}$ are freely independent and their moments are known. Then we can obtain mixed moments as $\varphi(a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k})$ as poylnomials in the moments of a and b.

$\star$ Since $\varphi(a - \varphi(a)1) = 0$ and $\varphi(b - \varphi(b)1) = 0$, then we obtain by freeness that

$$\begin{aligned}
0 &= \varphi((a - \varphi(a)1)(b - \varphi(b)1)) \\
&= \varphi(ab - \varphi(b)a - \varphi(a)b + \varphi(a)\varphi(b)1) = \varphi(ab) - \varphi(b)\varphi(a).
\end{aligned}$$

Also, the freeness implies that

$$\begin{aligned}
0 &= \varphi((a - \varphi(a)1)(b - \varphi(b)1)(a - \varphi(a)1)) \\
&= \varphi(\{ab - \varphi(b)a - \varphi(a)b + \varphi(a)\varphi(b)1\}(a - \varphi(a)1)) \\
&= \varphi(\{aba - \varphi(b)a^2 - \varphi(a)ba + \varphi(a)\varphi(b)a\}) \\
&\quad - \varphi(\{\varphi(a)ab - \varphi(b)\varphi(a)a - \varphi(a)^2 b + \varphi(a)^2\varphi(b)1\}) \\
&= \varphi(aba) - \varphi(b)\varphi(a^2) - \varphi(a)\varphi(ba) + \varphi(a)^2\varphi(b) \\
&\quad - \varphi(a)\varphi(ab) - \varphi(b)\varphi(a)^2 + \varphi(a)^2\varphi(b) - \varphi(a)^2\varphi(b) \\
&= \varphi(aba) - \varphi(b)\varphi(a^2).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
0 &= \varphi((a-\varphi(a)1)(b-\varphi(b)1)(a-\varphi(a)1)(b-\varphi(b)1)) \\
&= \varphi(\{aba-\varphi(b)a^2-\varphi(a)ba+\varphi(a)\varphi(b)a\}(b-\varphi(b)1)) \\
&\quad -\varphi(\{\varphi(a)ab-\varphi(b)\varphi(a)a-\varphi(a)^2b+\varphi(a)^2\varphi(b)1\}(b-\varphi(b)1)) \\
&= \varphi(abab)-\varphi(b)\varphi(a^2b)-\varphi(a)\varphi(bab)+\varphi(a)^2\varphi(b)^2 \\
&\quad -\varphi(aba)\varphi(b)+\varphi(b)^2\varphi(a^2)+\varphi(a)\varphi(ba)\varphi(b)-\varphi(a)^2\varphi(b)^2 \\
&\quad -\varphi(a)\varphi(ab^2)+\varphi(b)\varphi(a)\varphi(ab)+\varphi(a)^2\varphi(b^2)-\varphi(a)^2\varphi(b)^2 \\
&\quad +\varphi(a)\varphi(ab)\varphi(b)-\varphi(b)^2\varphi(a)^2-\varphi(a)^2\varphi(b)^2+\varphi(a)^2\varphi(b)^2 \\
&= \varphi(abab)-\varphi(b)\varphi(a^2b)+\varphi(b)^2\varphi(a^2)-\varphi(a)\varphi(ab^2)+\varphi(a)^2\varphi(b^2)-\varphi(a)^2\varphi(b)^2.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
0 &= \varphi((a^2-\varphi(a^2)1)(b-\varphi(b)1)) \\
&= \varphi(a^2b-\varphi(b)a^2-\varphi(a^2)b+\varphi(a^2)\varphi(b)1) \\
&= \varphi(a^2b)-\varphi(a^2)\varphi(b).
\end{aligned}
$$

It then follows by our computation that $\varphi(abab)=\varphi(a)^2\varphi(b)^2$. This seems to be correct. □

2.3 Operator like distributions

Let $(\mathfrak{A},\varphi)$ be an operator like probability space. For elements $a_1,\cdots,a_n \in \mathfrak{A}$, their joint distribution is defined to be the set of all joint moments $\varphi(a_{i_1}\cdots a_{i_m})$ for $1 \leq i_k \leq n$, $1 \leq k \leq m$, and $m \in \mathbb{N}$.

We denote by $\mathbb{C}^*[X_1,\cdots,X_n]$ the algebra of all polynomials over $\mathbb{C}$ in mutually non-commuting variables $X_1,\cdots,X_n$.

The joint distribution for the elements $a_1,\cdots,a_n \in \mathfrak{A}$ may be also defined to be the linear functional $\mu:\mathbb{C}^*[X_1,\cdots,X_n]\to\mathbb{C}$ given by

$$\mu(p(X_1,\cdots,X_n))=\varphi(p(a_1,\cdots,a_n)).$$

Similarly, the joint $*$-distribution for $a_1,\cdots,a_n \in \mathfrak{A}$ with their adjoints in $\mathfrak{A}$ is defined to be the joint distribution for $a_1,\cdots,a_n \in \mathfrak{A}$ as well as their adjoints involved.

For a normal $a \in \mathfrak{A}$, i.e. $aa^*=a^*a$, there is a compactly supported measure μ on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} z^k\overline{z}^l d\mu(z)=\varphi(a^k(a^*)^l), \quad k,l\in\mathbb{N}.$$

If $a=a^*$ is self-adjoint, then the measure is compactly supported on $\mathbb{R}$.

This is an analytic interpretation as a distribution of moments in one variable.

We now consider operator probability spaces $(\mathfrak{A}, \varphi)$ and $(\mathfrak{A}_n, \varphi_n)$ for $n \in \mathbb{N}$. Let J be some index set. A family $(a_{n,j})_{n\in\mathbb{N}, j\in J}$ of a family $(a_{n,j})_{n\in\mathbb{N}}$ for $j \in J$ and $a_{n,j} \in \mathfrak{A}_n$ converges in distribution to a family $(a_j)_{j\in J}$ with $a_j \in \mathfrak{A}$ if

$$\lim_{n\to\infty} \varphi_n(a_{n,i_1} \cdots a_{n,i_k}) = \varphi(a_{i_1} \cdots a_{i_k})$$

for any $k \in \mathbb{N}$ and $i_1, \cdots, i_k \in J$. Similarly, convergence in $*$-distribution is defined as that the same limit holds for joint $*$-moments involved. As well, such a family of operator random variables is said to be asymptotically free if the family converges in distribution to some family of free elements.

2.4 Operator like distributions by examples

Example 2.4.1. Let (X, μ) be a classical probability space with $\mu(X) = 1$. The expectation functional $\varphi = E$ on $L^\infty(X, \mu) = L^\infty(X)$ is defined to be

$$\varphi(f) = E(f) = \int_X f(x)d\mu(x), \quad f \in L^\infty(X).$$

Suppose that $X \subset \mathbb{R}$ and $f \in L^\infty(X)$ with $f : X \to \mathbb{R}$. We may refer to [33]. Then the distribution function F for f on $\mathbb{R}$ is defined to be

$$F(y) = \mu(\{x \in X \mid f(x) \leq y\}), \quad y \in \mathbb{R}.$$

A distribution function F is said to be absolutely continuous if there is a non-negative measurable density function g on $\mathbb{R}$ to $\mathbb{R}$ such that

$$F(y) = \int_{-\infty}^{y} g(t)dt$$

the Legesgue or Riemann broad (or wide) sense integral.

The (Lebesque) integral of f with respect to F is defined to be $\int_{\mathbb{R}} f(x)dF(x)$ as a Lebesgue-Stieltjes integral with respect to real-valued functions with bounded variation. The expectation for f with respect to F is defined to be this integral. $\square$

$\star$ We have $\mu(a < f \leq b) = F(b) - F(a)$ for $a < b \in \mathbb{R}$. The function $F(y)$ is monotone non-descreasing and continuous from the right on $\mathbb{R}$, taking values 0 and 1 at infinities $-\infty$ and ∞ respectively.

Example 2.4.2. Let $\varphi = \frac{1}{n}\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ be the normalized trace with $\varphi(1) = 1$. Let A be a normal (or self-adjoint) matrix in $M_n(\mathbb{C})$. Let $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ (or real $\mathbb{R}$) be the eigenvalues of A with multiplicity counted. We can diagonalize the normal matrix A by a unitary matrix U by using Linear Algebra.

$\star$ Namely,

$$U^*AU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

It then follows that

$$\varphi(A) = \varphi(U^*AU) = \frac{1}{n}\sum_{j=1}^{n}\lambda_j = \frac{1}{n}\sum_{j=1}^{n}\lambda_j \int_{\mathbb{C}} d\delta_{\lambda_j}(z)$$
$$= \int_{\mathbb{C}} d\left(\frac{1}{n}\sum_{j=1}^{n}\lambda_j\delta_{\lambda_j}\right)(z) \equiv \int_{\mathbb{C}} d\mu(z)$$

where δ_{λ_j} is the Dirac measure on the point λ_j, and μ is a complex measure defined so above. □

Example 2.4.3. A semi-circle or a semi-circular element with variance σ^2 is given by an self-adjoint element s of an operator probability space $(\mathfrak{A}, \varphi)$ such that its moments are given by

$$\varphi(s^{2m}) = \sigma^{2m}\frac{1}{m+1}\binom{2m}{m} \equiv \sigma^{2m}\mathrm{Ct}_m, \quad \varphi(s^{2m+1}) = 0$$

for $m \in \mathbb{N}$, where we denote by Ct_m the m-th Catalan number as a binomial coefficient defined so as in combinatorics. If the variance $\sigma^2 = 1$, then s is said to be standard.

⋆ We have the Catalan numbers computed as $\mathrm{Ct}_1 = 1$, and

$$\mathrm{Ct}_2 = \frac{1}{3}\binom{4}{2} = \frac{1}{3}\frac{4!}{2!2!} = 2, \quad \mathrm{Ct}_3 = \frac{1}{4}\frac{6!}{3!3!} = 5,$$
$$\mathrm{Ct}_4 = \frac{1}{5}\frac{8!}{4!4!} = \frac{8\cdot 7\cdot 6}{4\cdot 3\cdot 2} = 14, \quad \mathrm{Ct}_5 = \frac{1}{6}\frac{10!}{5!5!} = \frac{10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2} = 42,$$
$$\mathrm{Ct}_m = \frac{1}{m+1}\frac{2m!}{m!m!} = \frac{2m(2m-1)(2m-2)\cdots(m+2)}{m(m-1)(m-2)\cdots 2}$$

with cancellation such as $\frac{2m}{m\cdot 2} = 1$, $\frac{2m-2}{m-1} = 2$ and so.

The Catalan numbers are either the numbers of non-crossing (any) pair (like) partitions of the set of $2m$ elements like $\{1, 2, \cdots, 2m\}$, denoted as $NCr_2(2m)$, or the numbers of non-crossing partitions of the set of m elements, denoted as $NCr(m)$.

⋆ Note that the numbers $NCr_2(2m)$ are given like by

$$\{1,2\} = \{1,2\},$$
$$\{1,2,3,4\} = \{1,2,3,4\} = \{1,2\} \sqcup \{3,4\},$$
$$\{1,2,3,4,5,6\} = \{1,2,3,4,5,6\}$$
$$= \{1,2\} \sqcup \{3,4\} \sqcup \{5,6\} = \{1,2,3,4\} \sqcup \{5,6\}$$
$$= \{1,2\} \sqcup \{3,4,5,6\} = \{1,2,3\} \sqcup \{4,5,6\}.$$

Also the numbers $NCr(m)$ are given like by

$$
\begin{aligned}
&\{1\} = \{1\}, \quad \{1,2\} = \{1,2\} = \{1\} \sqcup \{2\}, \\
&\{1,2,3\} = \{1,2,3\} = \{1\} \sqcup \{2\} \sqcup \{3\} \\
&= \{1,2\} \sqcup \{3\} = \{1\} \sqcup \{2,3\} = \{1,3\} \sqcup \{2\}, \\
&\{1,2,3,4\} = \{1,2,3,4\} = \{1\} \sqcup \{2\} \sqcup \{3\} \sqcup \{4\} \\
&= \{1,2\} \sqcup \{3\} \sqcup \{4\} = \{1\} \sqcup \{2,3\} \sqcup \{4\} = \{1\} \sqcup \{2\} \sqcup \{3,4\} \\
&= \{1,3\} \sqcup \{2\} \sqcup \{4\} = \{1,4\} \sqcup \{2\} \sqcup \{3\} = \{1\} \sqcup \{2,4\} \sqcup \{3\} \\
&= \{1,2\} \sqcup \{3,4\} = \{1,4\} \sqcup \{2,3\} = \{1,2,3\} \sqcup \{4\} \\
&= \{1\} \sqcup \{2,3,4\} = \{1,2,4\} \sqcup \{3\} = \{1,3,4\} \sqcup \{2\}.
\end{aligned}
$$

The (operator) cumulants $k_n : \mathfrak{A}^n \to \mathbb{C}$ of such a semi-circle element s is given by $k_1 = \varphi$,

$$k_2(s,s) = \sigma^2, \quad k_n(s, \cdots, s) = 0, \quad n \geq 3.$$

The corresponding measure to φ has the (upper) semi-circle density function

$$\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - t^2}, \quad t \in [-2\sigma, 2\sigma].$$

$\star$ Note that the semi-circle disk denoted as D_{sc} has the volume known as

$$|D_{sc}| = \int_{-2\sigma}^{2\sigma} \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - t^2}dt = \frac{1}{4\pi\sigma^2}\pi(2\sigma)^2 = 1.$$

A semi-circle in operator probability plays the role of the Gaussian (function or measure) in classical probability. That is the limit distribution by the central limit theorem. □

$\star$ The Gauss function is defined to be

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

for some non-negative constants m and σ.

