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TWO-VARIABLE ZETA FUNCTIONS FOR GRAPHS AND TUTTE POLYNOMIALS

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Abstract. We prove that the difference between the two-variable zeta function of a graph and a certain rational function defined using the Tutte polynomial is always a polynomial. We also proposed a conjecture specifying concrete conditions under which the two-variable zeta function can be determined solely by the Tutte polynomial.

1. Introduction

There are various invariants associated with finite undirected (connected) graphs. Among the invariants related to the chip-firing game on graphs or the theory of divisors on graphs are the Tutte polynomial, the sandpile group (Jacobian group), and the two-variable zeta function. The Tutte polynomial is a two-variable polynomial that satisfies the so-called deletion-contraction relation and includes specializations such as the chromatic polynomial and the level polynomial of critical configurations. The sandpile group is a finite abelian group formed by the critical configurations on a graph. The two-variable zeta function of a graph, introduced by Lorenzini [6], is a generating function for the ranks of divisors on the graph and can also be regarded as an analog of the local zeta function for algebraic curves over finite fields.

These three invariants are, to some extent, independent of each other in the sense that no pair among them uniquely determines the third. For instance, there exist graphs with the same Tutte polynomial and the same two-variable zeta function but non-isomorphic Jacobian groups (see [2]). On the other hand, certain relationships exist among these invariants: for example, a specialization of the numerator of the two-variable zeta function can be expressed in terms of the Tutte polynomial, and the total number of spanning trees of a graph is equal to the order of the Jacobian group or a special value of the Tutte polynomial.

In our previous paper [4], we provided concrete examples of infinite sequences of graphs with increasing genus, for which the two-variable zeta function can be explicitly determined. Notably, we observed a remarkable phenomenon: in all such examples, the two-variable zeta function could be uniformly represented by a certain rational function defined using the Tutte polynomial. Unfortunately, the reason why such a representation holds remains unclear at this time.

In this paper, for a given finite connected loopless graph G, we prove that the difference between the two-variable zeta function of G and a certain rational function defined using the Tutte polynomial of G is always a polynomial. This polynomial satisfies the same functional equation as the two-variable zeta function and has a

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degree at most $2g-4$, where g denotes the genus of the graph G. Furthermore, we propose a conjecture outlining specific conditions under which the two-variable zeta function coincides with the rational function defined using the Tutte polynomial.

2. Divisors and linear relations on graphs

We quickly review the definition and basic properties of divisors on graphs and related notions. We refer to the textbooks [5] and [3] for detailed information.

2.1. **Definitions and basic facts.** Let $G=(V,E)$ be an undirected connected finite (multi)graph without loops. The *genus* g of G is defined by $g := |E|-|V|+1$. When G is connected, we have $g \geq 0$, and $g=0$ if and only if G is a tree. For an edge $e \in E$, $G-e$ is the graph obtained from G by deleting e, and G/e is the graph obtained from G by contracting e. The degree of $v \in V$ is denoted by $d(v)$, and the number of edges between v and w $(v, w \in V)$ is denoted by $\nu(v,w)$. An edge $e \in E$ is called a *bridge* of G if $G-e$ is not connected. $T(G,x,y)$ denotes the *Tutte polynomial* of a G. It is known that $T(G,1,1)$ is the number of spanning trees of G. For later use, we define $c_i(G)$ for $i=0,1,\dots,g$ by

$$T(G,1,y) = \sum_{i=0}^{g} c_i(G) y^i. \tag{2.1}$$

Notice that $c_g(G)=1$.

We denote by $\mathrm{Div}(G)$ the divisor group on G. We often express the coefficient of v in $D \in \mathrm{Div}(G)$ by $D(v)$ as

$$D = \sum_{v \in V} D(v) v.$$

We say that $E \in \mathrm{Div}(G)$ is *effective* and write $E \geq 0$ if $E(v) \geq 0$ for all $v \in V$. The sum of all the coefficients $D(v)$ of $D \in \mathrm{Div}(G)$ is called the *degree* of D, and is denoted by $\deg D$. For convenience, we put

$$\mathrm{Div}^k(G) := \left\{ D \in \mathrm{Div}(G) \,\middle|\, \deg D = k \right\}.$$

Let $\mathrm{Prin}(G)$ be a subgroup of $\mathrm{Div}^0(G)$ generated by the elements of the form

$$\sum_{w \in V} \nu(v,w)(v-w) \quad (v \in V).$$

Two divisors $D, D' \in \mathrm{Div}(G)$ are called *linearly equivalent* if and only if $D - D' \in \mathrm{Prin}(G)$. We write $D \sim D'$ to mean that D and D' are linearly equivalent. Define

$$L(D) := \left\{ E \in \mathrm{Div}(G) \,\middle|\, E \geq 0,\ D \sim E \right\}.$$

