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A basic learning transformation of some string like theory by duality and C^*-algebras

Takahiro SUDO

Abstract

We study some string like theory by duality and C^*-algebras involved.

C*-algebra, K-theory, string theory, twisted K-theory, duality
46L80, 46L05

1 Introduction

Following Rosenberg [52] we as beginners, outsiders, fools or not begin to study the string like theory basics involving duality and C^*-algebras.

We made some considerable effort to understand the theory with limited time and space, as a back to the past for a return to the future ahead or beyond.

Let us go flying and reviewing over a mathematical crop field (or mou) half like with some distance. Without rapidly loosing speed before crashing of despair, we would like to keep doing over the math field for some possible while.

We do begin with contents as follows:

With notation used or changed by our taste.

2 Motivation and more

2.1 Some theories in physics and mathematics

Some or most theories in physics can be expressed in terms of fields in the following senses.

The fields

• Functions or vectors. Or scalar-valued functions on some spaces. Real or complex-valued functions.

• Sections of vector bundles. Or vector-valued functions.

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• Connections on principal bundles, as special cases of gauge fields.
• Sections of spinor bundles, called spinors.

A space E is said to be a principal G-bundle over a space X for a topological group G if there is a continuous free G-action on E with X the quotient space and a continuous surjective open map $p : E \to X$ such that X has a covering by open subsets U_j with each $p^{-1}(U_j)$ G-isomorphic to the product space $U_j \times G$.

$\star$ Note that

$$\begin{array}{ccc} E = \cup_j p^{-1}(U_j) \cong \cup_j (U_j \times G) & \longleftarrow & G \\ \Big\downarrow p & & \\ X = \cup_j U_j & & \end{array}$$

as a locally trivial bundle. □

A connection on such a principal G-bundle E where G is a Lie group and E, X are smooth manifolds is choosing a horizontal subspace H_ξ of the tangent space $T_\xi E$ at each $\xi \in E$ such that the differential $dp : H_\xi \to T_{p(\xi)}X$ is an isomorphisms for any $\xi \in E$.

$\star$ Note that

$$\begin{array}{ccccc} E & \xrightarrow{p} & X & & \\ T_\star E = H_\star \oplus H_\star^\perp & \xrightarrow{dp} & T_\star X & \xrightarrow{dp^{-1}} & H_\star \end{array}$$

as an x-axis. □

The fields in classical physics satisfy a variational principle of an action S, at least critical points. The (least) action S is given by the integral of a local functional L called Lagrangian.

The Euler-Lagrange equations for critical points of the action are equations of motion.

We may refer to [23] as well as [40] and more in what follows.

Example 2.1.1. $\star$ With $P = mv$ as momentum with $v = v(t)$ velocity at a time t of a particle p (at $x = x(t)$ as position) with mass m, and $\frac{dP}{dt} = F$ as force as momentum equation, we have the principle

$$S = P = \int F dt = \int L dt$$

as a case. As well, with W as work of F,

$$S = W = \int F v dt = \frac{1}{2} m v^2 = \int P$$

with $L = Fv$ as another case.

The variational principle says that

$$\int L = \int^{\sim} L$$

which means that the integrals along a path and any near path are the same, so that integrations for L among paths have critical values at L. That is also denoted as

$$\Delta \int L = \int L - \int^{\sim} L = 0.$$

□

Example 2.1.2. ⋆ Let $q_j = q_j(t)$, $t \in \mathbb{R}$ be generalized coordinates in a (dynamical) system of free degree $1 \leq j \leq f$. The Lagrange motion equation is defined to be

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad 1 \leq j \leq f$$

with $\dot{q}_j = \frac{dq_j}{dt}$, where

$$L = L(q_j, \dot{q}_j, 1 \leq j \leq f) = T - U$$

is a Lagrangian function, or Lagrangian, where T is motion energy and U is position (or potential) energy.

□

Example 2.1.3. ⋆ Let $T = \frac{1}{2}mv^2$ and $U = U(x)$ such that $-\mathrm{grad}(U) = F = ma$ with $a = \frac{dv}{dt}$ acceleration. Then we have

$$\begin{aligned} &\frac{d}{dt}\frac{\partial}{\partial v}(T - U) - \frac{\partial}{\partial x}(T - U) \\ &= \frac{d}{dt}mv - F = ma - F = 0. \end{aligned}$$

That's it. □

Example 2.1.4. In the theory of optics by Fermat, as governed by variational principles, a light travels along the curve which minimizes its traveling time. This is a functional of the path space of curves. On the other hand, when a light passes from one medium to another, with different speeds in the two media, the light does not necessarily travel along a straight line.

Example 2.1.5. Let M be a real 4-dimensional manifold. Let M be compact and represent our space time (or time and space).

The Yang-Mills theory is given as follows. The field in this theory is a connection on a principal G-bundle (over M), where G is come compact (linear) Lie group. The field strength F is the curvature, that is, a $\mathfrak{g}$-valued 2-form, where $\mathfrak{g}$ is the (matrix) Lie algebra of G. The action is given by

$$S = \int_M \mathrm{tr}(F \wedge *F)$$

where $F \wedge *F$ is a 4-form with values in $\mathfrak{g}$, composed with tr trace,and $\int_M$ integrated over M. If the bundle is non-trivial, then F does not vanish usually, since F is related to the characteristic classes of the bundle by the Chern-Weil theory.

General relativity in Euclidean signature is given as follows. The field in this theory is a Riemannian metric g on M. The action is given as

$$S = \int_M R dv$$

where R is the scalar curvature of the metric. Associated to this field is the Einstein equation.

Example 2.1.6. If no matter in a region under gravity, we have

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 0.$$

With matter, we have $G_{ij} = \kappa T_{ij}$ where κ is gravity constant and T_{ij} is energy momentum tensor representing dynamical state of a matter such as energy and momentum.

Example 2.1.7. $\star$ Let $E = \mathfrak{g} \times M$ over M. Then a section of the bundle $E \otimes \wedge^r T^* M$ over M is said to be a $\mathfrak{g}$-valued differential r-form.

For any $X \in \mathfrak{g}$, there corresponds to the subgroup $\exp tX$ of G.

There also corresponds to $*X$ the bundle over E, that is vertical as $H_*^{\perp}$.

Possibly, that's it like $*$. We may denote it as $X^{\perp}$.

There is a connection form ω over M with valued in E, that is defined to be a $\mathfrak{g}$-valued differential 1-form satisfying that at any $p \in E$, $\omega_p(*X_p) = X_p$ for $X \in \mathfrak{g}$, and $\omega_p(Y_p) = 0$ for $Y_p \in H_p$.