Let $f(x)$ be a random variable on a probability space $(\mathbb{R}, \mu)$. Suppose that

$$\mu(a \leq f \leq b) = \int_a^b G(y)dy.$$

In this case, we say that the distribution for f is Gaussian or normal distribution denoted as $N(m, \sigma^2)$ with m mean and σ^2 variance, where

$$m = \int_\alpha^\beta yG(y)dy, \quad \sigma^2 = \int_\alpha^\beta (y-m)^2 G(y)dy,$$

where $\alpha = \inf_{x\in\mathbb{R}} f(x)$ and $\beta = \sup_{x\in\mathbb{R}} f(x)$.

Let $f^{\sim} = \frac{1}{\sigma}(f - m)$. Then the distribution for $f^{\sim}$ is the standard normal distribution $N(0, 1^2)$ with density function as Gaussian $\frac{1}{2\pi}e^{-\frac{1}{2}x^2}$.

The central limit theorem is stated as follows. Suppose that a sequence of mutually independent random variables (f_j) on some probability space $(\mathbb{R}, \mu)$ have the same distribution and that $E(f_j^2) < \infty$, so that $\mu = E(f_j)$ and $\sigma^2 = V(f_j) = E(f_j^2) - E(f_j)^2$ for any $j \in \mathbb{N}$. For mean functions $M_n = \frac{1}{n}\sum_{j=1}^n f_j$, the following holds as $n \to \infty$,

$$\mu(\frac{M_n - E(M_n)}{\sqrt{V(M_n)}} \le y) = \mu(\frac{M_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} \le y) \to \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

The proof is omitted. In that case we note that

$$E(M_n) = \frac{1}{n}E(\sum_{j=1}^n f_j) = \frac{1}{n}\sum_{j=1}^n E(f_j) = \mu,$$

$$V(M_n) = \frac{1}{n^2}V(\sum_{j=1}^n f_j) = \frac{1}{n^2}\sum_{j=1}^n V(f_j) + \frac{1}{n^2}\sum_{j<k}\mathrm{Cov}(f_j, f_k) = \frac{\sigma^2}{n},$$

with covariance

$$\begin{aligned}
&\mathrm{Cov}(f_j, f_k) = E((f_j - \mu)(f_k - \mu)) \\
&= E(f_j f_k) - \mu E(f_k) - \mu E(f_j) + \mu^2 E(1) \\
&= E(f_j)E(f_k) - \mu^2 = 0. \quad \square
\end{aligned}$$

Example 2.4.4. Let $l^2(\mathbb{N})$ be the Hilbert space of square summable complex-valued functions on $\mathbb{N}$. Let (e_n) be the canonical orthonormal basis for $l^2(\mathbb{N})$. Namely, $e_n(n) = 1$ and $e_n(m) = 0$ for $m \ne n \in \mathbb{N}$. The unilateral shift S on $l^2(\mathbb{N})$ is defined by $S(e_n) = e_{n+1}$ for $n \ge 1$. Let $\varphi : \mathbb{B}(l^2(\mathbb{N})) \to \mathbb{C}$ be the state on the C^*-algebra $\mathbb{B}(l^2(\mathbb{N}))$ defined by $\varphi(x) = \langle xe_1, e_1\rangle$ for $x \in \mathbb{B}(l^2(\mathbb{N}))$.

The shift S is an isometry, so $S^*S = 1$. Thus, $\|S\|^2 = \|S^*S\| = 1$.

The sum $S + S^*$ is a (standard) semi-circle with respect to φ.

Let $F(H)$ be the Fock space by some Hilbert space H, with ω the vacuum vector. Let $l(\xi)$ be the left creation operator on $F(H)$ for $\xi \in H$. Let $\varphi : \mathbb{B}(F(H)) \to \mathbb{C}$ be defined by $\varphi(x) = \langle x\omega, \omega\rangle$. Then $l(\xi) + l(\xi)^*$ is a semi-circle with respect to φ, with $\sigma^2 = \|\xi\|$.

Let $\{\xi_1, \cdots, \xi_n\} \subset H$ be an orthonormal system. Then the operators $l(\xi_j) + l(\xi_j)^*$ for $1 \le j \le n$ are standard semi-circular elements that are free.

$\star$ We compute that

$$\begin{aligned}
&\varphi((S + S^*)^2) = \varphi(S^2 + SS^* + S^*S + (S^*)^2) \\
&= \langle S^2 e_1, e_1\rangle + \langle SS^* e_1, e_1\rangle + 1 + \langle S^* 0, e_1\rangle \\
&= 1 = \sigma^2 \mathrm{Ct}_1 = \sigma^2.
\end{aligned}$$

Also,

$$
\begin{aligned}
&\varphi((S+S^*)^3) = \varphi((S^2+SS^*+S^*S+(S^*)^2)(S+S^*)) \\
&= \varphi(S^3+S+S^*S^2+(S^*)^2S) \\
&= \langle e_4, e_1\rangle + \langle e_2, e_1\rangle + \langle e_2, e_1\rangle + \langle 0, e_1\rangle = 0.
\end{aligned}
$$

⋆ We also compute that

$$
\begin{aligned}
&\varphi(l(\xi_j)+l(\xi_j)^*) = \langle 0\oplus \xi_j, \omega\oplus 0\rangle + \langle 0,\omega\rangle = 0, \\
&\varphi((l(\xi_j)+l(\xi_j)^*)(l(\xi_k)+l(\xi_k)^*)) \\
&= \langle \xi_j\otimes\xi_k, \omega\rangle + \langle l(\xi_j)^*\xi_k, \omega\rangle = \begin{cases} 0 & j\neq k, \\ 1 & j=k. \end{cases} \qquad \square
\end{aligned}
$$

Example 2.4.5. Let $s_1 = s_1^*, s_2 = s_2^*$ be free standard semi-circle operators with respect to φ. A circular element for these is defined to be $c = \frac{1}{\sqrt{2}}(s_1+is_2)$. Then c is not normal.

Note that $\|\frac{1}{\sqrt{2}}(x+iy)\| = \sqrt{\frac{x^2+y^2}{2}}$ for $x+iy\in\mathbb{C}$, $x,y\in\mathbb{R}$. In particular, we have $\|\frac{1}{\sqrt{2}}(1+i1)\| = |\frac{1}{\sqrt{2}}(1+i)|\|1\| = 1$.

We compute that

$$
\begin{aligned}
c^*c &= \frac{1}{2}(s_1-is_2)(s_1+is_2) = \frac{1}{2}(s_1^2+s_2^2+i(s_1s_2-s_2s_1)). \\
cc^* &= \frac{1}{2}(s_1+is_2)(s_1-is_2) = \frac{1}{2}(s_1^2+s_2^2-i(s_1s_2-s_2s_1)).
\end{aligned}
$$

Thus, c is normal if and only if s_1 and s_2 commute.

Being free implies being non-commutative? Or being commutative implies being non-free? If so, that's the reason for being non-normal. It seems to be difficult at this moment.

The answer to the question above is false in general. Any operator and the identity operator are freely independent and commute. Because for any state φ, we have $\varphi(1) = 1 \neq 0$ and the algebra generated by 1 is $\mathbb{C}$. But if not constant, how much about the question? As well, the identity operator is not a semi-circle, since $\varphi(1^n) = 1\neq 0$ for any $n\in\mathbb{N}$.

We consider moment expressions like

$$
\varphi(c^{a_1}(c^*)^{b_1}c^{a_2}\cdots(c^*)^{b_n}), \quad a_j, b_j \in \{0\}\cup\mathbb{N}.
$$

If $\sum_{j=1}^n a_j \neq \sum_{j=1}^n b_j$, then the moment vanishes.

Since $\varphi(s_1) = \varphi(s_2) = 0$, then $\varphi(c) = \varphi(c^*) = 0$.

Since $\varphi(s_1s_2) = \varphi(s_2s_1) = 0$, then

$$
\varphi(c^*c) = \varphi(cc^*) = \frac{1}{2}(\varphi(s_1^2)+\varphi(s_2^2)) = \sigma^2 = 1.
$$

We also have

$$\begin{aligned} cc^*c &= \frac{1}{\sqrt{2}}(s_1 + is_2)\frac{1}{2}(s_1^2 + s_2^2 + i(s_1s_2 - s_2s_1)) \\ &= \frac{1}{2\sqrt{2}}(s_1^3 + s_1s_2^2 + i(s_1^2s_2 - s_1s_2s_1)) \\ &\quad + \frac{1}{2\sqrt{2}}(i(s_2s_1^2 + s_2^3) - (s_2s_1s_2 - s_2^2s_1)), \end{aligned}$$

so that $\varphi(cc^*c) = 0$ by free independence for s_1 and s_2 with respect to φ.

The cumulants are given by

$$k_2(c, c^*) = k_2(c^*, c) = 1$$

and the other cumulants are equal to zero.

Note that $k_1 = \varphi$. Check that

$$k_2(c, c^*) = \varphi(cc^*) - \varphi(c)\varphi(c^*) = 1 - 0 = 1,$$

with $\varphi(a_1a_2) = k_2(a_1, a_2) + k_1(a_1)k_2(a_2)$.

Being non-normal of c implies that there does not exist such a measure corresponding to φ at $*$-multiples of c. Then to what it corresponds? A nice question? Anyhow, by normality we can use C^*-algebra representation theory for normal operators. Such a normal operator commuting with its adjoint is represented as the complex variable function on the spectrum contained in $\mathbb{C}$. □

Example 2.4.6. A Haar unitary is defined to be a unitary u in an operator probability space $\mathfrak{A}$ with φ such that the moments $\varphi(u^k)$ and $\varphi((u^*)^k)$ for integers $k \geq 1$ are zero. The cumulants non-vanishing only have the form

$$k_{2m}(u, u^*, \cdots, u, u^*) = k_{2m}(u^*, u, \cdots, u^*, u) = (-1)^{m-1}\mathrm{Ct}_{m-1}.$$

The measure corresponds to the normalized Lebesgue (or Haar unitary) measure μ on the unit circle S^1 in $\mathbb{C}$, in which the spectrum $\mathrm{sp}(u)$ of u is contained.

We have

$$k_1(u) = \varphi(u) = \int_{S^1} z d\mu(z) = 0.$$

Is the spectrum of u full? As well, with $\mathrm{Ct}_0 = 1$,

$$k_2(u, u^*) = \varphi(uu^*) - \varphi(u)\varphi(u^*) = 1 = (-1)^{1-1}\mathrm{Ct}_{1-1}. \quad \square$$

Example 2.4.7. A (mod) k-Haar unital for $k \in \mathbb{N}$ is defined to be a unitary $u \in \mathfrak{A}$ an operator probability space with φ such that $u^k = 1$ and the moments are given as $\varphi(u^m) = 0$ and $\varphi((u^*)^m) = 0$ for positive integers m which can not be divided by k, denoted as $k \nmid m$. The corresponding measure is the uniform (Dirac like?) measure μ_k on the set R_k of all k-th roots of the unity.

We have

$$k_1(u) = \varphi(u) = \frac{1}{k}\sum_{z\in R_k} z = 0 = \int_{R_k} z d\mu_k(z)$$

and $\varphi(1) = \frac{1}{k}k1 = 1$. Is this so? Note that

$$0 = z^k - 1 = (z-1)(z^{k-1} + \cdots + 1).$$

There are no nice formula of cumulants for u, except for the case $k = 2$.

If $u^2 = 1$, then $\varphi(u^{2m}) = 1$ with $2 \mid 2m$ and $\varphi(u^{2m+1}) = 0$ with $2 \nmid 2m+1$. □

Example 2.4.8. Let u be a Haar unitary with respect to φ. The distribution of $u + u^*$ is said to be the arcsine law.