We define the *rank* function $r\colon \mathrm{Div}(G) \to \mathbb{Z}_{\geq -1}$ by the following conditions:

(i) If $L(D) = \varnothing$, then $r(D) := -1$.
(ii) For any $s \in \mathbb{Z}_{\geq 0}$,

$$r(D) \geq s \iff L(D-E) \neq \varnothing, \quad \forall E \in \mathrm{Div}^s(G) \text{ s.t. } E \geq 0.$$

The following graph-analog of the *Riemann-Roch theorem* is known.

Theorem 2.1 (Baker-Norine [1]). *For any $D \in \mathrm{Div}(G)$, we have*

$$r(D) - r(K - D) = \deg D - g + 1.$$

Here K is given by

$$K := \sum_{v \in V} (d(v) - 2)v.$$

We further define the *Picard group* and *Jacobian group* of G by

$$\mathrm{Pic}(G) := \mathrm{Div}(G)/\,\mathrm{Prin}(G),$$
$$\mathrm{Jac}(G) := \mathrm{Div}^0(G)/\,\mathrm{Prin}(G).$$

$|\mathrm{Jac}(G)|$ is equal to the number of spanning trees of G. We put $[D] := D + \mathrm{Prin}(G) \in \mathrm{Pic}(G)$ for $D \in \mathrm{Div}(G)$.

For $q \in V$, a divisor $D \in \mathrm{Div}(G)$ is called *q-reduced* if

(1) $D(v) \geq 0$ for all $v \in V \setminus \{q\}$,

(2) for any $S \subset V \setminus \{q\}$, there exists a vertex $v \in S$ such that $D(v) - \mathrm{exdeg}_S(v) < 0$,

where $\mathrm{exdeg}_S(v)$ is the number of edges between v and vertices outside S. We denote by $\mathrm{Div}(G)_q$ the set of all q-reduced divisors on G, and put

$$\mathrm{Div}^i(G)_q := \mathrm{Div}(G)_q \cap \mathrm{Div}^i(G).$$

The following fact is useful (see Theorem 3.6 and Corollary 3.7 in [3]).

Theorem 2.2. *Let $q \in V$ be an arbitrary vertex. $\mathrm{Div}(G)_q$ is a complete system of representatives with respect to the linear equivalence. Further, if $D \in \mathrm{Div}(G)_q$, then*

$$r(D) \geq 0 \iff D(q) \geq 0.$$

3. Two-variable zeta functions of graphs

3.1. **Definition and basic facts.** For $D \in \mathrm{Div}(G)$, we put

$$h(D) := r(D) + 1.$$

Lorenzini [6] introduced the two-variable zeta function $Z(G, t, u)$ of G by

$$Z(G,t,u) := \sum_{[D] \in \mathrm{Pic}(G)} \frac{u^{h(D)} - 1}{u - 1} t^{\deg D} = \sum_{i=0}^{\infty} b_i(G,u) t^i, \tag{3.1}$$

where we put

$$b_i(G,u) := \sum_{\substack{[D] \in \mathrm{Pic}(G) \\ \deg(D) = i}} \frac{u^{h(D)} - 1}{u - 1}$$

for brevity. The following is the basic facts on the zeta functions.

Theorem 3.1 (Lorenzini [6]). (1) *There exists a polynomial $L(G, t, u) \in \mathbb{Z}[t, u]$ such that*

$$Z(G,t,u) = \frac{L(G,t,u)}{(1-t)(1-ut)}. \tag{3.2}$$

(2) *$Z(G, t, u)$ satisfies the functional equation*

$$Z(G, 1/ut, u) = (ut^2)^{1-g} Z(G,t,u). \tag{3.3}$$

(3) $L(G,0,u)=1$, $L(G,1,u)=|\mathrm{Jac}(G)|$.
(4) $L(G,t,0)=t^gT(G,1,1/t)$.

Lemma 3.2 ([4, Lemma 3.5])**.** *If e is a bridge of G, then $Z(G,t,u)=Z(G/e,t,u)$.*

Based on this lemma, in what follows, we always assume that G is a 2-*edge-connected graph* (i.e. a graph which has no bridges).