Namely, locally, for instance,

$$\omega_p = X_p \otimes \sum_j f_j(p)(dx_j)_p$$

where

$$X_p = \sum_j g_j(p)(\frac{\partial}{\partial x_j})_p$$

with

$$(dx_j)_p((\frac{\partial}{\partial x_k})_p) = \delta_{jk} = \begin{cases} 1 & (j = k), \\ 0 & (j \neq k) \end{cases}.$$

Let V be a finite dimensional vector space. For ω a V-valued differential k-from over E or M, the covariant differential $D\omega$ is defined by

$$(D\omega)(X_1, \cdots, X_{k+1}) = (d\omega)(hX_1, \cdots, hX_{k+1})$$

where each X_j is a vector field over E and h means the projection to the horizontal components, so that $D\omega$ is a V-valued differential $(k+1)$-form over E.

In particular,

$$(D\omega)_p = D(X_p \otimes \sum_j f_j(p)(dx_j)_p) = X_p \otimes \sum_j df_j(p) \wedge (dx_j)_p \circ (h, h).$$

In particular, for a connection 1-form ω over M with valued in E, the curvature form ω is defined to be the covariant differential $D\omega$, that is 2-form.

□

Example 2.1.8. ⋆ Let g be a symmetric tensor of $T^*M \otimes T^*M$. Then g_p is a symmetric bilinear form on T_pM. Namely,

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

such that $g_p(X, Y) = g_p(Y, X)$ and

$$g_p(\alpha_1 X_1 + \alpha_2 X_2, Y) = \alpha_1 g_p(X_1, Y) + \alpha_2 g_p(X_2, Y).$$

In particular, $g_p(0, 0) = g_p(0, 0) + g_p(0, 0)$. Thus, $g_p(0, 0) = 0$.

If g_p at any $p \in M$ is non-degenerate, then g is said to be a psudo Riemannian metric. Namely, $g_p(X, Y) = 0$ for any Y imply that $X = 0$.

If g_p is positive definite, then g is said to be a Riemannian metric. Namely, $g_p(X, X) > 0$ for any nonzero X. Then the norm for a tangent vector X in T_pM is defined as

$$\|X\| = \sqrt{g_p(X, X)}.$$

Locally,

$$g = \sum g_{ij} dx_i dx_j$$

where the matrix $G_p = (g_{ij,p})$ with $g_{ij,p}$ of g_{ij} at p as components is a positive definite symmetric matrix. Namely, with $(X_j) = (\frac{\partial}{\partial x_j})$ as a basis,

$$g_{ij,p} = g_p(X_i, X_j) = g_{ji,p}.$$

Also,

$$\begin{aligned} g_p(\sum_j \alpha_j X_j, \sum_k \alpha_k X_j) &= \sum_{j,k} \alpha_j \alpha_k g_p(X_j, X_k) \\ &= \sum_{j,k} \alpha_j \alpha_k g_{jk,p} = \sum_j (\sum_k g_{jk,p} \alpha_k) \alpha_j \\ &= \langle (g_{ij,p})(\alpha_k) | (\alpha_j) \rangle = \langle G_p \alpha | \alpha \rangle \end{aligned}$$

with $\alpha = (\alpha_j)$ as a vector in a real Euclidean space.

The curvature tensor R is defined as

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z)$$

with $-\nabla_{[X,Y]}(Z)$ to be added, where ∇ is the covariant derivative of the Riemannian or Levi-Civita connection with respect to the orthonormal frame bundle

$$\begin{array}{ccc} \cup_{p\in M, M=\sum_j \mathbb{R}e_j}(p,\{e_1,\cdots,e_n\}) & \longleftarrow & O(n) \\ \downarrow & & \\ M(\dim M = n) & & \end{array}$$

Locally,

$$R = \sum R^i_{jkl} dx_j dx_k dx_l \frac{\partial}{\partial x_i}.$$

where the (function) components of the curvature tensor are defined as

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x_k} - \frac{\partial \Gamma^i_{kj}}{\partial x_l} + \sum_m (\Gamma^m_{ij}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}).$$

where the Christoffel symbols Γ^k_{ij} are defined as

$$\sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}.$$

Or locally,

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g_{kl} \{ \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \}.$$

Or

$$\sum_i R^i_{jkl} \frac{\partial}{\partial x_i} = \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_l} - \nabla_{\frac{\partial}{\partial x_k}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_l}.$$

$$M \xrightarrow{R} (\otimes^3 T^* M) \otimes TM \longrightarrow M.$$

The Riemannian or sectional curvature is defined to be

$$K_p(Q) = g_p(R(X,Y)Y, X)$$

as an inner product, where Q is a 2-dimensional subspace of T_pM with basis of X and Y.

The Ricci curvature (tensor) (reduced) is defined to be

$$R_{ij} = -\sum_k R^k_{ijk}.$$

Or the minus may be removed. Namely,

$$R = \sum R_{ij} dx_i dx_j = \pm \sum_k R^k_{ijk} dx_i dx_j dx_k \frac{\partial}{\partial x_k}.$$

The Ricci curvature is defined as

$$\mathrm{Ric} = (R_{ij})_p : T_p \times T_p \to \mathbb{R}.$$

Also,

$$\mathrm{Ric}(L, L) = \sum_i g_p(R(X_i, L)L, X_i)$$

where (X_i) is an orthogonal basis of T_pM. As well, the scalar curvature is defined as

$$R = \sum_i \mathrm{Ric}(X_i, X_i).$$

$\square$

Example 2.1.9. $\star$ Let $X = \sum g_j \frac{\partial}{\partial x_j}$ be a vector field on M. Namely, $X : M \to TM$ as a section. For a function f on M, we have the Lie derivative locally

$$L_X f = Xf = \sum \frac{\partial f}{\partial x_j} g_j.$$

We have $L_X Y = [X, Y]$ for another vector field Y.

Also, $L_X L_Y - L_Y L_X = L_{[X,Y]}$.

The covariant differentiation (or connection) of a bundle E over M is defined to be a linear map between the spaces of sections

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that the Leibniz formula holds

$$\nabla(f\xi) = df \otimes \xi + f\nabla\xi.$$

In particular, for $X \in T_xM$, we have

$$\nabla\xi(X) = \nabla_X \xi.$$

Then ∇_X is a linear map from $\Gamma(E)$ to E_x satisfying

$$\nabla_X(f\xi) = (Xf)\xi + f\nabla_X\xi.$$

$\square$

In **quantum mechanics**, observable quantities are presented by non-commuting operators A represented on an (infinite dimensional) Hilbert space H (of square integrable measurable functions on a measured space).

Particles are represented by wave functions as unit vectors ξ in H

The state of ξ by an observable A is defined to be the inner product

$$\langle A\xi, \xi\rangle = \varphi(A) \in \mathbb{C}$$

as expectation, which is viewed as a linear functional $\varphi = \varphi_\xi$ also called state, defined so and on noncommutative operators, as integration.

Quantum field theory as quantization of classical theories is based on path integrals.

The partition function is given as

$$Z = \begin{cases} \int e^{iS(\varphi)} d\varphi, & \text{Lorentz signature} \\ \int e^{-S(\varphi)} d\varphi, & \text{Euclidean signature} \end{cases}$$

Quantum fields are $\varphi = \varphi_\xi$.
Expectation value of A is given by a path integral

$$\frac{1}{Z} \int \varphi(A) e^{iS(\varphi)} d\varphi.$$

Example 2.1.10. $\star$ Let A be a self-adjoint operator on a Hilbert space H as observable. Suppose that $A = \sum a_n P_n$ as a spectral decomposition where a_n are real eigenvalues of A and each P_n is the projection to the eigenspace of a_n.