The moments of $u + u^*$ are given as

$$\varphi((u+u^*)^{2m}) = \binom{2m}{m}, \quad \varphi((u+u^*)^{2m+1}) = 0.$$

We have $\varphi(u+u^*) = \varphi(u) + \varphi(u^*) = 0 + 0 = 0$. As well,

$$\varphi((u+u^*)(u+u^*)) = \varphi(u^2 + uu^* + u^*u + (u^*)^2) = 0 + 2 + 0 = \binom{2}{1}.$$

Moreover,

$$\varphi((u+u^*)^3) = \varphi(u^3 + u + u + (u^*)^2u) + \varphi(u^2u^* + u^* + u^* + (u^*)^3) = 0.$$

Furthermore,

$$\varphi((u+u^*)^4) = \varphi(u^4 + u^2 + u^2 + (u^*)^2u^2) + \varphi(u^2 + 1 + 1 + (u^*)^3u)$$
$$+ \varphi(u^2 + 1 + 1 + (u^*)^2) + \varphi(u^2(u^*)^2 + (u^*)^2 + (u^*)^2 + (u^*)^4) = 6 = \binom{4}{2}.$$

The cumulants of $u + u^*$ are given as

$$\begin{cases} k_{2m}(u+u^*, \cdots, u+u^*) = 2(-1)^{m-1}\mathrm{Ct}_{m-1}, \\ k_{2m+1}(u+u^*, \cdots, u+u^*) = 0. \end{cases}$$

Note that $k_1(u+u^*) = \varphi(u+u^*) = 0$. As well,

$$k_2(u+u^*, u+u^*) = \varphi((u+u^*)^2) - \varphi(u+u^*)^2$$
$$= 2 = 2(-1)^{1-1}\mathrm{Ct}_0.$$

The density function of the arcsine law is given by $\frac{1}{\pi\sqrt{4-t^2}}$ for $t \in (-2, 2)$. Note that $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$. As well,

$$\int_{-2}^{2} \frac{1}{\pi\sqrt{4-t^2}} dt = \int_{-2}^{2} \frac{1}{2\pi\sqrt{1-(\frac{t}{2})^2}} dt \quad (\frac{t}{2} = s),$$
$$= \int_{-1}^{1} \frac{1}{2\pi\sqrt{1-s^2}} 2ds = \frac{1}{\pi}[\arcsin s]_{s=-1}^{1} = 1. \qquad □$$

Example 2.4.9. An operator $a \in \mathfrak{A}$ an operator probability space with φ is said to be a free Poisson with rate $\lambda \geq 0$ and jump size $\alpha \in \mathbb{R}$ or to be the Marchenko-Pastur law, if the moments are given as

$$\varphi(a^n) = \alpha^n \sum_{k=1}^{n} \frac{\lambda^k}{n-k+1} \binom{n}{k} \binom{n-1}{k-1}.$$

In particular, $\varphi(a) = \alpha\lambda\binom{1}{1} = \alpha\lambda$.

The cumulants are given as $k_n(a, \cdots, a) = \lambda\alpha^n$. In particular, we have

$$\begin{aligned} k_2(a,a) &= \varphi(a^2) - \varphi(a)^2 \\ &= \alpha^2(\frac{\lambda}{2}\binom{2}{1} + \lambda^2\binom{2}{2}) - (\alpha\lambda)^2 = \alpha^2\lambda. \end{aligned}$$

The measure of the free Poisson law with rate $\lambda \geq 0$ is given by $(1-\lambda)\delta_0 + \nu$ if $0 \leq \lambda \leq 1$, and by ν if $\lambda > 1$, where δ_0 is the Dirac at 0 and ν has density

$$\frac{1}{2\pi\alpha t}\sqrt{4\lambda\alpha^2 - (t - \alpha(1+\lambda))^2}, \quad t \in [\alpha(1-\sqrt{\lambda})^2, \alpha(1+\sqrt{\lambda})^2].$$

The square s^2 of a semi-circle element s of variance σ^2 is a free Poisson element with rate $\lambda = 1$ and jump size $\alpha = \sigma^2$.

Note that $\varphi(s^2) = \sigma^2 = \alpha = \alpha\lambda$. As well,

$$\varphi((s^2)^2) = \sigma^4 \mathrm{Ct}_2 = 2\sigma^4 = \alpha^2(1+1).$$

The measure is ν with $\lambda = 1$ and $\alpha = \sigma^2$, so that it has density

$$\frac{1}{2\pi\alpha t}\sqrt{4\alpha^2 - (t-2\alpha)^2}, \quad t \in [0, 4\alpha].$$

We compute that

$$\begin{aligned} &\int_0^{4\alpha} \sqrt{4\alpha^2 - (t-2\alpha)^2} dt \quad (t - 2\alpha = s) \\ &= \int_{-2\alpha}^{2\alpha} \sqrt{4\alpha^2 - s^2} ds = \frac{1}{2}\pi(2\alpha)^2 = 2\pi\alpha^2. \end{aligned}$$

It seems that the factor $\frac{1}{2\pi\alpha t}$ in the density should be corrected as $\frac{1}{2\pi\alpha^2}$. □

Example 2.4.10. A self-adjoint operator b in $\mathfrak{A}$ with φ is said to be a symmetric Bernoulli variable if the moments are given by $\varphi(b^{2m}) = \alpha^{2m}$ with $\alpha > 0$ and $\varphi(b^{2m+1}) = 0$. The cumulants are given by

$$k_{2m}(b, \cdots, b) = (-1)^{m-1}\mathrm{Ct}_{m-1}\alpha^{2m}, \quad k_{2m+1}(b, \cdots, b) = 0.$$

The corresponding measure is $\frac{1}{2}(\delta_{-\alpha} + \delta_\alpha)$. □

Example 2.4.11. Let $p = p^* = \overline{p^2} \in \mathfrak{A}$ be a projection with $\varphi(p) = t \in [0,1]$.

We have $\varphi(p) = \varphi(p^*) = \overline{\varphi(p)} \in \mathbb{R}$ and $\varphi(p) = \varphi(p^*p) \geq 0$. As well, $\varphi(p) \leq \|\varphi\|\|p\| = 1 \cdot 1 = 1$.

The moments are $\varphi(p^n) = \varphi(p) = t = k_1(p)$ for any $n \in \mathbb{N}$.

The $k_2(p) = \varphi(p^2) - \varphi(p)^2 = t - t^2$.

The corresponding measure is $(1-t)\delta_0 + t\delta_1$. □

Example 2.4.12. The free Cauchy distribution is the distribution of an unbounded operator. This is the same as the classical Cauchy distribution. □

3 The third outlooking at the random matrices

3.1 Gaussian random matrices

Let (X, μ) be a classical probability space with measure $\mu(X) = 1$. A random matrix is defined to be a matrix $f = (f_{ij})$ with entries f_{ij} given by classical random variables $f_{ij} : X \to \mathbb{C}$ measurable functions on X.

The operator probability space of $n \times n$ random matrices of p-times integrable measurable functions on X for any $1 \leq p < \infty$ is given by

$$\mathfrak{A} = M_n(\mathbb{C}) \otimes \{\cap_{1 \leq p < \infty} L^p(X, \mu)\}$$

with $\varphi = \mathrm{tr} \otimes E$, where tr is the normalized trace on $M_n(\mathbb{C})$ and E is the expectation on $L^1(X, \mu)$.

For any $f = (f_{ij}) \in \mathfrak{A}$, we have

$$\varphi(f) = E(\mathrm{tr}(f)) = \frac{1}{n} \sum_{j=1}^{n} \int_X f_{ij}(x) d\mu(x).$$

The space as the infinite intersection of $L^p(X, \mu)$ for $1 \leq p < \infty$ in the operator tensor algebra $\mathfrak{A}$ above is that of random variables for which all power moments exist. We may denote it by $L_\infty(X, \mu)$.

Let $f \in \mathfrak{A}$ be a self-adjoint random matrix. Suppose that $u^* f u$ for some unitary random matrix u is a diagonal random matrix with diagonal entries g_{jj}. This is possible if $f = a \otimes 1$ with $a = a^* \in M_n(\mathbb{C})$ or with a normal as $a^* a = a a^*$. Then we have

$$\varphi(f^k) = E(\mathrm{tr}(f^k)) = E(\mathrm{tr}((u^* f u)^k)) = \frac{1}{n} \sum_{j=1}^{n} E[g_{jj}^k].$$

If $g_{jj} = \lambda_j \in \mathbb{R}$, then the right hand side above can be written as

$$\int_{\mathbb{R}} t^k d\mu(t), \quad \mu = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}(t).$$

Definition 3.1.1. A Gaussian random matrix is defined to be a self-adjoint $n \times n$ random matrix $f = (f_{ij})$ so that $f = f^* = (\overline{f}_{ji})$ such that f_{ij} for $1 \leq i \leq j \leq n$ are independent complex Gaussian random variables satisfying $E[f_{ij}] = 0$, $E[f_{ij}^2] = 0$ for $i \neq j$, and $E[f_{ij}f_{ji}] = E[f_{ij}\overline{f}_{ij}] = \frac{1}{n}$.

Such a G random matrix is called GUE (Gaussian unitay ensemble). Note that the distribution of the entries of f is invariant under unitary conjugates.

We may find such an example.

Example 3.1.2. Let $x + iy = x \otimes 1 + i(1 \otimes y) \in \mathbb{C} \otimes (\otimes^2 L_\infty(\mathbb{R}, \mu))$ with μ gaussian measure. Then $E[x + iy] = E[x] + iE[y] = 0$, and moreover,

$$E[(x + iy)^2] = E[x^2] - E[y^2] + E[2ixy] = 2iE[x]E[y] = 0.$$

That's it! As well, $E[(x+iy)(x-iy)] = E[x^2+y^2]$. Then we obtain the following Gaussian random matrix as an example.

$$\begin{pmatrix} \frac{1}{\sqrt{nE[x_{11}^2]}}x_{11} & \frac{1}{\sqrt{nE[x_{12}^2+y_{12}^2]}}(x_{12} + iy_{12}) & \cdots & \frac{1}{\sqrt{nE[x_{1n}^2+y_{1n}^2]}}(x_{1n} + iy_{1n}) \\ \frac{1}{\sqrt{nE[x_{12}^2+y_{12}^2]}}(x_{12} - iy_{12}) & \frac{1}{\sqrt{nE[x_{22}^2]}}x_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{\sqrt{nE[x_{1n}^2+y_{1n}^2]}}(x_{1n} - iy_{1n}) & \cdots & \cdots & \frac{1}{\sqrt{nE[x_{nn}^2]}}x_{nn} \end{pmatrix}$$

which is an element of $M_n(\mathbb{C}) \otimes (\otimes^k L_\infty(\mathbb{R}, \mu))$ with $k = n + 2\frac{(n-1)n}{2} = n^2$, where self-adjoint random variables $x_{jj} = x_{jj}^*$ and pairs $x_{ij} = x_{ij}^*$ with $y_{ij} = y_{ij}^*$ for $i < j$ belong respectively to different tensor factors $L_\infty(\mathbb{R}, \mu)$ of the k-fold tensor product. □

Definition 3.1.3. Random variables $x_1, \cdots, x_n$, $n \in \mathbb{N}$ make a Gaussian family if the Wick (in physics 1950) or the Isserlis (in probability theory 1918) formula holds as follows. For any $1 \leq i_1, \cdots, i_m \leq n$ with $m \in \mathbb{N}$ even, we have

$$E[x_{i_1} \cdots x_{i_m}] = \sum_{q \in P_2(m)} \Pi_{q:\sqcup\{r,s\}} E[x_{i_r} x_{i_s}],$$

where $P_2(m)$ denotes the set of pair-wise partitions of the set $\{1, \cdots, m\}$ of numbers, with $m = 2k \in \mathbb{N}$ even, like that

$$q : \{1, 2\} \sqcup \{3, 4\} \sqcup \cdots \sqcup \{m - 1, m\} = \sqcup_{j=1}^k \{2j - 1, 2j\} = \{1, \cdots, m\}.$$

This combinatorial formula says that all the (variable-wise) joint moments of such a Gaussian family can be expressed in terms of the pair moments.

Example 3.1.4. Let $x_1, \cdots, x_n$ be Gaussian random variables which are mutually independent. It then follows that with m odd, we have

$$E[x_{i_1} \cdots x_{i_m}] = \Pi_{j=1}^m E[x_{i_j}] = 0^m = 0.$$

Without independence, the Wick formula for $m = 2$ is the trivial identity $E[x_{i_1}x_{i_2}] = E[x_{i_1}x_{i_2}]$. The Wick formula for $m = 4$ is the following

$$\begin{aligned} E[x_{i_1}x_{i_2}x_{i_3}x_{i_4}] &= E[x_{i_1}x_{i_2}]E[x_{i_3}x_{i_4}] \\ &\quad + E[x_{i_1}x_{i_3}]E[x_{i_2}x_{i_4}] + E[x_{i_1}x_{i_4}]E[x_{i_2}x_{i_3}]. \end{aligned}$$

With independence, we have $E[x_i x_j] = \delta_{ij}\sigma^2$ with $\sigma^2 = E[x_j^2] - E[x_j]^2$.

Note also that for $m \in \mathbb{N}$, we have

$$\begin{aligned} E[x_j^m] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} t^m e^{-\frac{t^2}{2\sigma^2}}\,dt \\ &= \begin{cases} 0 & (m \text{ odd}), \\ \sigma^m(m-1)!! & (m \text{ even}). \end{cases} \end{aligned}$$

Note that the function integrated is an odd function if m is odd. If $m = 2$, then $E[x_j^2] = \sigma^2$ with $(m-1)!! = 1$. For $m = 4$, the Wick formula implies that

$$E[x_j^4] = 3\sigma^2\sigma^2 = 3!!\sigma^4$$

with $3!! = 3 \cdot 1 = 3 = |P_2(4)|$ the cardinal number of $P_2(4)$. For $m = 2k$ general, the Wick formula implies that $E[x_j^m] = |P_2(m)|\sigma^m$.