3.2. **A formula for two-variable zeta functions.** Here we recall a formula for two-variable zeta functions obtained in [4]. For a divisor $D\in\mathrm{Div}(G)$, we denote by $\mathrm{Red}_q(D)$ the unique q-reduced divisor which is linearly equivalent to D. Define

$$\mu_k(D):=\max\bigl\{-\mathrm{Red}_q(D-D')(q)\bigm| D'\in\mathrm{Div}^0(G)_q,\,-D'(q)\le k\bigr\}$$

for $k\ge 0$ and $D\in\mathrm{Div}^0(G)$. By definition, we have

$$0\le -D(q)=\mu_0(D)\le\mu_1(D)\le\mu_2(D)\le\ldots\le\mu_g(D)=\mu_{g+1}(D)=\ldots=g$$

and

$$\mu_k(D)=l\iff r(D+(k+l)q)=k. \tag{3.4}$$

The two-variable zeta function $Z(G,t,u)$ is then written as

$$Z(G,t,u)=\frac{1}{1-t}\sum_{k=0}^{\infty}(ut)^kL_k(G,t) \tag{3.5}$$

with

$$L_k(G,t):=\sum_{D\in\mathrm{Div}^0(G)_q}t^{\mu_k(D)}. \tag{3.6}$$

We notice that

$$\begin{aligned}L_0(G,t)&=t^gT(G,1,1/t)=\sum_{i=0}^{g}c_i(G)t^{g-i},\\ L_k(G,1)&=|\mathrm{Jac}(G)|\qquad(k\ge 0),\\ L_k(G,t)&=|\mathrm{Jac}(G)|t^g\qquad(k\ge g).\end{aligned}$$

The polynomial $L(G,t,u)$ in Theorem 3.1 is given by

$$L(G,t,u)=L_0(G,t)+\sum_{k=1}^{g}(ut)^k\bigl(L_k(G,t)-L_{k-1}(G,t)\bigr). \tag{3.7}$$

4. Two-variable zeta functions and Tutte polynomials

4.1. **A function defined by Tutte polynomials.** We put

$$\begin{aligned}\widetilde{Z}(G,t,u)&:=\frac{L_0(G,t)}{(1-t)(1-ut^2)}+\frac{ut^{g+1}T(G,1,ut)}{(1-ut)(1-ut^2)},\\ \widetilde{L}(G,t,u)&:=(1-t)(1-ut)\widetilde{Z}(G,t,u).\end{aligned}$$

It is straightforward to verify that $\widetilde{Z}(G,t,u)$ satisfies the functional equation

$$\widetilde{Z}(G,t,u)=(ut^2)^{g-1}\widetilde{Z}(G,1/ut,u). \tag{4.1}$$

Lemma 4.1. $\widetilde{L}(G,t,u)\in\mathbb{Z}[t,u]$.

Proof. Put $c_i = c_i(G)$ for simplicity. We show that

$$\widetilde{L}(G,t,u) = \widetilde{L}_0(G,t) + \sum_{k=1}^{g} (ut)^k \big(\widetilde{L}_k(G,t) - \widetilde{L}_{k-1}(G,t)\big), \tag{4.2}$$

where

$$\widetilde{L}_k(G,t) := \sum_{i=0}^{g} c_i t^{g-i+\min\{i,k\}} = \sum_{i=0}^{k} c_i t^g + \sum_{i=1}^{g-k} c_{k+i} t^{g-i}. \tag{4.3}$$

Let us calculate $\widetilde{L}(G,t,u)$:

$$\begin{aligned}
\widetilde{L}(G,t,u) &= \frac{1-ut}{1-ut^2} L_0(G,t) + \frac{1-t}{1-ut^2} ut^{g+1} \sum_{i=0}^{g} c_i (ut)^i \\
&= (1-ut) \sum_{j=0}^{\infty} (ut^2)^j L_0(G,t) + ut^{g+1}(1-t) \sum_{j=0}^{\infty} (ut^2)^j \sum_{i=0}^{g} c_i (ut)^i \\
&= \sum_{k=0}^{\infty} u^k t^{2k} L_0(G,t) - \sum_{k=0}^{\infty} u^{k+1} t^{2k+1} L_0(G,t) \\
&\qquad + \sum_{i=0}^{g} \sum_{j=0}^{\infty} c_i u^{j+1+i} t^{g+1+2j+i} - \sum_{i=0}^{g} \sum_{j=0}^{\infty} c_i u^{j+1+i} t^{g+2+2j+i} \\
&= L_0(G,t) + \sum_{k=1}^{\infty} u^k t^{2k} L_0(G,t) - \sum_{k=1}^{\infty} u^k t^{2k-1} L_0(G,t) \\
&\qquad + \sum_{k=1}^{\infty} u^k \sum_{\substack{0 \le i \le g \\ 0 \le j \\ k=i+j+1}} c_i t^{g+2j+i+1} - \sum_{k=1}^{\infty} u^k \sum_{\substack{0 \le i \le g \\ 0 \le j \\ k=i+j+1}} c_i t^{g+2j+i+2} \\
&= L_0(G,t) + \sum_{k=1}^{\infty} (ut)^k (t^k - t^{k-1}) \Big\{ L_0(G,t) - \sum_{\substack{0 \le i \le g \\ i \le k-1}} c_i t^{g-i} \Big\} \\
&= \widetilde{L}_0(G,t) + \sum_{k=1}^{g} (ut)^k (t^k - t^{k-1}) \sum_{i=k}^{g} c_i t^{g-i}.
\end{aligned}$$