When A is observed at a unit state $\xi \in H$ with norm one, which of eigenvalues of A, say a_n is measured and its probability is given by $p_n = \langle \xi, P_n \xi \rangle^2$ up to constant $e^{i\theta}$ in the circle. In this moment, the state is transformed to $P_n \xi$. $\square$

Example 2.1.11. $\star$ In quantum mechanics, we suppose that self-adjoint operators Q_k as coordinates and P_k as momentum satisfy the canonical commutation relations

$$[Q_k, P_l] = Q_k P_l - P_l Q_k = i\hbar \delta_{kl} 1, \quad [Q_k, Q_l] = [P_k, P_l] = 0$$

with $\hbar = \frac{h}{2\pi}$ and h the Planck constant. Quantum energy $E = h\nu$ with ν frequency.

The Schrödinger rerepsentation (densely defined and extended to be completed) on the Hilbert space $L^2(\mathbb{R}^n)$ is given by that Q_k are multiplication operators M_{x_k} of variables x_k of $\mathbb{R}^n$ and P_k are (closures of) partial differential operators $-i\hbar \frac{\partial}{\partial x_k}$.

For $f(x) \in L^2(\mathbb{R}^n)$ to be differentiable or smooth, with $k \neq l$ we have

$$\begin{aligned} [Q_k, P_l] f(x) &= (Q_k P_l f)(x) - (P_l Q_k f)(x) \\ &= -i\hbar x_k \frac{\partial f}{\partial x_l}(x) - (-i\hbar) x_k \frac{\partial f}{\partial x_l}(x) = 0. \end{aligned}$$

If $k = l$, then

$$\begin{aligned} [Q_k, P_k] f(x) &= (Q_k P_k f)(x) - (P_k Q_k f)(x) \\ &= -i\hbar x_k \frac{\partial f}{\partial x_k}(x) - (-i\hbar)(f(x) + x_k \frac{\partial f}{\partial x_k}(x)) = i\hbar f(x). \end{aligned}$$

As well, with $k \neq l$ or not, we have

$$\begin{aligned} [Q_k, Q_l] f(x) &= (Q_k Q_l f)(x) - (Q_l Q_k f)(x) \\ &= x_k x_l f(x) - x_l x_k f(x) = 0. \end{aligned}$$

Moreover, for f of C^2-class,

$$[P_k, P_l]f(x) = (P_k P_l f)(x) - (P_l P_k f)(x)$$
$$= (-i\hbar)^2 \frac{\partial^2 f}{\partial x_k \partial x_l}(x) - (-i\hbar)^2 \frac{\partial^2 f}{\partial x_l \partial x_k}(x) = 0.$$

□

Example 2.1.12. ⋆ Time evolution of a state $\xi \in H$ is given by $\xi_t = U_t \xi \in H$ by unitary operator U_t on H for $t \in \mathbb{R}$.

By weak continuity of U_t with respect to inner product, assumed, there is a self-adjoint operator H on H such that $U_t = \exp(-\frac{it}{\hbar}H)$, called Hamilton operator or Hamiltonian. The H operator represents total energy of such a state system as physical interpretation.

The time-dependent Schrödinger equation is given by

$$i\hbar \frac{\partial}{\partial t} \xi_t = H \xi_t$$

We have

$$\frac{\partial}{\partial t} \xi_t = \frac{\partial}{\partial t} \exp(-\frac{it}{\hbar} H)\xi = -\frac{i}{\hbar} H \xi_t.$$

□

Example 2.1.13. We have

$$g = \sum g_{ij} dx_i dx_j = ds^2.$$

Also,

$$ds = \sum \sqrt{g_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds}} ds.$$

In special theory of relativity,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

with $(x_0, x_1, x_2, x_3) = (ct, x, y, z)$ with c light velocity or speed, where $t \in \mathbb{R}$, $(x, y, z) \in \mathbb{R}^3$. The metric makes the Minkowski space time.

2.2 Some string like theory basics

Ideas. Replacing particles as points in physics with strings of dimension one is the basic idea in string theory.

A point x may be changed to an interval I of some dimension.

Strings in low energy are expected to be short or small, with respect to the Plank length

$$l_{pk} = \sqrt{\frac{\hbar G}{c^3}} \approx \frac{1.616}{10^{35}} [m].$$

Or powers.

Moving of a string on time and in space makes a 2-dimensional surface in space. This is said to be a world-sheet denoted as Σ. Why?

Fields in string theory are defined to be maps φ from Σ a 2-manifold to X the space time.

String theory may combine four forces in classical and quantum physics such as gravity, electromagnetic force, and nuclear weak and strong forces.

$\star$ It looks like respective two by two, isn't it? Certainly, two forces are visible as actions, but the other two are not in sight.

Σ **model.** Let Σ be sheet of a string and X the manifold of time and space.

String theory is based on a non-linear Σ model.

The fundamental field is given by a map $\varphi : \Sigma \to X$. The action as the leading term is given by

$$S(\varphi) = \frac{1}{4\pi\alpha} \int_\Sigma \|\nabla\varphi\|^2 dv$$

as the energy of the map φ in Euclidean signature.

This action is called the string Σ model action or Polyakov action in the lierature in physics. The constant α is called the Regge slope parameter, representing square of string length, and $\frac{1}{2\pi\alpha}$ is called the string tension.

Such various gauge fields give rise to fundamental particles, to be added. Such a gravity term involves the scalar curvature of the metric on X.

Also, super symmetry may be required. In this case, the theory involves both bosons and fermions, both of which are interchanged by symmetries.

$\star$ The domain Σ may be fixed or not. The maps $\varphi : \Sigma \to X$ as fields makes the world, do they? Like Ggle maps or YoTb movies do it.

Example 2.2.1. $\star$ Let

$$\varphi = \varphi(s,t) = \begin{pmatrix} \varphi_1(s,t) \\ \vdots \\ \varphi_4(s,t) \end{pmatrix} \in X = \mathbb{R}^3 \times \mathbb{R}$$

for $(s,t) \in \Sigma$ of real variables locally. Then

$$\nabla\varphi(s,t) = \begin{pmatrix} \frac{\partial\varphi}{\partial s}(s,t) \\ \frac{\partial\varphi}{\partial t}(s,t) \end{pmatrix} = (\varphi_s, \varphi_t)(s,t) \in X \times X$$

or in

$$T_{(s,t)}\Sigma \times T_{(s,t)}\Sigma \subset (\Sigma \times X) \times (\Sigma \times X).$$

We have

$$\|\nabla\varphi\|^2 = \langle \nabla\varphi, \nabla\varphi \rangle = \langle \varphi_s, \varphi_s \rangle + \langle \varphi_t, \varphi_t \rangle.$$

□

B**-field and** H**-flux.** Another term called the Wess-Zumino term is added to the action $S(\varphi)$ above, defined as

$$\frac{1}{4\pi\alpha}\int_\Sigma \varphi^* B,$$

where B is a 2-form on X defined locally and called the B-field. It need not be closed or defined globally, since strings are small at a moment. But the differential $dB = H$ should be a well-defined closed 3-form on X called the H-flux.