By induction, we suppose that $|P_2(m)| = (m-1)!!$. We then have

$$|P_2(m+2)| = (m+1)|P_2(m)| = (m+1)!! \qquad \square$$

Example 3.1.5. Let $A = (a_{ij})$ be an $n \times n$ GUE. Then the entries $\mathrm{Re}(a_{ij})$ and $\mathrm{Im}(a_{ij})$ for $1 \leq i, j \leq n$ make a Gaussian family.

Note that $\mathrm{Re}(a_{jj}) = a_{jj}$ and $\mathrm{Im}(a_{jj}) = 0$. As well, with $i \neq j$,

$$\mathrm{Re}(a_{ij}) = \frac{a_{ij} + a_{ij}^*}{2} = \frac{a_{ji}^* + a_{ji}}{2} = \mathrm{Re}(a_{ji}).$$

Also,

$$\mathrm{Im}(a_{ij}) = \frac{a_{ij} - a_{ij}^*}{2i} = \frac{a_{ji}^* - a_{ji}}{2i} = -\mathrm{Im}(a_{ji}).$$

By independence and being Gaussian, we have

$$E[a_{ij}a_{kl}] = \frac{1}{n}\delta_{il}\delta_{jk}, \quad 1 \leq i, j, k, l \leq n.$$

We can compute the even moments of A as with m even,

$$\varphi(A^m) = E[\mathrm{tr}(A^m)].$$

Suppose that $m = 2$. Then

$$A^2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix}^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{12}^* & a_{11}a_{12} + a_{12}a_{22} \\ a_{12}^*a_{11} + a_{22}a_{12}^* & a_{12}^*a_{12} + a_{22}^2 \end{pmatrix}$$

so that

$$\begin{aligned}
\varphi(A^2) &= E[\mathrm{tr}(A^2)] \\
&= \frac{1}{2}(E[a_{11}^2] + E[a_{12}a_{12}^*] + E[a_{12}^*a_{12}] + E[a_{22}^2]) \\
&= \frac{1}{2}(E[a_{11}^2] + E[a_{12}a_{21}] + E[a_{21}a_{12}] + E[a_{22}^2]) \\
&= \frac{1}{2}\sum_{i_1,i_2=1}^{2} E[a_{i_1i_2}a_{i_2i_1}] = \frac{1}{2}\cdot\frac{4}{2} = 1.
\end{aligned}$$

Moreover, we compute

$$A^4 = \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix}^4 = \begin{pmatrix} a_{11}^2 + a_{12}a_{12}^* & a_{11}a_{12} + a_{12}a_{22} \\ a_{12}^*a_{11} + a_{22}a_{12}^* & a_{12}^*a_{12} + a_{22}^2 \end{pmatrix}^2$$

with the diagonal entries

$$\begin{aligned}
&(a_{11}^2 + a_{12}a_{12}^*)^2 + (a_{11}a_{12} + a_{12}a_{22})(a_{11}a_{12} + a_{12}a_{22})^* \quad \text{and} \\
&(a_{11}a_{12} + a_{12}a_{22})^*(a_{11}a_{12} + a_{12}a_{22}) + (a_{12}^*a_{12} + a_{22}^2)^2
\end{aligned}$$

so that

$$\begin{aligned}
&\varphi(A^4) = E[\mathrm{tr}(A^4)] \\
&= \frac{1}{2}(E[a_{11}^4] + E[a_{11}^2a_{12}a_{21}] + E[a_{12}a_{21}a_{11}^2] + E[a_{12}a_{21}a_{12}a_{21}]) \\
&+ \frac{1}{2}(E[a_{11}a_{12}a_{21}a_{11}] + E[a_{11}a_{12}a_{22}a_{21}] + E[a_{12}a_{22}a_{21}a_{11}] + E[a_{12}a_{22}^2a_{21}]) \\
&+ \frac{1}{2}(E[a_{21}a_{11}^2a_{12}] + E[a_{21}a_{11}a_{12}a_{22}] + E[a_{22}a_{21}a_{11}a_{12}] + E[a_{22}a_{21}a_{12}a_{22}]) \\
&+ \frac{1}{2}(E[a_{21}a_{12}a_{21}a_{12}] + E[a_{21}a_{12}a_{22}^2] + E[a_{22}^2a_{21}a_{12}] + E[a_{22}^4]) \\
&= \frac{1}{2}(3\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + 2\frac{1}{2^2}) + \frac{1}{2}(\frac{1}{2^2} + 0 + 0 + \frac{1}{2^2}) \\
&+ \frac{1}{2}(\frac{1}{2^2} + 0 + 0 + \frac{1}{2^2}) + \frac{1}{2}(2\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + 3\frac{1}{2^4}) \\
&= \frac{7}{2^2} + \frac{2}{2^2} = \frac{9}{4} = 2 + \frac{1}{2^2}.
\end{aligned}$$

As well, we have, with i_1, i_2 mod 2,

$$\begin{aligned}
\varphi(A^4) &= \frac{1}{2}(E[a_{11}^4] + E[a_{12}a_{21}a_{12}a_{21}] + E[a_{21}a_{12}a_{21}a_{12}] + E[a_{22}^4]) \\
&\quad + \frac{1}{2}(E[a_{11}^2 a_{12}a_{21}] + E[a_{12}a_{21}a_{11}^2] + E[a_{11}^2 a_{12}a_{21}] + E[a_{12}a_{21}a_{11}^2]) \\
&\quad + \frac{1}{2}(E[a_{11}a_{12}a_{21}a_{11}] + E[a_{22}a_{21}a_{12}a_{22}]) + \frac{1}{2}(E[a_{12}a_{22}^2 a_{21}] + E[a_{21}a_{11}^2 a_{12}]) \\
&= \frac{1}{2}\sum_{i_1,i_2=1}^{2} E[a_{i_1i_2}a_{i_2i_1}a_{i_1i_2}a_{i_2i_1}] + \frac{1}{2}\sum_{i_1,i_2=1}^{2} E[a_{i_1i_2}a_{i_2i_1}a_{i_1i_2+1}a_{i_2+1i_1}] \\
&\quad + \frac{1}{2}\sum_{i_1=1}^{2} E[a_{i_1i_1}a_{i_1i_1+1}a_{i_1+1i_1}a_{i_1i_1}] + \frac{1}{2}\sum_{i_1=1}^{2} E[a_{i_1+1i_1}a_{i_1i_1}a_{i_1i_1}a_{i_1i_1+1}].
\end{aligned}$$

Is it possible to have such a formula for $\varphi(A^m)$ in general? □

The limits of the moments $\varphi(A^m)$ of an $n \times n$ GUE $A = (a_{ij})$ as $n \to \infty$ are equal to the number of the set of non-crossing pair partitions of even m elements set. The numbers are also equal to the Catalan numbers $\mathrm{Ct}_{\frac{m}{2}} = \frac{2}{m+2}\binom{m}{\frac{m}{2}}$. Namely, with any integer $m \geq 0$,

$$\lim_{n\to\infty} \varphi(A^{2m}) = \mathrm{Ct}_m.$$

On the other hand, the Catalan numbers Ct_m are the moments for the semi-circle law in the sense that (for m even?)

$$\mathrm{Ct}_m = \frac{1}{2\pi}\int_{-2}^{2} t^m \sqrt{4-t^2}\,dt.$$

Example 3.1.6. We compute

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-2}^{2} \sqrt{4-t^2}\,dt = \frac{1}{\pi}\int_{0}^{2} \sqrt{4-t^2}\,dt \quad (t = 2\sin\theta) \\
&= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} 2\cos\theta(2\cos\theta d\theta) = \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} 2(1+\cos 2\theta)d\theta \\
&= \frac{2}{\pi}[\theta + \frac{1}{2}\sin 2\theta]_{\theta=0}^{\frac{\pi}{2}} = 1 = \mathrm{Ct}_0.
\end{aligned}$$

We also compute

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_{-2}^{2} t\sqrt{4-t^2}\,dt \quad (t = 2\sin\theta) \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin\theta(2\cos\theta)(2\cos\theta d\theta) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^2\theta\sin\theta d\theta \\
&= \frac{1}{2\pi}[-\frac{8}{3}\cos^3\theta]_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0 \neq 1 = \mathrm{Ct}_1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-2}^{2} t^2\sqrt{4-t^2}dt \quad (t = 2\sin\theta)\\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 4\sin^2\theta(2\cos\theta)(2\cos\theta d\theta) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 4\sin^2 2\theta d\theta\\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 2(1-\cos 4\theta)d\theta = 1 = \mathrm{Ct}_1.
\end{aligned}$$

Is the above formula correct?
Furthermore,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-2}^{2} t^4\sqrt{4-t^2}dt \quad (t = 2\sin\theta)\\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 2^4\sin^4\theta(2\cos\theta)(2\cos\theta d\theta) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 2^6(\sin^4\theta - \sin^6\theta)d\theta\\
&= \frac{2^6}{\pi}\frac{\pi}{2}(\frac{3\cdot 1}{4\cdot 2} - \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}) = 2^5\frac{3}{6\cdot 4\cdot 2} = 2 = \mathrm{Ct}_2.
\end{aligned}$$

The corrected formula is the following. For any integer $m \geq 0$,

$$\mathrm{Ct}_m = \frac{1}{2\pi}\int_{-2}^{2} t^{2m}\sqrt{4-t^2}dt.$$

Check that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-2}^{2} t^6\sqrt{4-t^2}dt \quad (t = 2\sin\theta)\\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 2^6\sin^6\theta(2\cos\theta)(2\cos\theta d\theta) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 2^8(\sin^6\theta - \sin^8\theta)d\theta\\
&= \frac{2^8}{\pi}\frac{\pi}{2}(\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2} - \frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}) = 2^7\frac{5\cdot 3}{8\cdot 6\cdot 4\cdot 2} = 5 = \mathrm{Ct}_3. \quad \square
\end{aligned}$$

Theorem 3.1.7. *The asymptotic eigenvalue distribution of an $n\times n$ GUE A is given by the Wigner semi-circle law. Namely, the measures μ_A as $n\to\infty$ converge weakly to μ_S (or as moments), where $d\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}dt$ for $t \in [-2,2]$. It says that for any integer $m \geq 0$,*

$$\begin{aligned}
&\lim_{n\to\infty}(\mathrm{tr}\otimes E)(A^{2m}) = \lim_{n\to\infty}\int_{\mathbb{R}} t^{2m}d\mu_A(t)\\
&= \int_{-2}^{2} t^{2m}\frac{1}{2\pi}\sqrt{4-t^2}dt = \mathrm{Ct}_m
\end{aligned}$$

with $\mu_A = \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_{j,A}}$ with $\lambda_{j,A}$ eigenvalues for A with multiplicity (if possible as in the case of complex matrices).

3.2 The Central Limit Theorem analogues

We may refer to [16].

Let $\mathfrak{A}$ be an operator probability space with φ. Let (a_n) be a sequence of self-adjoint elements of $\mathfrak{A}$ as random variables which are either independent or freely independent. Assume that the variables a_n are centered, so that $\varphi(a_n) = 0$ for $n \in \mathbb{N}$, and the common variance of the variables is $\sigma^2 = \varphi(a_n^2) \geq 0$ for $n \in \mathbb{N}$.

A central limit theorem (CLT) says that the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} a_j$$

is valued in some sense with certain convergence involved.

Let $(\mathfrak{A}, \varphi)$ and $(\mathfrak{A}_n, \varphi_n)$ for $n \in \mathbb{N}$ be operator probability spaces. We say that random variables $a_n \in \mathfrak{A}_n$ converge in distribution to $a \in \mathfrak{A}$ as $n \to \infty$ if for any $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \varphi_n(a_n^k) = \varphi(a^k) \in \mathbb{C}.$$

This convergence in distribution is weaker in general than the usual convergence in the classical central limit theorems. So the classical CLT is stronger than the quantum CLT. But the convergence in distribution is said to be the weak convergence, so the classical convergence may be called the strong convergence. Anyhow, we may distinguish these convergences.

The classical convergence in distribution (or convergence in law) for probability measures μ_n on compact spaces $X_n \subset \mathbb{R}$ to μ on a compact space $X \subset \mathbb{R}$ means the weak convergence as the following limit

$$\lim_{n\to\infty} \int_{X_n} f(t)d\mu_n(t) = \int_X f(t)d\mu(t)$$

for any bounded continuous functions f on $\mathbb{R}$. Also, the Stone-Weierstrass theorem implies that f can be replaced with polynomials in t. It just looks like that $t \in C(X_n)$ converge in distribution to $t \in C(X)$ in the sense above as the commutative case. Is this correct?

The classical CLT is the following.