It is immediate to see that

$$(t^k - t^{k-1}) \sum_{i=k}^{g} c_i t^{g-i} = \widetilde{L}_k(G,t) - \widetilde{L}_{k-1}(G,t). \qquad \square$$

We have observed in [4] that $Z(G,t,u) = \widetilde{Z}(G,t,u)$, that is, the zeta function $Z(G,t,u)$ is completely determined by the Tutte polynomial $T(G,x,y)$, when (i) the genus g of G is at most 2, (ii) G is either a dipole graph, a doubled tree or a friendship graph.

In what follows, we assume that $g \geq 3$. Let us look at the difference

$$f(G,t,u) := \widetilde{Z}(G,t,u) - Z(G,t,u).$$

Theorem 4.2. *$f(G,t,u) \in \mathbb{Z}[t,u]$, and*

$$f(G,t,u) = (ut^2)^{g-1} f(G, 1/ut, u). \tag{4.4}$$

Proof. It is elementary to verify that

$$L(G,1,u) = \widetilde{L}(G,1,u) = |\mathrm{Jac}(G)|, \quad L(G,t,1/t) = \widetilde{L}(G,t,1/t) = L_g(G,t).$$

This implies that $\widetilde{L}(G,t,u) - L(G,t,u)$ is divisible by $(1-t)(1-ut)$, and hence $f(G,t,u) \in \mathbb{Z}[t,u]$. Since $Z(G,t,u)$ and $\widetilde{Z}(G,t,u)$ satisfy the same functional equation ((4.1) and (3.3)), $f(G,t,u)$ also satisfies the functional equation of the same form. □

Lemma 4.3. $L_1(G,0) = \widetilde{L}_1(G,0) = 0$.

Proof. $\widetilde{L}_1(G,0) = 0$ is trivial by the definition (4.3). To prove $L_1(G,0) = 0$, it suffices to see that $\mu_1(D) \geq 1$ (or $r(D+q) \leq 0$) for any $D \in \mathrm{Div}^0(G)_q$. Let $D \in \mathrm{Div}^0(G)_q$. $D+q$ is q-reduced and $(D+q)(q) = D(q)+1$. If $D(q) < -1$, then $r(D+q) = -1$. If $D(q) = -1$, then $D = v - q$ for some $v \in V \setminus \{q\}$. In this case, we have $r(D+q) = 0$ since $L((D+q)-q) = L(D) = \varnothing$. If $D(q) = 0$, then $D = 0$. For any $v \in V \setminus \{q\}$, $(D+q) - v = -v + q$ is *v-reduced* and $(D+q)(v) = -1 < 0$. Hence $r(D+q) = 0$. □

Thus $\widetilde{L}(G,t,u) - L(G,t,u)$ is divisible by ut^2, and so is $f(G,t,u)$. By this fact and (4.4), the degree of $f(G,t,u)$ in t is at most $2g-4$. Thus $f(G,t,u)$ is of the form

$$f(G,t,u) = ut^2 \sum_{i=0}^{2g-6} a_i(G,u) t^i \tag{4.5}$$

and its coefficients satisfy the relation

$$a_{2g-6-i}(G,u) = u^{g-3-i} a_i(G,u) \quad (0 \leq i \leq g-3).$$

4.2. **The lowest coefficient $a_0(G,u)$.** Since

$$\begin{aligned}
\widetilde{L}(G,t,u) - L(G,t,u) &= (1-t)(1-ut)ut^2 \sum_{i=0}^{2g-6} a_i(G,u)t^i \\
&= ut\Big(\widetilde{L}_1(G,t) - L_1(G,t)\Big) \\
&\quad + (ut)^2\Big(\widetilde{L}_2(G,t) - L_2(G,t) - \widetilde{L}_1(G,t) + L_1(G,t)\Big) + \dots,
\end{aligned}$$

it follows that $a_0(G,u)$ is the coefficient of t in $\widetilde{L}_1(G,t) - L_1(G,t)$, which is equal to

$$1 - \left|\left\{D \in \mathrm{Div}^0(G)_q \,\middle|\, \mu_1(D) = 1\right\}\right| = 1 - \left|\left\{D \in \mathrm{Div}^0(G)_q \,\middle|\, r(D+2q) = 1\right\}\right|.$$