$\star$ Flux means lattice or bundle. Note that $\varphi^* B = B \circ \varphi$.

The 3-form corresponds to an integral or torsion free cohomology class as in $H^3(X)$, which is also called the H-flux, with overlooking the torsion part.

Let $\varphi(\Sigma)$ be an embedded surface in X by φ, like a tubed circle. Suppose that there are two different 3-manifolds M and M' which bound the surface, which look like solid handle bodies. Namely, the boudnary $\partial M = \partial M' = \varphi(\Sigma)$.

The Stokes theorem implies that

$$\int_M H = \int_M dB = \int_{\partial M} B = \int_\Sigma \varphi^* B.$$

Similarly, this holds for M' replacing M. Therefore, we have

$$\int_M H = \int_{M'} H.$$

Let N be a closed 3-dimensional submanifold of X obtained by gluing M and M' along their boundary surface, where N is oriented by the orientation on M and the reverse orientation on M'. Namely, $N = M \cup_{\varphi(\Sigma)} (-M')$. Then the equation above implies that $\int_N H = 0$.

For this reason, we may need H to be paired with every closed 3-dimensional submanifold of X with paired integration to be zero.

Example 2.2.2. $\star$ Let S be a smooth surface of dimension m, contained in $\mathbb{R}^n$, and let ∂S be the boundary surface of S of dimension $m-1$. Let ω be a $(m-1)$-form of C^1-class. Then the Stokes formula holds as

$$\int_{\partial S} \omega = \int_S d\omega.$$

□

Example 2.2.3. $\star$ Let $\varphi = \varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$ be a continuous map from the interval $I = [a, b] \subset \mathbb{R}$ to $\mathbb{R}^n$. For a function $f(x_1, \cdots, x_n)$ defined on a neighbourhood of the image $\varphi(I) = C$, the curvilinear integral of f with respect to x_j along C is defined to the Stieltjes integral as

$$\int_a^b f(\varphi_1(t), \cdots, \varphi_n(t)) d\varphi_j(t),$$

which is denoted by $\int_C f dx_j$.

The arc length of C is defined to be

$$s(t) = \int \sqrt{(\varphi_1'(t))^2 + \cdots + (\varphi_n'(t))^2} dt$$

up to a constant. The Stieltjes integral of f with respect to the arc length s is the curvilinear integral of f with respect to line, denoted as $\int_C f ds$.

For a 0-form f, we have $df = \sum_j \frac{\partial f}{\partial x_j} dx_j$.

For a 1-form $\omega = f_1 dx_1 + \cdots + f_n dx_n$, its curvilinear integral is defined to be

$$\int_C \omega = \sum_{j=1}^{n} \int_C f_j dx_j.$$

□

Example 2.2.4. ⋆ Let $\varphi = \varphi(u) = (\varphi_1(u_1, \cdots, u_m), \cdots, \varphi_n(u_1, \cdots, u_m))$ be a class C^1 regular map from a domain $G \subset \mathbb{R}^m$ to $\mathbb{R}^n$. The image S of φ is said to be an m-dimensional smooth surface. For a continuous function $f(x_1, \cdots, x_n)$ defined on a neighbourhood of $S \subset \mathbb{R}^n$, the surface integral of f with respect to $x_{i_1}, \cdots, x_{i_m}$ of $x_1, \cdots, x_n$ on S is defined as

$$\int_G f(\varphi(u)) \frac{\partial(x_{i_1}, \cdots, x_{i_m})}{\partial(u_1, \cdots, u_m)} du_1 \cdots du_m.$$

This is denoted as $\int_S f dx_{i_1} \cdots dx_{i_m}$.

When the Jacobian $\frac{\partial(\cdots)}{\partial(\cdots)}$ is replaced with

$$\sqrt{\sum_{1 \le i_1 < \cdots < i_m \le n} (\frac{\partial(x_{i_1}, \cdots, x_{i_m})}{\partial(u_1, \cdots, u_m)})^2}$$

the surface integra with respect to surface is defines so, denoted as $\int_S f dS$ or $\int_S f d\sigma$. □

Example 2.2.5. The Wes-Zumino-Witten model (WZW). Let $X = G$ be a compact semi-simple Lie group like $SU(2)$. Let H be the canonical 3-form corresponding to the tripling map as

$$(X, Y, Z) \mapsto \langle [X, Y], Z \rangle$$

on the Lie algebra $\mathfrak{g}$ of G, with respect to an invariant inner product on $\mathfrak{g}$ such as from the killing form. Then H is closed but not exact, with B not defined globally.

Note that $\pi_3(X) \cong \mathbb{Z}$ for any compact simple Lie group G. Also, H normalized has an integral de Rham class dual to the image of the generator of $\pi_3(X)$ under the Hurewicz map.

Example 2.2.6. $\star$ Let K be a commutative ring with unit such as real $\mathbb{R}$ or complex $\mathbb{C}$. A Lie algebra $\mathfrak{g}$ over K is defined to be a K-module satisfying the following structure. There is a bracket product as a K-bilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ such that

$$[\sum_i \alpha_i X_i, \sum_j \beta_j Y_j] = \sum_{i,j} \alpha_i \beta_j [X_i, Y_j] \quad \alpha_i, \beta_j \in K, \quad X_i, Y_j \in \mathfrak{g}$$

and $[X, X] = 0$ so that $[X + Y, X + Y] = 0$ implies $[X, Y] = -[Y, X]$, and the Jacobi identity holds as

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad X, Y, Z \in \mathfrak{g}.$$

An algebra A over K can be a Lie algebra with bracket defined as $[x, y] = xy - yx$ for $x, y \in A$.

In particular, the general linear Lie algebra $\mathfrak{gl}(n, K)$ is defined to be the $n \times n$ matrix algebra $M_n(K)$ with such bracket.

Let V be a linear space over K. Let $L(V)$ denote the algebra of all linear maps from V to V. Then the Lie algebra $\mathfrak{gl}(V)$ is defined to the Lie algebra of $L(V)$ with bracket as commutator.

If $\dim V = m$, then $\mathfrak{gl}(V) \cong \mathfrak{gl}(m, K)$.