Theorem 3.2.1. *Let $\mathfrak{A}$ be an operator probability space with φ. Let (a_n) be a sequence of self-adjoint random variables of $\mathfrak{A}$ which are independent. Assume that the variables are centered so that $\varphi(a_n) = 0$ for $n \in \mathbb{N}$ and the common variance of the variables is $\sigma^2 = \varphi(a_n^2)$. Then we have that there is a normally distributed random variable x on $\mathbb{R}$ with variance σ^2 in the sense that*

$$\begin{aligned}
&\lim_{n\to\infty} \varphi((\frac{1}{\sqrt{n}} \sum_{j=1}^{n} a_j)^k) = \varphi(x^k) = E[t^k] \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} t^k e^{-\frac{t^2}{2\sigma^2}} dt = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \sigma^k (k-1)!! & \text{if } k \text{ is even} \end{cases}
\end{aligned}$$

with $(k-1)!! = (k-1)(k-3)\cdots 3\cdot 1 = P_2(k)$ the number of partitions of the set of even k elements by pairs.

The free or quantum CLT is the following.

Theorem 3.2.2. *Let $\mathfrak{A}$ be an operator probability space with φ. Let (a_n) be a sequence of self-adjoint random variables of $\mathfrak{A}$ which are freely independent. Assume that the variables are centered so that $\varphi(a_n) = 0$ for $n \in \mathbb{N}$ and the common variance of the variables is $\sigma^2 = \varphi(a_n^2)$. Then we have that there is a semi-circlular self-adjoint element s with variance σ^2 in the sense that*

$$\lim_{n\to\infty} \varphi((\frac{1}{\sqrt{n}}\sum_{j=1}^{n} a_j)^k) = \varphi(s^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \sigma^k \frac{2}{k+2}\binom{k}{\frac{k}{2}} = \sigma^k \mathrm{Ct}_{\frac{k}{2}} & \text{if } k \text{ is even} \end{cases}$$

with $\mathrm{Ct}_{\frac{k}{2}}$ the Catalan number defined so. With $\sigma^2 = 1$ we also have

$$\varphi(s^k) = \frac{1}{2\pi}\int_{-2}^{2} t^k\sqrt{4-t^2}dt$$

for any $k \in \mathbb{N}$, and as well $\varphi(2k) = NCr_2(2k) = \mathrm{Ct}_k$.

Note as well that the number (or family) of non-crossing partitions of the set of even $2k$ elements by pairs, denoted as $NCr_2(2k)$ by us is equal to the Catalan number Ct_k. This is also equal to the number (or family) of non-crossing partitions of the set of k elements, denoted as $NCr(k)$ by us.

By the way, a partition of the set X_n of even or not n elements 1 to n by pairs is said to be non-crossing if any two pairs $\{p_1, p_2\}$ and $\{q_1, q_2\}$ of elements of X_n with $1 \le p_1 < p_2 \le n$ and $1 \le q_1 < q_2 \le n$ dose not satisfy the inequality

$$p_1 < q_1 < p_2 < q_2.$$

Example 3.2.3. We have $NCr_2(2) = 1$.

We have $NCr_2(4) = 2$. The non-crossing partitions of X_4 by pairs are given by

$$X_4 = \{1,2\} \sqcup \{3,4\} = \{1,4\} \sqcup \{2,3\}.$$

There is only one crossing partition $\{1,3\} \sqcup \{2,4\}$. □

Theorem 3.2.4. *Let $\{(a_{n,j}))\,|\, n \in \mathbb{N}, j \in J\}$ be a sequence of families $(a_{n,j})_{j\in J}$ indexed by an index set J of freely independent random variables $a_{n,j} \in \mathfrak{A}$ for $j \in J$ such that $\varphi(a_{n,j}) = 0$ and $\varphi(a_{n,j}^2) = 1$. Then we have*

$$\lim_{n\to\infty} \varphi((\frac{1}{\sqrt{n}}\sum_{i=1}^{n} a_{i,j})^k) = \varphi(s_j^k), \quad k \in \mathbb{N}$$

where $(s_j)_{j\in J}$ is a family of semi-circular elements of covariance $(c_{ij})_{i,j\in J}$ with $c_{ij} = \varphi(a_{n,i}a_{n,j})$ so that

$$\varphi(s_{i_1}\cdots s_{i_m}) = \sum_{P\in NCr_2(m)} \Pi_{\{r,p\}\subset P} c_{i_r i_p}$$

for any even $m \in \mathbb{N}$.

3.3 Operator cumulants and more

Definition 3.3.1. Let $(\mathfrak{A}, \varphi)$ be an operator probability space. Then the first operator cumulant $k_1 : \mathfrak{A} \to \mathbb{C}$ is defined to be $k_1 = \varphi$ on $\mathfrak{A}$.

The second operator cumulant $k_2 : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ is defined by the equation

$$\varphi(a_1 a_2) = k_2(a_1, a_2) + k_1(a_1)k_1(a_2), \quad a_1, a_2 \in \mathfrak{A}$$

the terms of which correspond to non-crossing partitions of the set $\{1, 2\}$. Namely,

$$k_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2) = k_1(a_1 a_2) - k_1(a_1)k_2(a_2).$$

The third operator cumulant $k_3 : \mathfrak{A}^3 \to \mathbb{C}$ is defined by

$$\begin{aligned}\varphi(a_1 a_2 a_3) = k_3(a_1, a_2, a_3) &+ k_1(a_1)k_2(a_2, a_3) + k_1(a_2)k_2(a_1, a_3) \\ &+ k_1(a_3)k_2(a_1, a_2) + k_1(a_1)k_1(a_2)k_1(a_3)\end{aligned}$$

the terms of which correspond to non-crossing partitions of the set $\{1, 2, 3\}$.

Inductively, the n-th operator cumulant $k_n : \mathfrak{A}^n \to \mathbb{C}$ is defined by

$$\varphi(a_1 \cdots a_n) = \sum_{\{p_1, \cdots, p_l\} \in NCr(n)} k_{p_1}(a_{j_{11}}, \cdots, a_{j_{1p_1}}) \cdots k_{p_l}(a_{j_{l1}}, \cdots, a_{j_{lp_l}})$$

where $NCr(n)$ is the number as well as the set of non-crossing partitions of the set $\{1, \cdots, n\}$, and $\{p_1, \cdots, p_l\}$ corresponds to the numbers of elements of parts in a non-crossing partition. Namely, k_n corresponds to the trivial partition, and the other terms are multiples of cummulants of lower degree, corresponding to non-trivial non-crossing partitions.

Definition 3.3.2. Let $NCr(n)$ denote the set of non-crossing partitions of the set $\{1, \cdots, n\}$ of n elements. A partial order on this set is denoted as $P_1 \leq P_2$ and is defined that if any part of the partition P_1 is contained in a part of P_2.

Example 3.3.3. As for $NCr(3)$, we have the following (total) ordering.

$$\{1\} \sqcup \{2\} \sqcup \{3\} \leq \begin{cases} \{1, 2\} \sqcup \{3\} \\ \{1\} \sqcup \{2, 3\} \\ \{1, 3\} \sqcup \{2\} \end{cases} \leq \{1, 2, 3\}. \qquad \square$$

There is a lattice structure of the ordered set $NCr(n)$. Namely, for any two partitions $P, Q \in NCr(n)$, there is a minimal partition (or sup or join) $P \vee Q$ such that $P, Q \leq P \wedge Q$, and there is a maximal partition (or inf or meet) $P \wedge Q$ such that $P \wedge Q \leq P, Q$.

Namely, decreasing in that order means being finer in separation. Also, increasing means being non-finer.

Example 3.3.4. As for the ordered set $NCr(4)$, we have

$$[\{1,2,3\} \sqcup \{4\}] \wedge [\{1\} \sqcup \{2,3,4\}] = \{1\} \sqcup \{2,3\} \sqcup \{4\}.$$

As well,

$$[\{1,2,3\} \sqcup \{4\}] \vee [\{1\} \sqcup \{2,4\} \sqcup \{3\}] = \{1,2,3,4\}.$$

The lattice $NCr(n)$ has the largest element $\{1, \cdots, n\} = \mathfrak{N}_n$ and the smallest element $\{1\} \sqcup \cdots \sqcup \{n\} = \mathfrak{n}_n$.

Note that for any partition $P \in NCr(n)$, we have $P \vee \mathfrak{N}_n = \mathfrak{N}_n$ and $P \wedge \mathfrak{N}_n = P$. Also, $P \wedge \mathfrak{n}_n = \mathfrak{n}_n$ and $P \vee \mathfrak{n}_n = P$. That's it!

Example 3.3.5. With multiples $b_1 = a_1a_2$ and $b_2 = a_3$ such that $b_1b_2 = a_1a_2a_3$ with $2 \leq 3$, we compute that

$$\begin{aligned} k_2(b_1, b_2) = k_2(a_1a_2, a_3) &= \varphi((a_1a_2)a_3) - \varphi(a_1a_2)\varphi(a_3) \\ &= \varphi(a_1a_2a_3) - (k_2(a_1, a_2) + k_1(a_1)k_1(a_2))k_1(a_3) \\ &= k_3(a_1, a_2, a_3) + k_1(a_1)k_2(a_2, a_3) + k_1(a_2)k_2(a_1, a_3). \end{aligned}$$

The terms correspond to

$$\begin{cases} P_1 = \{1\} \sqcup \{2,3\} \\ P_2 = \{2\} \sqcup \{1,3\} \end{cases} \leq \{1,2,3\} = \mathfrak{N}_3$$

with

$$\begin{aligned} &[\{1\} \sqcup \{2,3\}] \vee [\{1,2\} \sqcup \{3\} = P_3] = \{1,2,3\}, \\ &[\{2\} \sqcup \{1,3\}] \vee [\{1,2\} \sqcup \{3\} = P_3] = \{1,2,3\}. \end{aligned}$$

Namely, the above summation Σ for $k_2(a_1a_2, a_3)$ just corresponds to the sum Σ with respect to partitions P of $NCr(n)$ such that $P \vee P_3 = \mathfrak{N}_3$.

By the way, the factors $b_1 = a_1a_2$ and $b_2 = a_3$ of b_1b_2 just correspond to P_3. □

There is a formula as a theorem (of Speicher) that fully generalizes the example above to operator cumulants involving multiples $b_1, \cdots, b_m$ of $a_1, \cdots, a_n \in \mathfrak{A}$ such that the product $b_1 \cdots b_m = a_1 \cdots a_n$ with $m \leq n$, as the sum decompositions of operator cumulants with respect to partitions P of $NCr(n)$ such that $P \vee Q = \mathfrak{N}_n$ for some same partition Q (chan). The Q corresponds to the factors $b_1, \cdots, b_m$ of $b_1 \cdots b_m$.

Recall that two unital subalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of $(\mathfrak{A}, \varphi)$ are free if $\varphi(a_1 \cdots a_n) = 0$ for $a_j \in \mathfrak{A}_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ in $\{1, 2\}$ and $\varphi(a_j) = 0$ for $1 \leq j \leq n$.

In such a case, it implies that $k_2(a_1, a_2) = \varphi(a_1a_2) = 0$ and $k_3(a_1, a_2, a_3) = \varphi(a_1a_2a_3) = 0$. Inductively, $k_n(a_1, \cdots, a_n) = \varphi(a_1 \cdots a_n) = 0$. Namely, the operator cumulants vanish for such elements.

There is a theorem (of Speicher) that the converse of the implication by freeness holds under a suitably weakened condition on vanishing of operator cumulants, where the two of two unital subalgebras can be taken to be an arbitrary finite number. Moreover, subalgebras can be replaced with elements or operators.

3.4 Operator operations by certain transformations

Let $(\mathfrak{A}, \varphi)$ be an operator probability space. Let $a, b \in \mathfrak{A}$ that are self-adjoint and free.

The question is that how the distribution of $a + b$ can be described in terms of the distributions of a and b?

We may calculate the moments of $a + b$ in terms of the moments of a and b. But this seems to be somewhat complicated as in the case of higher powers of $a + b$. We may use cumulants instead.

Let denote the cumulant $k_n(a, \cdots, a) = k_n(a)$. Being free of a and b and distributive like law for k_n implies that

$$k_n(a + b) = k_n(a) + k_n(b).$$

Because the cumulants for mixed elements of a and b such as $(a, b, \cdots, b, a)$ vanish by freeness.

By the way, in such a case, it seems that the moments for $a + b$ are also computable.

Example 3.4.1. In general, we have

$$\begin{aligned}\varphi((a+b)^2) &= \varphi(a^2 + ab + ba + b^2) \\ &= \varphi(a^2) + \varphi(ab) + \varphi(ba) + \varphi(b^2).\end{aligned}$$

As well,

$$\begin{aligned}&k_2(a+b, a+b) = \varphi((a+b)^2) - \varphi(a+b)^2 \\ &= \varphi(a^2) + \varphi(ab) + \varphi(ba) + \varphi(b^2) - \{\varphi(a)^2 + 2\varphi(a)\varphi(b) + \varphi(b)^2\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&k_2(a, a) + k_2(a, b) + k_2(b, a) + k_2(b, b) \\ &= \varphi(a^2) - \varphi(a)^2 + \varphi(ab) - \varphi(a)\varphi(b) + \varphi(ba) - \varphi(b)\varphi(a) + \varphi(b^2) - \varphi(b)^2.\end{aligned}$$

Therefore, we obtain that

$$k_2(a+b, a+b) = k_2(a, a) + k_2(a, b) + k_2(b, a) + k_2(b, b).$$

This is a distributive law that we obtained. It seems to be interesting to consider more general cases as a question. □

Let a be an element of $(\mathfrak{A}, \varphi)$. The series of moments of $a \in \mathfrak{A}$ is defined to be a formal power series with respect to a variable z such that

$$sm(z) = \sum_{n=0}^{\infty} \varphi(a^n) z^n$$

where $\varphi(a^0)z^0 = \varphi(1) = 1$. The series of cumulants of $a \in \mathfrak{A}$ is defined to be

$$sc(z) = \sum_{n=0}^{\infty} k_n(a) z^n$$

where $k_0(a)z^0 = 1$.