Let us count $\left|\left\{D \in \mathrm{Div}^0(G)_q \,\middle|\, r(D+2q) = 1\right\}\right|$. Suppose that $D \in \mathrm{Div}^0(G)_q$ and $r(D+2q) = 1$. Then we should have $r(D+q) = 0$, so that we have $D = v - q$ for some $v \in V$ (possibly equal to q). Thus we have

$$\left|\left\{D \in \mathrm{Div}^0(G)_q \,\middle|\, r(D+2q) = 1\right\}\right| = \left|\left\{v \in V \,\middle|\, r(v+q) = 1\right\}\right|.$$

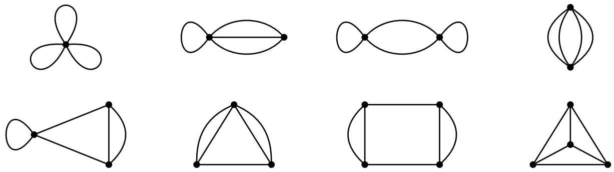
If $r(v+q) = 1$, then $r(v+q-w) = 0$ for any $w \in V$. This implies that $v+q-w$ is not *w-reduced* for any $w \in V \setminus \{v,q\}$. By an argument using Dhar's burning algorithm (see, for example, §2.6.7 of [5]), there can be at most one such v. Hence we have

$$a_0(G,u) = 0 \text{ or } 1. \tag{4.6}$$

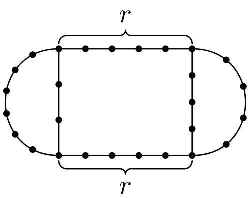
When G is of the form as in (4.7) below, we have $a_0(G,u)=0$. Indeed, for any fixed sink vertex q, if we take v to be a vertex located 'symmetrically opposite' to q, then $r(v+q)=1$.

4.3. **Examples and conjectures.**

Example 4.4 (genus 3 case)**.** Let G be a connected graph of genus 3 which does not have loops and bridges. Such a graph G is obtained as a refinement of one of the following eight graphs:

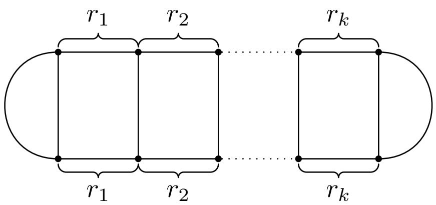


If G is a graph of the form shown below, or any degenerate form thereof, then $f(G,t,u)=0$:

$G=$ 

Otherwise, $f(G,t,u)=ut^2$. See [7].

Conjecture 4.5. *If G is of the form shown below, or any degenerate form thereof, then $f(G,t,u)=0$:*

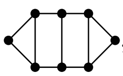
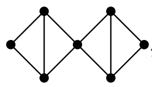
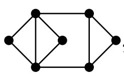
(4.7) $G=$ 

Conjecture 4.6. *If $f(G,t,u)\neq 0$, then every nonzero coefficient of $f(G,t,u)$ is positive.*

Example 4.7 (genus 4 case)**.** Let G be a connected graph of genus 4 which does not have loops and bridges. Then we have

$$f(G,t,u)=ut^2(a_0(G,u)+a_1(G,u)t+ua_0(G,u)t^2)$$

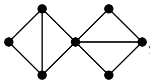
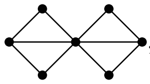
in general. For

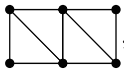
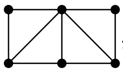
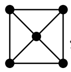
$X_1=$ , $X_2=$ , $X_3=$ ,

we have

$$f(X_1,t,u)=f(X_2,t,u)=f(X_3,t,u)=0.$$

For

$Y_1 =$ , $Y_2 =$ ,

$Y_3 =$ , $Y_4 =$ , $Y_5 =$ ,

we have

$$
\begin{aligned}
f(Y_1,t,u) &= ut^2(1+3t+ut^2),\\
f(Y_2,t,u) &= ut^2(1+5t+ut^2).\\
f(Y_3,t,u) &= ut^2(1+5t+ut^2),\\
f(Y_4,t,u) &= ut^2(1+4t+ut^2),\\
f(Y_5,t,u) &= ut^2(1+3t+ut^2).
\end{aligned}
$$

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