A representation ρ of a Lie algebra $\mathfrak{g}$ is defined to be a homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ such that

$$\rho([X, Y]) = [\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \rho(Y)\rho(X) \quad X, Y \in \mathfrak{g}.$$

In particular, for taking $\mathfrak{g}$ as V, the adjoint representation of $\mathfrak{g}$ is defined as

$$\mathrm{ad}(X)Y = [X, Y] \quad X, Y \in \mathfrak{g}.$$

Note that the Jacobi identity implies that

$$\begin{aligned} &\mathrm{ad}([X, Y])Z = [[X, Y], Z] = [X, [Y, Z]] + [Y, [Z, X]] \\ &= \mathrm{ad}(X)\mathrm{ad}(Y)Z - \mathrm{ad}(Y)\mathrm{ad}(X)Z = [\mathrm{ad}(X), \mathrm{ad}(Y)]Z. \end{aligned}$$

For a representation ρ of $\mathfrak{g}$ on V, we define a symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$ by trace on $\mathfrak{gl}(V)$ as

$$B_\rho(X, Y) = \mathrm{tr}(\rho(X)\rho(Y)) \in K, \quad X, Y \in \mathfrak{g}.$$

It then holds as invariance that

$$B_\rho([X, Z], Y) = B_\rho(X, [Z, Y]).$$

Note that traceness implies that

$$\begin{aligned} &\mathrm{tr}(\rho([X, Z])\rho(Y)) = \mathrm{tr}([\rho(X), \rho(Z)]\rho(Y)) \\ &= \mathrm{tr}(\rho(X)\rho(Z)\rho(Y)) - \mathrm{tr}(\rho(Z)\rho(X)\rho(Y)) \\ &= \mathrm{tr}(\rho(X)\rho(Z)\rho(Y)) - \mathrm{tr}(\rho(X)\rho(Y)\rho(Z)) \\ &= \mathrm{tr}(\rho(X)[\rho(Z), \rho(Y)]) = B_\rho(X, [Z, Y]). \end{aligned}$$

In particular, the Killing form of $\mathfrak{g}$ is defined to be B_{ad}. Namely,

$$B_{\mathrm{ad}}(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

This should be the inner product with invariance defined as $\langle\cdot,\cdot\rangle_{\mathrm{ad}} = B_{\mathrm{ad}}(\cdot,\cdot)$, in general? We may refer to [35] in what follows.

But for the Lie algebra $\mathfrak{g}$ of a compact Lie group, the Killing form is negative definite on the semi-simple part $\mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}]$, so that $\langle X, X\rangle_{\mathrm{ad}} < 0$ for nonzero $X \in \mathfrak{g}_1$, and as well $\langle X, Z\rangle_{\mathrm{ad}} = 0$ for $X \in \mathfrak{g}$ and $Z \in \mathfrak{z}$ the center.

Note that we compute that

$$\begin{aligned}
&\{\mathrm{ad}([X_1,X_2])\mathrm{ad}([Y_1,Y_2])\}^* = [\mathrm{ad}(Y_1),\mathrm{ad}(Y_2)]^*[\mathrm{ad}(X_1),\mathrm{ad}(X_2)]^* \\
&= [\mathrm{ad}(Y_2)^*,\mathrm{ad}(Y_1)^*][\mathrm{ad}(X_2)^*,\mathrm{ad}(X_1)^*] \\
&= \mathrm{ad}([Y_1^*,Y_2^*])\mathrm{ad}([X_1^*,X_2^*]).
\end{aligned}$$

For the Lie algebra $\mathfrak{g}$ of a reductive Lie group, there is a Lie algebra direct sum decomposition as $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1$.

The Hermite inner product on a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}^* = \mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ is defined as $\langle X, Y\rangle = \mathrm{tr}(X^*Y)$. Then we have

$$\langle \mathrm{ad}(X)Y, Z\rangle = \langle Y, \mathrm{ad}(X^*)Z\rangle.$$

Indeed, we check that

$$\begin{aligned}
&\langle \mathrm{ad}(X)Y, Z\rangle = \mathrm{tr}((XY - YX)^*Z) \\
&= \mathrm{tr}(Y^*X^*Z) - \mathrm{tr}(X^*Y^*Z) \\
&= \mathrm{tr}(Y^*X^*Z) - \mathrm{tr}(Y^*ZX^*) = \langle Y, \mathrm{ad}(X^*)Z\rangle.
\end{aligned}$$

Hence, $\mathrm{ad}(X)^* = \mathrm{ad}(X^*)$. □

Example 2.2.7. ⋆ The Hurewicz homomorphism τ from the homotopy $\pi_n(X,A)$ to the homology $H_n(X,A)$ is defined by $\tau([f]) = f_*(\varepsilon_n)$, where ε_n is the generator of $H_n(I^n, \partial(I^n))$.

Let X be a topological space with a base point $x \in X$. The n-dimensional homotopy group $\pi_n(X)$ with x omitted is defined as the group of homotopy classes $[f]$ of continuous maps f from $(I^n, \partial(I^n))$ to (X, x) respectively, where $I^n = \Pi^n[0,1]$ the n-dimensional cube and $\partial(I^n)$ is the boundary of I^n.

When $n = 1$, the homotopy group is the fundamental group of X.

When $n \geq 2$, the homotopy group is abelian.

Let X be a topological space and $A \subset X$ a subspace with a common base point $x \in A$. The n-dimensional relative homotopy group $\pi_n(X,A)$ of (X,A) for $n \geq 2$ is defined to be the group of homotopy classes $[f]$ of continuous maps f from $(I^n, \partial(I^n), \overline{\partial(I^n) \setminus I^{n-1}})$ to (X, A, x) respectively.

When $n \geq 3$, the relative homotopy group is abelian.

We have the following diagram

$$\begin{array}{ccccc}
(I^n, \partial(I^n)) & \xrightarrow{f} & (X,A) & \xrightarrow{\textit{homotopy}} & \pi_n(X,A) \\
\downarrow{\scriptstyle \textit{homology}} & & \downarrow{\scriptstyle \textit{homology}} & & {\scriptstyle \mathrm{Hu}}\downarrow{\scriptstyle \tau} \\
H_n(I^n, \partial(I^n)) & \xrightarrow{f_*} & H_n(X,A) & = & H_n(X,A)
\end{array}$$

□

Example 2.2.8. For a compact topological group G, the following are equivalent.

(1) G is a subgroup of the general linear group over $\mathbb{R}$ or $\mathbb{C}$ as a topological group.

(2) G is a Lie group.

For this, we use the Peter-Weyl theorem.

For a non-compact Lie group G like $SL_2(\mathbb{R})$, the universal covering group $G^\sim$ of the group can not be such a subgroup.

Let $\mathfrak{g}$ be the Lie algebra of a compact semi-simple Lie group G. There is a finite direct sum decomposition of $\mathfrak{g}$ by simple Lie algebras $\mathfrak{g}_j$. There corresponds to each $\mathfrak{g}_j$ the simply connected Lie group $G_j^\sim$ with center Z_j. Then the direct product group of $G_j^\sim$ is a simply connected compact Lie group $G^\sim$ with $\mathfrak{g}$ the Lie algebra and finite center Z the product of Z_j. There is a finite subgroup Γ of Z such that $G = G^\sim/\Gamma$, and Γ is the same as the fundamental group of G.

Example 2.2.9. The fundamental group of $SO(2) \approx S^1$ is $\mathbb{Z}$.