Theorem 3.4.2. *The relation among moments $\varphi(a^n)$ and cumulants $k_n(a)$ such that $\varphi(a^n)$ is equal to the sum of certain multiples $\Pi_{j=1}^{l} k_{n_j}(a)$ for $1 \leq n_j \leq n$ with $\sum_{j=1}^{l} n_j = n$, with respect to non-crossing partitions of the set $\{1, \cdots, n\}$, is equivalent to the equation*

$$sm(z) = sc(z \cdot sm(z)).$$

Proof. (Sketch of the proof). We compute the series $sm(z)$ of the moments by inserting the moments as sums of the cumulants term-wise. We then convert the series to those of the cumulants with $z \cdot sm(z)$ as a variable by manipulating sums of multiples of the cumulants to the corresponding multiples of the moments, with summations changed. □

By the way, for the classical cumulants (c_n) for a random variable f, there is the formula among moments and cumulants so defined such that

$$\varphi(f^n) = E[f^n] = \sum_{\{k_1, \cdots, k_j\} \in P(n)} c_{k_1} \cdots c_{k_j}$$

where $P(n)$ denotes the set of partitions $p_1 \sqcup \cdots \sqcup p_j$ of the set $\{1, \cdots, n\}$ with $|p_1| = k_1, \cdots, |p_j| = k_j$ cardinal numbers identified.

In particular, $E[f] = c_1(f)$. Also, $E[f^2] = c_2(f, f) + c_1(f)^2$. As well,

$$E[f^3] = c_3(f, f, f) + 3c_1(f)c_2(f, f) + c_1(f)^3.$$

Is this correct?

We define the following two exponential like formal power series of moments and cumulants

$$em(z) = \sum_{n=0}^{\infty} \frac{1}{n!} E[f^n] z^n \quad \text{and} \quad ec(z) = \sum_{n=0} \frac{1}{n!} c_n z^n.$$

We then have the following equation $ec(z) = \log em(z)$.

Proof. We may refer to [5]. Recall that we have the following cumulant formula.

$$\log E[e^{itx}] = \sum_{n=1}^{\infty} \frac{1}{n!} (it)^n c_n.$$

We let $z = it$. Then we have

$$E[e^{zx}] = \int \sum_{n=0}^{\infty} \frac{1}{n!} z^n x^n d\mu(x)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} E[x^n] z^n$$

by integration by terms. Therefore, the cumulant formula implies the desired equation above. □

That equation may be deduced from the formula among moments and cumulants. How? Possibly, we insert the formula to $em(z)$ term-wise, and then covert to another series involving summations changed, and then take log operation to become $ec(z)$. Right?

The Cauchy transform with respect to $a \in (\mathfrak{A}, \varphi)$ is a map $Cu(z)$ defined to be

$$Cu(z) = \varphi(\frac{1}{z-a}) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \varphi(a^n)$$

where

$$\frac{1}{z-a} = \frac{1}{z} \frac{1}{1 - \frac{a}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{a^n}{z^n}$$

for $\|\frac{a}{z}\| < 1$, so $|z| > \|a\|$. As well, $|\varphi(a^n)| \leq \|a\|^n$. It then follows that

$$sm(\frac{1}{z}) = zCu(z) \quad \text{or} \quad Cu(z) = \frac{1}{z} sm(\frac{1}{z}).$$

The Voiculescu R-transform with respect to $a \in (\mathfrak{A}, \varphi)$ is a map $R(z)$ defined to be

$$R(z) = \sum_{n=0}^{\infty} k_{n+1}(a) z^n.$$

It then follows that

$$sc(z) = 1 + zR(z) \quad \text{or} \quad R(z) = \frac{1}{z}(sc(z) - 1).$$

If $\mathfrak{A}^n$ has the supremum norm, then we have $|k_{n+1}(a)z^n| \leq \|k_{n+1}\| \|a\| |z|^n$, so that it is sufficient to have the series of $\|k_{n+1}\| |z|^n$ convergent, to define $R(z)$.

The relation $sm(z) = sc(z \cdot sm(z))$ between sm and sc can be converted in terms of $Cu(z)$ and $R(z)$ to

$$\begin{aligned} \frac{1}{z} Cu(\frac{1}{z}) &= sm(z) = sc(z \cdot sm(z)) \\ &= 1 + (z \cdot sm(z)) R(z \cdot sm(z)) \\ &= 1 + Cu(\frac{1}{z}) R(Cu(\frac{1}{z})). \end{aligned}$$

Replacing $\frac{1}{z}$ with z implies that

$$zCu(z) = 1 + Cu(z) R(Cu(z)).$$

Therefore, dividing the equation by $Cu(z)$ implies

$$z = \frac{1}{Cu(z)} + R(Cu(z)).$$

It then follows that the composition of the maps $Cu(z)$ and $\frac{1}{z} + R(z)$ is the identity map with respect to z. Since the maps are inverses each other, we also have

$$Cu(\frac{1}{z} + R(z)) = z.$$

Since we have

$$Cu(z) = \varphi(\frac{1}{z-a}) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

for some measure with respect to $a = a^*$, we can define an analytic function $Cu(z)$ from $\mathbb{C}^+$ to $\mathbb{C}^-$.

Note that the integration above can be viewed as the convolution $\frac{1}{t} * d\mu(t)$ at z.

The measure μ can be recovered from $Cu(z)$ by the Stieltjes inversion formula. Namely, we have

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0+0} \mathrm{Im}(Cu(t + i\varepsilon)).$$

Note that for $z = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$ with $b > 0$, we have

$$\frac{1}{z-t} = \frac{1}{a-t+ib} = \frac{1}{(a-t)^2 + b^2}(a - t - ib).$$

Its imaginary part is negative!

We have that for $s \in \mathbb{R}$,

$$Cu(s + i\varepsilon) = \int_{\mathbb{R}} \frac{1}{s - t + i\varepsilon} d\mu(t) = \int_{\mathbb{R}} \frac{1}{(s-t)^2 + \varepsilon^2}(s - t - i\varepsilon) d\mu(t).$$

Thus, we have

$$\mathrm{Im}(Cu(s + i\varepsilon)) = -\int_{\mathbb{R}} \frac{\varepsilon}{(s-t)^2 + \varepsilon^2} d\mu(t) \equiv -I_\varepsilon.$$

The integral I_ε above is computed, as in the Lebesgue or Riemann measure case, but not compactly supported,

$$\begin{aligned} I_\varepsilon &= \int_{\mathbb{R}} \frac{\varepsilon}{x^2 + \varepsilon^2} d\mu(x) \quad (x = t - s) \\ &= \frac{1}{\varepsilon} \int_{\mathbb{R}} \frac{1}{(\frac{x}{\varepsilon})^2 + 1} d\mu(x) \quad (y = \frac{x}{\varepsilon}) \\ &= \frac{1}{\varepsilon} \int_{\mathbb{R}} \frac{1}{y^2 + 1} \varepsilon d\mu(y) = [\arctan y]_{y=-\infty}^{\infty} = \pi. \end{aligned}$$

In this case, we then have $-\frac{1}{\pi} \lim_{\varepsilon \to 0+0}(-I_\varepsilon) = 1$. But for the formula in general, involved as a weak limit is any bounded continuous function f on $\mathbb{R}$ as a function multiplied to the measure in the integration. Namely, in that case,

$$\int_{\mathbb{R}} f(t) d\mu(t) = \int_{\mathbb{R}} f(t) dt.$$

Example 3.4.3. Suppose that we have the cumulant $k_2 = 1$ but $k_n = 0$ for $n \neq 2$. In this case, we have

$$R(z) = \sum_{n=0}^{\infty} k_{n+1}(a) z^n = z.$$

It then follows that

$$z = \frac{1}{Cu(z)} + Cu(z).$$

The equation implies $Cu(z)^2 - zCu(z) + 1 = 0$. This is solved as

$$Cu(z) = \frac{z \pm \sqrt{z^2 - 4}}{2}.$$

Since

$$\begin{aligned} |Cu(z)| &= |\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} \varphi(a^n)| \\ &\leq |\frac{1}{z}| \sum_{n=0}^{\infty} (\frac{\|a\|}{|z|})^n = |\frac{1}{z}| e^{\frac{\|a\|}{|z|}}, \end{aligned}$$

then $Cu(z)$ is approximated closely to $\frac{1}{z}$ as $|z|$ large enough. It implies that

$$Cu(z) = \frac{z - \sqrt{z^2 - 4}}{2} = \frac{4}{2(z + \sqrt{z^2 - 4})} = \frac{1}{\frac{1}{2}(z + \sqrt{z^2 - 4})}.$$

Then we compute

$$Cu(s + i\varepsilon) = \frac{s + i\varepsilon - \sqrt{(s + i\varepsilon)^2 - 4}}{2} = \frac{s + i\varepsilon - \sqrt{s^2 - \varepsilon^2 - 4 + i2s\varepsilon}}{2}.$$

Note that for $z \in \mathbb{C}$ with $z = |z|e^{i\theta}$, we have $\sqrt{z} = \sqrt{|z|} e^{i\frac{\theta}{2}}$. Then the imaginary part $\mathrm{Im}(\sqrt{z})$ is $\sqrt{|z|} \sin \frac{\theta}{2}$, with $\sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos \theta)$.

It then follows that the imaginary part

$$\begin{aligned} &\mathrm{Im}(\sqrt{s^2 - \varepsilon^2 - 4 + i2s\varepsilon}) \\ &= ((s^2 - \varepsilon^2 - 4)^2 + 4s^2\varepsilon^2)^{\frac{1}{4}} \frac{1}{\sqrt{2}} \sqrt{1 - \frac{s^2 - \varepsilon^2 - 4}{\sqrt{(s^2 - \varepsilon^2 - 4)^2 + 4s^2\varepsilon^2}}} \\ &= \frac{1}{\sqrt{2}} \sqrt{\sqrt{(s^2 - \varepsilon^2 - 4)^2 + 4s^2\varepsilon^2} - (s^2 - \varepsilon^2 - 4)}. \end{aligned}$$

Therefore, we obtain that

$$\mathrm{Im}(Cu(s + i\varepsilon)) = \frac{\varepsilon}{2} - \frac{1}{2\sqrt{2}} \sqrt{\sqrt{(s^2 - \varepsilon^2 - 4)^2 + 4s^2\varepsilon^2} - (s^2 - \varepsilon^2 - 4)}.$$

Taking the limit as $\varepsilon \to 0+0$ and multiplying $-\frac{1}{\pi}$ implies that

$$\begin{cases} 0 & (s^2 \geq 4) \\ \frac{1}{2\pi\sqrt{2}}\sqrt{4-s^2} & (s^2 < 4). \end{cases}$$

This Stieltjes inversion formula form is corrected in part. $\square$

The free convolution of two distributions μ_1 and μ_2 is defined by the following way by 4 steps.

(1) We make the Cauchy transforms C_1 and C_2 and the R-transforms R_1 and R_2 for μ_1 and μ_2 respectively.

(2) The R-transform R_3 of the free convolution is defined to be $R_1 + R_2$ by additivity of the cumulants of free variables.

(3) We compute the Cauchy transform C_3 of the free convolution from R_3 by using the composition $Cu(R(z) + \frac{1}{z}) = z$.

(4) We obtain the free convolution measure μ from C_3 by the Stieltjes inversion formula.

We may denote μ by $\mu_{1C} *_R \mu_2$ or $\mu_1 \boxplus \mu_2$ the additive (box) convolution.

Suppose that

$$\int_{\mathbb{R}} x^k d\mu_j(x) = \varphi(a_j^k), \quad k \in \mathbb{N}, j = 1, 2$$

where μ_j are compactly supported probability measures on $\mathbb{R}$ and a_j are self-adjoint operator random variables in relation free with respect to φ. Then we have

$$\int_{\mathbb{R}} x^k d(\mu_1 \boxplus \mu_2)(x) = \varphi((a_1 + a_2)^k), \quad k \in \mathbb{N}.$$

Example 3.4.4. Suppose that

$$\int_{\mathbb{R}} x^k d\delta_{t_j}(x) = t_j^k = \varphi(a_j^k), \quad k \in \mathbb{N}, j = 1, 2$$

It then follows that

$$\varphi((a_1 + a_2)^k) = (t_1 + t_2)^k = \int_{\mathbb{R}} x^k d\delta_{t_1+t_2}(x)$$

so that $\delta_{t_1} \boxplus \delta_{t_2} = \delta_{t_1+t_2}$. $\square$

Let μ_1 and μ_2 be compactly supported probability measures on $\mathbb{R}$ that corresponds respectively to self-adjoint operators a_1 and a_2 which are free with respect to φ on a C^*-algebra $\mathfrak{A}$. The multiplicative (box) free convolution μ of μ_1 and μ_2 is defined to be the (spectral) measure corresponding to $a_1^{\frac{1}{2}} a_2 a_1^{\frac{1}{2}}$, where a_1 is assumed to be positive. We denote it by $\mu_1 \boxtimes \mu_2$.