The fundamental group of $SO(m)$ for $m \geq 3$ is $\mathbb{Z}_2$.

The $SO(3)$ is homeomorphic to the real projective space $P_3(\mathbb{R}) = (\mathbb{R}^4 \setminus \{0\})/\sim$ of real dimension 3, so that the fundamental group of $SO(3)$ is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

The universal covering group $SO(m)^\sim$ of $SO(m)$ is called the spinor group denoted by $\mathrm{Spin}(m)$.

Table 1: Classical simply connected compact simple Lie groups

Type	Group	Center	Dimension	n, k
A_n	$SU(n+1)$	$\mathbb{Z}_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$	$n(n+2)$	$n \geq 1$
B_n	$SO(2n+1)^\sim$	$\mathbb{Z}_2$	$n(2n+1)$	$n \geq 2$
C_n	$Sp(n)$	$\mathbb{Z}_2$	$n(2n+1)$	$n \geq 3$
D_n	$SO(2n)^\sim$	$\mathbb{Z}_4$ $(n = 2k+1)$	$n(2n-1)$	$k \geq 2$
	$SO(2n)^\sim$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ $(n = 2k)$	$n(2n-1)$	$k \geq 2$

Example 2.2.10. The universal covering group $G^\sim$ of a connected Lie group G is defined to be the group of homotopy classes $[f]$ of continuous functions f from the closed interval $[0, 1]$ to G such that $f(0)$ is the identity 1 of G.

The covering map $q : G^\sim \to G$ is defined by $q([f]) = f(1) \in G$.

Note that two continuous functions $[0, 1] \to G$ with the same homotopy class have the same end points at 0 and 1! in their curves in G.

The fundamental group $\pi_1(G)$ of G is defined to be the discrete subgroup $q^{-1}(1)$ of $G^\sim$. This group is a subgroup of the center of $G^\sim$ so that it is abelian.

If the fundamental group of G is trivial, then G is called simply connected.

Example 2.2.11. We have the following topological space isomorphisms (cf. [58]).

$$SU(n)/SU(n-1) \cong S^{2n-1}, \quad SO(n)/SO(n-1) \cong S^{n-1}, \quad Sp(n)/Sp(n-1) \cong S^{4n-1}$$

for $n \geq 2$. We also have that $SU(1) = \{1\}$, $SO(1) = \{1\}$, and $Sp(1) \cong S^3$ so that $\pi_1(Sp(1)) \cong 0$.

It then follows that with dim real dimension,

$$\dim SU(n) = 0 + 3 + 5 + \cdots + (2n-1) = \frac{(n-1)(3+(2n-1))}{2} = (n-1)(n+1).$$
$$\dim SO(n) = 0 + 1 + 2 + \cdots + (n-1) = \frac{(n-1)n}{2}.$$
$$\dim Sp(n) = 3 + 7 + 11 + \cdots + (4n-1) = \frac{n(3+(4n-1))}{2} = n(2n+1).$$

By induction, there is the following exact sequence of homotopy groups.

$$\begin{array}{ccccccc} \pi_2(S^{n-1}) & \longrightarrow & \pi_1(SO(n-1)) & \longrightarrow & \pi_1(SO(n)) & \longrightarrow & \pi_1(S^{n-1}) \\ \| & & \| & & \| & & \| \\ 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0. \end{array}$$

By induction, there is the following exact sequence of homotopy groups.

$$\begin{array}{ccccccc} \pi_2(S^{2n-1}) & \longrightarrow & \pi_1(SU(n-1)) & \longrightarrow & \pi_1(SU(n)) & \longrightarrow & \pi_1(S^{2n-1}) \\ \| & & \| & & \| & & \| \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

By induction, there is the following exact sequence of homotopy groups.

$$\begin{array}{ccccccc} \pi_2(S^{4n-1}) & \longrightarrow & \pi_1(Sp(n-1)) & \longrightarrow & \pi_1(Sp(n)) & \longrightarrow & \pi_1(S^{4n-1}) \\ \| & & \| & & \| & & \| \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Example 2.2.12. For any unitary $A \in U(n)$, there is a unitary $B \in U(n)$ such that

$$B^{-1}AB = e^{i\theta_1} \oplus \cdots \oplus e^{i\theta_n}, \quad \theta_j \in \mathbb{R}$$

as a diagonal sum. This holds for $A \in SU(n)$ with $B \in SU(n)$ by replacing B with $\frac{1}{(\det B)^{\frac{1}{n}}}B$. If A is in the center of $SU(n)$, then $A = B^{-1}AB = \oplus_{j=1}^n e^{i\theta_j}$. Note that we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}.$$

It then follows that $A = \oplus_{j=1}^n e^{i\theta}$ for $\theta = \theta_j$ with $\det A = (e^{i\theta})^n = 1$. Therefore, the center of $SU(n)$ is isomorphic to $\mathbb{Z}_n$.

Example 2.2.13. For any special orthogonal $A \in SO(n)$, there is $B \in SO(n)$ such that with 1_n the $n \times n$ identity matrix,

$$B^{-1}AB = -1_{2k} \oplus 1_l \oplus \{\oplus_j \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}\}$$

where $k \geq 0$, and $l \geq 1$ is odd if n is odd, and $l \geq 0$ is even if n is even. By the invariance for the adjoint operation by $-1_2 \oslash 1_2$ as the off diagonal sum as given above as $-1 \oslash 1$, we have that if A is in the center of $SO(n)$, then A is equal to $-1_{2k'} \oplus 1_{l'}$, with $2k' + l' = n$.

D-Branes.

Both closed and open strings are given by compact manifolds, but with boundary in the open case, as with Dirichlet or Neumann conditions on some submanifold $\varphi(\Sigma)$ of X.

A p-brane has space dimension p, or has dimension $p+1$ with time.

Example 2.2.14. A (-1)-brane is some points of X. Such points are called instantons, which means instants in time and space.

An open string in $X = \mathbb{R}^3$ may be a relatively open curve $C(t) \subset \mathbb{R}^2$: $f(t,x) \in \mathbb{R}$ for on $x \in (a,b) \subset \mathbb{R}$ with lines $x = a$ and $x = b$ as boundary in $\mathbb{R}^2$, which can move free to slide up and down in the region $[a,b] \times \mathbb{R}$ with respect to t. This is a 1-brane in this sense.

Example 2.2.15. We may as well assume that $f(t,a) = 0 = f(t,b)$ for $t \in \mathbb{R}$ and some $g(x) = f(0,x)$ for $x \in \mathbb{R}$ as a boundary and initial conditions for a Dirichlet problem of the Lapace equation $\Delta f = f_{tt} + f_{xx} = 0$.

The p-branes are viewed as fundamental objects in the string theory playing fundamental roles. They are coupled to the Ramond-Ramond fields.

Fields representing particles in string theory lie in different sectors, depending on worldsheet boundary conditions which they satisfy ([8], [48], or [59]). These are called Ramond sectors and Neveu-Schwarz sectors. Putting the(holomorphic:) left-moving sectors and the (anti-holomorphic:) right-moving sectors together, explained are terms like Ramond-Ramond or NS-NS.