Note that $a_1 a_2$ is not self-adjoint in general. Also, if φ is a trace, then

$$\begin{aligned}\varphi((a_1 a_2)^n) &= \varphi(a_1^{\frac{1}{2}}(a_1^{\frac{1}{2}} a_2 a_1^{\frac{1}{2}})^{n-1}(a_1^{\frac{1}{2}} a_2)) \\ &= \varphi((a_1^{\frac{1}{2}} a_2 a_1^{\frac{1}{2}})^n).\end{aligned}$$

There is a theorem of Speicher that for $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ two finite subsets of $\mathfrak{A}$ which are free by φ, the formulae such that the cumulants of $(a_1 b_1, \cdots, a_n b_n)$ can be written as the sum of products of cumulants of $(a_1, \cdots, a_n)$ and (induced) cumulants $(b_1, \cdots, b_n)$ over $NCr(n)$ of non-crossing partitions, and as well the moments of $a_1 b_1 \cdots a_n b_n$ can be written as the sum of products of cumulants of $(a_1, \cdots, a_n)$ and (induced) moments of $(b_1, \cdots, b_n)$ over $NCr(n)$.

The Voiculescu S-transform S for an operator $a \in \mathfrak{A}$ is defined to be

$$S(z) = S_a(z) = \frac{1+z}{z} sm^{-1}(z)$$

where $sm^{-1}(z)$ means the inverse of the moment series function $sm(z)$ for a by φ with respect to composition.

It then holds multiplicatively that

$$S_{bc}(z) = S_b(z) S_c(z)$$

for $b, c \in \mathfrak{A}$ that are free with respect to φ.

3.5 Asymptotic freeness of random matrices

Two sequences (A_n) and (B_n) of matrices of operator random variables are said to be asymptotically free if they converge in distribution respectively to some operators a, b in an operator probability space $\mathfrak{A}$ by φ, namely,

$$\lim_{m \to \infty} \varphi(A_n^m) = \varphi(a^m), \quad \lim_{m \to \infty} \varphi(B_n^m) = \varphi(b^m),$$

where $\varphi = E \circ \mathrm{tr}$ (or $E \otimes \mathrm{tr}$) ($E = \varphi$ on $\mathfrak{A}$), and a, b are free with respect to φ.

Equivalently, the convergence in distribution means that

$$\lim_{n \to \infty} \varphi(p(A_n, B_n)) = \varphi(p(a, b))$$

for any complex polynomial $p(x, y)$ in non-commuting variables x, y.

There is a theorem (of Speicher) that elements of a semi-circular family $s_1, \cdots, s_n$ with diagonal covariance such that

$$\varphi(s_{i_1} \cdots s_{i_m}) = \sum_{P \in NCr_2(m)} \Pi_{\{r,p\} \subset P} \varphi(s_{i_r} s_{i_p})$$

and $\varphi(s_i s_j) = \delta_{ij}$ for $i, j = 1, \cdots, n$ are free. It then follows that independent Gaussian GUE random matrices are asymptotically free.

A sequence of complex matrices $(D_n)_{n \in \mathbb{N}}$ with $D_n \in M_n(\mathbb{C})$ is said to be deterministic if the limits of $\mathrm{tr}(D_n^m)$ for $m \in \mathbb{N}$ as $n \to \infty$ exist.

Namely, the sequence (D_n) converges in distribution to some $D \in \mathfrak{A}$, $\cup_{n=1}^{\infty} M_n(\mathfrak{A})$, or $\mathfrak{A} \otimes \mathbb{K}$ so that

$$\varphi(D^m) = \lim_{n \to \infty} \mathrm{tr}(D_n^m).$$

4 Appendixes

4.1 Appendix to moments

We may refer to [12].

The moment generating function with respect to a density function f on a space X is defined to be the integration function

$$M(\theta) = \int_X e^{\theta x} f(x)dx = E[e^{\theta x}].$$

In particular, $M(0) = \int_X f(x)dx = E[1] = 1$.

The k-th moments as with $\mu = \int_X xf(x)dx$ are obtained by differentiating in integration as

$$\mu_k = \int_X x^k f(x)dx = \frac{d^k}{d\theta^k} M(\theta)|_{\theta=0}.$$

We define as $g(t) = M(it)$ with $\theta = it$ for $t \in \mathbb{R}$ as a distribution characteristic function.

We can have the function g Taylor expanded around zero as

$$\varphi(t) = \varphi(0) + \sum_{k=1}^{m} \frac{c_k}{k!}(it)^k + o(|t|^m)$$

if and only if $c_k = \mu_k$ for $k \leq m$ the moments exist.

Note that

$$g'(t)|_{t=0} = \int_X ixe^{itx} f(x)dx|_{t=0} = i\mu_1.$$

The cumulant generating function associated to $M(\theta)$ is defined to be $K(\theta) = \log M(\theta)$.

We have the Taylor expansion for K as

$$K(\theta) = \sum_{j=0} \frac{k_j}{j!}\theta^j$$

with coefficients k_j named as cumulants, each of which can be written as polynomials of the moments μ_s for $s \leq j$.

$\star$ Note that

$$k_0 = K(0) = \log M(0) = \log 1 = 0.$$

Also,

$$\begin{aligned} k_1 &= K'(\theta)|_{\theta=0} = (\log M(\theta))'|_{\theta=0} \\ &= \frac{M'(\theta)}{M(\theta)}|_{\theta=0} = M'(0) = \mu = \mu_1. \end{aligned}$$

Moreover,

$$\begin{aligned}
k_2 = K''(\theta)|_{\theta=0} &= (\log M(\theta))''|_{\theta=0} \\
&= \frac{M''(\theta)M(\theta) - (M'(\theta))^2}{M(\theta)^2}|_{\theta=0} \\
&= M''(0) - (M'(0))^2 = \mu_2 - \mu_1^2. \quad \square
\end{aligned}$$

For the normal distribution, $M(\theta)$ is given as $e^{\mu\theta+\frac{\sigma^2}{2}\theta^2}$, so that $K(\theta) = \mu\theta + \frac{\sigma^2}{2}\theta^2$ with the cumulants of degree more than two vanishing.

$\star$ The normal distribution is defined to be $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $x \in \mathbb{R}$. Then with $\theta = it$, the Fourier transform implies that

$$\begin{aligned}
M(\theta) &= \int_{\mathbb{R}} e^{itx}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \quad (\frac{x-\mu}{\sqrt{2}\sigma} = -v) \\
&= \int_{\mathbb{R}} e^{it(-\sqrt{2}\sigma v+\mu)}\frac{1}{\sqrt{2\pi}\sigma}e^{-v^2}\sqrt{2}\sigma dv \\
&= \frac{e^{it\mu}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-v^2}e^{-i(\sqrt{2}\sigma t)v}dv \\
&= \frac{1}{\sqrt{\pi}}e^{it\mu}e^{-2\sigma^2t^2} = \frac{1}{\sqrt{\pi}}e^{\mu\theta}e^{2\sigma^2\theta^2} = \frac{1}{\sqrt{\pi}}e^{\mu\theta+2\sigma^2\theta^2}
\end{aligned}$$

(corrected as so). $\square$

4.2 Appendix to distributions

We may refer to [7]. As well we may refer to [8].

• The binomial distribution $B(n,p)$ has density $f(k) = {}_nC_k p^k q^{n-k}$ at $0 \le k \le n$ with $0 < p < 1$ and $p + q = 1$.

Note that $\sum_{k=0}^n f(k) = (p+q)^n = 1$.

We have binomial expansion $(q+px)^n = \sum_{k=0}^n {}_nC_k p^k q^{n-k} x^k$ with $x \in \mathbb{R}$.

Differentiating both sides with respect to x implies

$$np(q+px)^{n-1} = \sum_{k=1}^n {}_nC_k p^k q^{n-k} k x^{k-1}.$$

Evaluating both sides at $x = 1$ we obtain $np = \sum_{k=0}^n kf(k) = E[k]$.

Multiplying both sides by x and differentiating implies

$$\begin{aligned}
&np(q+px)^{n-1} + n(n-1)p^2x(q+px)^{n-2} \\
&= np(q+px)^{n-2}(q+npx) = \sum_{k=1}^n {}_nC_k p^k q^{n-k} k^2 x^{k-1}
\end{aligned}$$

Evaluating both sides at $x = 1$ we obtain $np(q+np) = \sum_{k=0}^n k^2 f(k) = E[k^2]$. It then follows that with $k = X$,

$$\sigma^2 = V(X) = E(X^2) - E(X)^2 = npq.$$

The moment generating function is given by

$$E[e^{tk}] = \sum_{k=0}^{n} e^{tk} f(k) = (q + pe^t)^n.$$

• The Poisson distribution $Poi(\lambda)$ with $\lambda > 0$ is the limit distribution of $B(n, \frac{\lambda}{n})$ as $n \to \infty$. It has density $f(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for integers $k \geq 0$.

Note that $\sum_{k=0}^{n} {}_nC_k(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} = (\frac{\lambda}{n} + (1-\frac{\lambda}{n}))^n = 1$.

We have $\sum_{k=0}^{\infty} f(k) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = 1$. We compute

$$E[k] = \sum_{k=0}^{\infty} kf(k) = e^{-\lambda}\sum_{k=1}^{\infty}\lambda\frac{\lambda^{k-1}}{(k-1)!} = \lambda.$$

We also have

$$\begin{aligned} E[k^2] &= e^{-\lambda}\sum_{k=1}^{\infty} k\lambda\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{t=0}^{\infty}(t+1)\frac{\lambda^t}{t!} \\ &= \lambda(E[t]+1) = \lambda(\lambda+1). \end{aligned}$$

Therefore, $\sigma^2 = V(k) = E(k^2) - E(k)^2 = \lambda$.

The moment generating function is given by

$$\begin{aligned} E[e^{tk}] &= \sum_{k=0}^{\infty} e^{tk-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(e^t\lambda)^k}{k!} \\ &= e^{-\lambda}e^{e^t\lambda} = e^{\lambda(e^t-1)}. \end{aligned}$$

By the way, the density limit is obtained as follows. With $p = \frac{\lambda}{n}$,

$$\begin{aligned} {}_nC_kp^kq^{n-k} &= \frac{n!}{k!(n-k)!}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} \\ &= \frac{\lambda^k}{k!}\frac{n(n-1)\cdots(n-k+1)}{n^k}(1-\frac{\lambda}{n})^{-k}(1-\frac{\lambda}{n})^n. \end{aligned}$$

As $n \to \infty$ we have the second fraction factor and the third factor $(1-\frac{\lambda}{n})^{-k}$ going to 1. As well, we have

$$(1-\frac{\lambda}{n})^n = (1+\frac{1}{\frac{n}{-\lambda}})^{\frac{n}{-\lambda}(-\lambda)}$$

going to $e^{-\lambda}$ as $n \to \infty$. Therefore the binomial density goes to the limit as the Poisson density in the way as $p = \frac{\lambda}{n}$.

• What is the Bernoulli distribution? The Bernoulli (local) density is p^kq^{n-k} (or ${}_nC_kp^kq^{n-k}$) for $n \in \mathbb{N}$ with $p+q=1$. With $0 < p < 1$ as a constant, we have

$$1 = \lim_{n\to\infty}(p+q)^n = \lim_{n\to\infty}\sum_{k=0}^{n}{}_nC_kp^kq^{n-k}.$$

Is this the meaning?

• What is the Cauchy distribution? This is $Cau(\mu, \sigma)$ that is $\mu + \sigma \frac{X}{Y}$ with $\sigma > 0$, where $Cau(0, 1)$ is the distribution of $\frac{X}{Y}$ where X and Y are independent under $N(0, 1)$. The Cauchy density is $f(x) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (x-\mu)^2}$. In particular, $f(\mu) = \frac{1}{\pi\sigma}$. We have the derivative of f at $x \in \mathbb{R}$ as

$$f'(x) = -\frac{\sigma}{\pi} \frac{2(x - \mu)}{(\sigma^2 + (x - \mu)^2)^2}.$$

It then follows that $f(\mu)$ is the maximal value and the maximum. We compute the integral

$$\begin{aligned}\int_{\mathbb{R}} f(x)dx &= \frac{1}{\pi\sigma} \int_{\mathbb{R}} \frac{1}{(\frac{x-\mu}{\sigma})^2 + 1} dx \quad (\frac{x - \mu}{\sigma} = t) \\ &= \frac{1}{\pi\sigma} \int_{\mathbb{R}} \frac{1}{t^2 + 1} \sigma dt = \frac{1}{\pi} [\arctan t]_{t=-\infty}^{\infty} = 1.\end{aligned}$$

Note also that

$$\begin{aligned}\int \frac{x}{(\frac{x-\mu}{\sigma})^2 + 1} dx &= \int \frac{\sigma t + \mu}{t^2 + 1} \sigma dt \\ &= \sigma^2 \frac{1}{2} \log(t^2 + 1) + \mu\sigma \arctan t + C.\end{aligned}$$

It then follows that the Cauchy mean $E[x]$ does not exist.