Example 2.2.16. What is a sector? This seems to be like a section symbol.

There are cyclic conditions for functions or fields ψ^μ as particles such that Ramond (R): $\psi^\mu(w + 2\pi) = \psi^\mu(w)$ and Neveu-Schwarz (NS): $\psi^\mu(w + 2\pi) = -\psi^\mu(w)$ for $w = \sigma^1 + i\sigma^2$ as cylinder coordinate with $\sigma^1 \in \mathbb{R}/2\pi\mathbb{Z}$ and $\sigma^2 \in \mathbb{R}$. Namely, as in one formula,

$$\psi^\mu(w + 2\pi) = e^{2\pi i\nu}\psi^\mu(w), \quad \nu \in \{0, \frac{1}{2}\}.$$

As well, for $\psi^{\mu,\sim}$, the conditions are given by $\psi^{\mu,\sim}(\overline{w} + 2\pi) = e^{-2\pi i\nu}\psi^{\mu,\sim}(\overline{w})$ ([48]). It then follows that there are closed super-strings of 4 kinds represented by R-R, R-NS, NS-R, NS-NS.

For example, $\psi^\mu(w) = e^{iw}$ of R. Also, $\psi^\mu(w) = e^{\frac{1}{2}iw}$ of NS.

As well, $\psi^{\mu,\sim}(\overline{w}) = e^{i\overline{w}}$ of R. Also, $\psi^{\mu,\sim}(w) = e^{\frac{1}{2}i\overline{w}}$ of NS.

⋆ But those respective conditions for ψ^μ and $\psi^{\mu,\sim}$ on the cylinder look like the same. Note that the function w is holomorphic, but $\overline{w}$ is not. Indeed,

$$w' = 1 = (\sigma^1)_x + i(\sigma^2)_x = \frac{1}{i}((\sigma^1)_y + i(\sigma^2)_y)$$

so that $(\sigma^1)_x = 1 = (\sigma^2)_y$ and $(\sigma^1)_y = 0 = -(\sigma^2)_x$ satisfying the Cauchy Riemman equation. As for $\overline{w} = \sigma^1 - i\sigma^2$, we have

$$(\sigma^1)_x = 1 \neq -1 = (-\sigma^2)_y \quad \text{and} \quad (\sigma^1)_y = 0 = -(-\sigma^2)_x.$$

The string theories of 5 types.

There are five (super-symmetric) string theories, which have slightly different fields and Lagrangians. The differences are given by such as the gauge groups for some of the gauge fields, closed or open strings, orientation of the strings, and chirality like left or right handedness.

• Type I. The theory involves unoriented strings, so that the stirng world-sheet Σ can be a non-orientable surface like a Klein bottle.

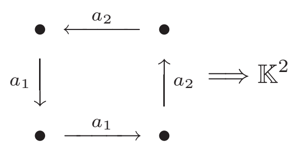
• Type IIA. In the theory, left-moving and right-moving spinors of oriented strings have opposite handedness.

• Type IIB. In the theory, left-moving and right-moving spinors of oriented strings have the same handedness.

• Type Heterotic E_8. In the theory, left-movers behave as in bosonic theory and right-movers do as in super-symmetric theory, the gauge group is the product of two copies of the exceptional Lie group E_8. Why 8?

• Type Heterotic $SO(32)$. In the theory, left-movers behave as in bosonic theory and right-movers do as in super-symmetric theory, the gauge group is a Lie group locally isomorphic to $SO(32)$. Why 32?

Example 2.2.17. The Klein bottle $(\mathbb{K}^2)$ is obtained as $a_1a_1a_2a_2$ by the real 2-dimensional closed interval where the boundary of the interval is directed or oriented one way so that two adjoining edges a_j, a_j, $j = 1, 2$ of four directed edges in the boundary as their union are identified respectively along the path way to make a circle, with 4 vertices • of the edges to be the common point. Namely,



On the other hand, the 2-torus $\mathbb{T}^2$ is described as $a_1a_2a_1^{-1}a_2^{-1}$, where the orientation of each a_j^{-1} is reverse to a_j.

Example 2.2.18. Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.

Topological groups are said to be locally isomorphic if there is an isomorphism between some neighbourhoods of the respective identity.

For a Lie subgroup G of $GL_n(\mathbb{C})$, the Lie algebra of G is defined to be the set of all matrices X in $M_n(\mathbb{C})$ such that $\exp(tX) \in G$ for any $t \in \mathbb{R}$.

The multiplicative group $\mathbb{R}^+$ of all positive reals is a Lie group such that the Lie algebra is $\mathbb{R}$. Note that $e^{tx} \in \mathbb{R}^+$ for any $t, x \in \mathbb{R}$.

The (multiplicative) 1-torus $\mathbb{T}$ of all complex numbers with absolute value 1 is also a Lie group such that the Lie algebra is $\mathbb{R}$. Note that $e^{tix} \in \mathbb{T}$ for any $t, x \in \mathbb{R}$, where ix may be identified with x?

Therefore, $\mathbb{R}^+$, $\mathbb{T}$, and as well $\mathbb{R}$ as a Lie group are locally isomorphic, with $\mathbb{R}^+ \cong \mathbb{R}$. Any commutative Lie group is the product of some copies of them.

Note that $SO(n+1)/SO(n) \cong S^n$, $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$, and $\pi_1(SO(2)) \cong \mathbb{Z}$. It then follows that $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$, and as well, $\pi_1(SL_n(\mathbb{R})) \cong \mathbb{Z}_2$ for $n \geq 3$, but $\mathbb{Z}$ for $n = 2$.

In particular, $SO(3)$ has dimension $1 + 2 = 3$. Moreover, $SO(n)$ has dimension

$$1 + 2 + \cdots + (n-1) = \frac{(n-1)n}{2}.$$

In particular, $SO(32)$ has dimension $31 \cdot 16 = 496$.

The exceptional group E_6 has center $\mathbb{Z}_3$ and dimension 78.

The exceptional group E_7 has center $\mathbb{Z}_2$ and dimension 133.

The exceptional groups E_8, F_4, G_2 have respective center trivial and respective dimension 248, 52, 14.

We then have

$$\dim E_8 \times E_8 = 2 \cdot 248 = 496 = \dim SO(32).$$

That's it.

By the way, it looks like that up to some conjugacy,

$$SO(n+1) = \begin{pmatrix} SO(n) & \mathbb{R}^n \\ 0 & 1 \end{pmatrix}$$

so that

$$\begin{aligned} SO(n+1)^t SO(n+1) &= \begin{pmatrix} SO(n)^t & 0^t \\ (\mathbb{R}^n)^t & 1 \end{pmatrix} \begin{pmatrix} SO(n) & \mathbb{R}^n \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} SO(n)^t SO(n) & SO(n)^t \mathbb{R}^n \\ (\mathbb{R}^n)^t SO(n) & (\mathbb{R}^n)^t \mathbb{R}^n + 1 \end{pmatrix}. \end{aligned}$$

We may refer to [47], [48], [59], [8] as the points of view in physics and also to [15], [31], [27] as more mathematical points of view.