The Cauchy density for $\frac{X}{Y}$ under $N(0, 1)$ is computed as follows.

We take the transformation $u = \frac{x}{y}$ and $v = y$. Then $x = uv$ and $y = v$. The Jacobian for this transformation is

$$J = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

The vectors $(1, s)$ and $(0, 1)$ in the uv-plane are mapped respectively to (s, s) and $(0, 1)$ in the xy-plane. The volume of the parallelogram in (u, v) is 1 and that of (x, y) is $|s|$.

The density with respect to (u, v) is given by

$$\begin{aligned}g(x)g(y)|J| &\equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} |J| \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}(uv)^2} e^{-\frac{1}{2}v^2} |v|.\end{aligned}$$

Note that we have, because of independence,

$$1 = \iint_{\mathbb{R}^2} g(x)g(y)dxdy = \int_{\mathbb{R}\setminus\{0\}} dv \int_{\mathbb{R}} \frac{1}{2\pi} e^{-\frac{1}{2}(uv)^2} e^{-\frac{1}{2}v^2} |v| du.$$

The density for $u = \frac{x}{y}$ is given by the integral, by changing order of integrations, with replacing $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$ up to measure zero sets,

$$\int_{\mathbb{R}} \frac{1}{2\pi} e^{-\frac{1}{2}(uv)^2} e^{-\frac{1}{2}v^2} |v| dv = \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}(u^2+1)v^2} v dv$$
$$= \frac{1}{\pi}[-\frac{1}{u^2+1} e^{-\frac{1}{2}(u^2+1)v^2}]_{v=0}^{\infty} = \frac{1}{\pi}\frac{1}{u^2+1}.$$

The density for $\mu + \sigma\frac{x}{y}$ is given by

$$\frac{1}{\sigma}\frac{1}{\pi}\frac{1}{(\frac{u-\mu}{\sigma})^2+1} = \frac{1}{\pi}\frac{\sigma}{(u-\mu)^2+\sigma^2}.$$

That's it!

4.3 Appendix to the central limit

We may refer to [8].

The classical CLT with respect to binomial distribution is the following.

Theorem 4.3.1. *By the transformation* $t = \frac{k-m_n}{\sigma_n}$, *the binomial distribution* $B(n,p)$ *with respect to* k *as a variable converges to the normal distribution* $N(0,1)$ *with respect to* t *as* $n \to \infty$.

Proof. Recall that for the binomial $B(n,p)$, the mean $m_n = np$ and the variance $\sigma_n^2 = npq$.

The variables t and k are discrete, but they become continuous like in the limit so that with n large enough,

$$dt = \frac{1}{\sigma_n} dk$$

Let $f_n(t)$ be the distribution transformed from $B(n,p)(k)$ so that

$$f_n(t)dt = B(n,p)(k)dk$$

as change of variables. It then follows that with $p + q = 1$,

$$f_n(t) = \sqrt{npq}\binom{n}{k} p^k q^{n-k} = \sqrt{n}\frac{n!}{k!(n-k)!} p^{k+\frac{1}{2}} q^{n-k+\frac{1}{2}}.$$

On the other hand, the Stirling limit formula implies that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2n\pi} n^n e^{-n}} = 1.$$

The Stirling limit formula is

$$\lim_{x\to\infty} \frac{\Gamma(x)}{\sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}} = 1$$

where the Gamma function is defined to be

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0.$$

Namely, $n!$ is equivalent to $\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$ as $n \to \infty$. The equivalence is denoted as $\sim$. Therefore, equivalently, we have

$$\begin{aligned} f_n(t) &= \sqrt{n}\frac{n!}{k!(n-k)!}p^{k+\frac{1}{2}}q^{n-k+\frac{1}{2}} \\ &\sim \frac{\sqrt{n}}{\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}}\frac{\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}}{\sqrt{2\pi}(n-k)^{n-k+\frac{1}{2}}e^{-n+k}}p^{k+\frac{1}{2}}q^{n-k+\frac{1}{2}} \\ &= \frac{1}{\sqrt{2\pi}}(\frac{np}{k})^{k+\frac{1}{2}}(\frac{nq}{n-k})^{n-k+\frac{1}{2}} \end{aligned}$$

Note that since $t = \frac{k-m_n}{\sigma_n} = \frac{k-np}{\sqrt{npq}}$ then we have $k = np + \sqrt{npq}t$. Thus, if t is fixed and n is large, then k is large and positive. Indeed, if t is positive, then $k > np > M$ any positive with some $n > \frac{M}{p}$. If negative, then for any positive M, if

$$k = \sqrt{n}(\sqrt{n}p + \sqrt{pq}t) > M,$$

there is some $n > \frac{q}{p}(-t)^2$ such that $p(\sqrt{n})^2 + (\sqrt{pq}t)\sqrt{n} - M > 0$ so that $\sqrt{n} > \frac{\sqrt{pq}(-t)+\sqrt{pqt^2+4pM}}{2p}$ with $pqt^2 + 4pM > 0$.

It then follows that $\frac{k}{np} = 1 + \sqrt{\frac{q}{np}}t$. Also,

$$n - k = n - np - \sqrt{npq}t = nq - \sqrt{npq}t$$

so that $\frac{n-k}{nq} = 1 - \sqrt{\frac{p}{nq}}t$. Inserting these equalities into the limit equivalence above we obtain

$$f_n(t) \sim \frac{1}{\sqrt{2\pi}}(1 + \sqrt{\frac{q}{np}}t)^{-np-\sqrt{npq}t-\frac{1}{2}}(1 - \sqrt{\frac{p}{nq}}t)^{-nq+\sqrt{npq}t-\frac{1}{2}}.$$

It then follows that

$$\begin{aligned} -\log(\sqrt{2\pi}f_n(t)) &\sim (np + \sqrt{npq}t + \frac{1}{2})\log(1 + \sqrt{\frac{q}{np}}t) \\ &+ (nq - \sqrt{npq}t + \frac{1}{2})\log(1 - \sqrt{\frac{p}{nq}}t) \\ &= (np + \sqrt{npq}t + \frac{1}{2})[\sqrt{\frac{q}{np}}t - \frac{1}{2}\frac{q}{np}t^2 + O(n^{-\frac{3}{2}})] \\ &+ (nq - \sqrt{npq}t + \frac{1}{2})[-\sqrt{\frac{p}{nq}}t - \frac{1}{2}\frac{p}{nq}t^2 + O(n^{-\frac{3}{2}})] \end{aligned}$$

where $\log(1+x) = x - \frac{x^2}{2} + O(x^3)$ for x small enough. Indeed,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

with $0 < \theta = \theta(x) < 1$. Therefore, with $1 + \theta x > \frac{1}{2}$ for x small enough,

$$|\frac{\log(1+x) - x + \frac{x^2}{2}}{x^3}| = \frac{1}{3|1+\theta x|^3} < \frac{2^3}{3}.$$

As well,

$$(\sqrt{\frac{q}{np}})^3 = (\frac{q}{p})^{\frac{3}{2}} n^{-\frac{3}{2}} \quad (-\sqrt{\frac{p}{nq}})^3 = (\frac{p}{q})^{\frac{3}{2}} n^{-\frac{3}{2}}.$$

It then follows by expanding and summing the right hand side in the limit equivalence above converted to that

$$\sqrt{npq}t - \frac{1}{2}qt^2 + npO(n^{-\frac{3}{2}}) + qt^2 - \frac{1}{2}\frac{q^{\frac{3}{2}}}{\sqrt{np}}t^3 + \frac{1}{2}[\sqrt{\frac{q}{np}}t - \frac{1}{2}\frac{q}{np}t^2 + O(n^{-\frac{3}{2}})]$$

$$- \sqrt{npq}t - \frac{1}{2}pt^2 + nqO(n^{-\frac{3}{2}}) + pt^2 + \frac{1}{2}\frac{p^{\frac{3}{2}}}{\sqrt{nq}}t^3 + \frac{1}{2}[-\sqrt{\frac{p}{nq}}t - \frac{1}{2}\frac{p}{nq}t^2 + O(n^{-\frac{3}{2}})]$$

$$= -\frac{1}{2}t^2 + (n+1)O(n^{-\frac{3}{2}}) + t^2 - \frac{1}{4}\frac{p^2+q^2}{npq}t^2 + \frac{1}{2}\frac{p^2-q^2}{\sqrt{npq}}t^3 + \frac{1}{2}\frac{q-p}{\sqrt{npq}}t$$

$$= \frac{1}{2}t^2 + O(n^{-\frac{1}{2}})$$

for n large enough, where with some positive constant M,

$$|\frac{(n+1)O(n^{-\frac{3}{2}})}{n^{-\frac{1}{2}}}| \leq (n+1)Mn^{-1} \leq 2M$$

as $n \to \infty$, and as well, with $\frac{1}{\sqrt{n}} \leq 1$,

$$|-\frac{1}{4}\frac{p^2+q^2}{npq}t^2 + \frac{1}{2}\frac{p^2-q^2}{\sqrt{npq}}t^3 + \frac{1}{2}\frac{q-p}{\sqrt{npq}}t|\sqrt{n}$$

$$\leq \frac{1}{4}\frac{p^2+q^2}{pq}t^2 + \frac{1}{2}\frac{|p^2-q^2|}{\sqrt{pq}}|t|^3 + \frac{1}{2}\frac{|q-p|}{\sqrt{pq}}|t|.$$

It then follows that

$$\lim_{n\to\infty} \frac{\log(\sqrt{2\pi}f_n(t))}{-\frac{1}{2}t^2 + O(n^{-\frac{1}{2}})} = 1.$$

Moreover, taking exponential in the limit equivalence termwise if allowed, then we obtain

$$\lim_{n\to\infty} \frac{\sqrt{2\pi}f_n(t)}{\exp(-\frac{1}{2}t^2 + O(n^{-\frac{1}{2}}))} = 1.$$

By the way, $O(n^{-\frac{1}{2}})$ looks like $Mn^{-\frac{1}{2}}$ so that this function is vanishing as $n \to \infty$.

Namely, obtained is that

$$\lim_{n\to\infty} f_n(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}.$$

Note that if $f(x)$ and $g(x)$ are differentiable, with both limits as $x \to \infty$ zero or $\pm\infty$, and $f(x) \sim g(x)$ as $x \to \infty$, then $\log f(x) \sim \log g(x)$ as $x \to \infty$? In fact, the l'Hospital theorem implies that

$$\lim_{x\to\infty} \frac{\log f(x)}{\log g(x)} = \lim_{x\to\infty} \frac{f'(x)}{f(x)} \frac{g(x)}{g'(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided that $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, with respective limits of f' and g' indefinite as $x \to \infty$. If so, and if the limit is equal to 1, then this can be applied for that case above.

Similarly, the limit equivalence $f(x) \sim g(x)$ as $x \to \infty$ implies the limit equivalence $e^{f(x)} \sim e^{g(x)}$? The l'Hospital theorem implies that

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to\infty} \frac{e^{f(x)} f'(x)}{e^{g(x)} g'(x)}$$

with $e^{f(x)-g(x)} = e^{g(x)(1-\frac{f(x)}{g(x)})}$, provided that the limit in the right hand side exists. If so, and if the limit is equal to 1, with $g(x)$ vanishing to 0 as $x \to \infty$, then this cane be applied for that case.

But this is a general case. In that case, $e^{\log f(x)} = f(x)$, that's enough. □

4.4 Appendix to covariance

We may refer to [7].

The variance of a (classical) random variable f is defined to be

$$v(f) = E[(f - E[f])^2].$$

It then follows that

$$v(f) = E[f^2 - 2E[f]f + E[f]^2] = E[f^2] - E[f]^2.$$

The covariance of two (classical) random variables f and g is defined to be

$$c(f, g) = E[(f - E[f])(g - E[g])].$$

It then follows that

$$\begin{aligned} c(f, g) &= E[fg - E[g]f - E[f]g + E[f]E[g]] \\ &= E[fg] - E[f]E[g] = c(g, f). \end{aligned}$$

In particular, if $E[f] = 0$ or $E[g] = 0$, then $c(f, g) = E[fg] = c(g, f)$. Moreover, we have

$$\begin{aligned} v(f+g) &= E[(f+g)^2] - E[f+g]^2 \\ &= E[f^2 + 2fg + g^2] - E[f]^2 - 2E[f]E[g] - E[g]^2 \\ &= v(f) + v(g) + 2c(f, g). \end{aligned}$$

We may continue to investigate the next stage, but do not at this moment.

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