We may also refer to [30].

Example 2.2.19. Open strings of dimension one such as open intervals have left or right orientation or direction. Closed strings of dimension one such as circles have left or right round orientation.

An open string can make a closed string. Namely, for $a < c < d < b$,

$$(a, b) = (a, c) \sqcup [c, d] \sqcup (d, b)$$

as a disjoint union of intervals. But closed strings can not make open ones by taking disjoint union operation.

Why not consider half open strings?

But it in fact seems that an open string in the theory is just a closed interval.

A closed interval is changed to a circle by identifying or gluing the end points.

A circle is not deformed to a closed interval. But it seems that it is changed to a half open interval by cutting the circle.

The type I theory has open and closed strings with no orientation of string world sheets and does gauge group $SO(32)$.

The type II theory has closed strings with left or right round orientation and open strings as extreme points of D-branes with left or right orientation and has gauge group $U(n)$, with super-symmetry.

The Heterotic theory has only closed strings with left round of 9 dimension and right round of 25 dimension of boson string as a string making space and has gauge group $SO(32)$ or $E_8 \times E_8$.

A string theory is discovered by Green and Schwarz in 1984 as a theory with no quantum anomaly term when gauge group is $SO(32)$. There is also a string theory with quantum anomaly pointed out by Álvarez-Gaumé and Witten in 1983. There is some more interesting story about the theory.

2.3 Some duality theory related to string theory

Duality.

A duality is meant to be a transformation between different looking physical or mathematical theories, having magically the same observable physics or mathematics. Such a duality is given by cyclic groups $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_4$, or the discrete group $SL_2(\mathbb{Z})$ or so, as symmetry or periodicity.

Example 2.3.1. Duality between electric and magnetic. The Maxwell equations in a free space are given by inner products and outer products of vectors as

$$\langle \nabla, E \rangle = 0, \quad \frac{\partial}{\partial t} E = c\nabla \times B,$$
$$\langle \nabla, B \rangle = 0, \quad \frac{\partial}{\partial t} B = -c\nabla \times E.$$

There is a symmetry σ of the Maxwell given by sending E to $-B$ and B to E. This duality has order 4.

Namely,

$$(E, B) \xrightarrow{\sigma} (-B, E) \xrightarrow{\sigma} (-E, -B) \xrightarrow{\sigma} (B, -E) \xrightarrow{\sigma} (E, B).$$

Note that $\frac{\partial}{\partial t}(-B) = c\nabla \times E$ is the same as $\frac{\partial}{\partial t} B = -c\nabla \times E$ by linearity. As well,

$$\frac{\partial}{\partial t} E = -c\nabla \times (-B) = c\nabla \times B.$$

Thus, the Maxwell system is preserved under the symmetry σ.

We may refer to [22].

Example 2.3.2. The Maxwell fundamental equations which are independent of matter and always hold are given by

$$\begin{aligned} \operatorname{div} D = \rho, \quad & \frac{\partial}{\partial t} D = \operatorname{rot} H - J, \\ \operatorname{div} B = 0, \quad & \frac{\partial}{\partial t} B = -\operatorname{rot} E \end{aligned}$$

with $D = \varepsilon E$, $B = \mu H$, and $J = \sigma E$.

A free space is a space with ε, μ, σ constants and $\rho = 0$.

It seems that this free space is slightly different from that free space by J term.

Note that for magnetron charge m with respect H or B which may varies, $I = \frac{dm}{dt}$ [ampere] is electric flow strength and J is electric flow density per section volume. But it seems that it is a scalar.

Note that divergence $\operatorname{div}(\cdot)$ as a linear map is the same as the inner product operation $\langle \nabla, \cdot \rangle$. Namely,

$$\operatorname{div} B = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} B = \langle \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \rangle = \langle \nabla, B \rangle.$$

Also, rotation $\operatorname{rot}(\cdot)$ is the same as the outer product $\nabla \times (\cdot)$ as a linear map. Namely,

$$\operatorname{rot} E = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial x_2} & E_2 \\ \frac{\partial}{\partial x_3} & E_3 \end{vmatrix} \\ \begin{vmatrix} \frac{\partial}{\partial x_3} & E_3 \\ \frac{\partial}{\partial x_1} & E_1 \end{vmatrix} \\ \begin{vmatrix} \frac{\partial}{\partial x_1} & E_1 \\ \frac{\partial}{\partial x_2} & E_2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \nabla \times E.$$

By the way, E is an electric field (strength) as a vector in a space such as $\mathbb{R}^3$. Force between two point electrons e, e' of q and q' [coulomb] with distance r [m] between e and e' as vectors is given by $F = \frac{1}{4\pi\varepsilon_0} \frac{qq'}{r^2}$ [newton]. Also, H is a magnetic field (strength) as a vector in a space. Force between two point magnetrons g, g' of m and m' [weber] with distance r [m] between g and g' as vectors is given by $F = \frac{1}{4\pi\mu_0} \frac{mm'}{r^2}$ [newton].

We may assume that $q' = 1$ and $m' = 1$ respectively, and then, for instance, the fields E and H are defined by F respectively as $F \frac{1}{r} x$ with norm $\|x\| = r$ the distance between x and either e or g as in $\mathbb{R}^3$. In this case, E and H are constants (strength) on each sphere with e and g as origins respectively.

That classical duality has a quantum extension as given by Dirac [16], and generalized by Goddard, Nuyts, and Olive [21] and by Montonen and Olive [44]. That duality is also applied to gauge fields with non-abelian gauge group, as appearing in elementary particle theory. Then the duality switches a Lie group with its Langlands dual, as in the Langlands program in representation theory and automorphic forms (cf. [10], [32]).

Example 2.3.3. Fourier duality as configuration space and momentum space duality. The quantum harmonic oscillator in one dimension is another example from standard quantum mechanics. For an object with mass m and a force with constant k, the Hamiltonian is given by $H = \frac{k}{2}x^2 + \frac{1}{2m}p^2$, where p is the momentum and $x = x(t)$ for $t \in \mathbb{R}$ is the usual coordinate on the real $\mathbb{R}$.

In classical mechanics, $p = m\frac{d}{dt}x$. But in quantum mechanics, $[x,p] = i$ with $\hbar = 1$.

We have a duality of

$$H = \frac{k}{2}x^2 + \frac{1}{2m}p^2 \quad \text{and} \quad [x,p] = i$$

given by sending as $m \mapsto \frac{1}{k}$, $k \mapsto \frac{1}{m}$, $x \mapsto p$, and $p \mapsto -x$. This duality has order 4 again. This is related to the Fourier transform. In the Schrödinger representation on the Hilbert space $L^2(\mathbb{R})$ of square integrable measurable complex-valued functions on the real, p is represented by the differential operator $-i\frac{d}{dx}$ whose Fourier transform is x. If Lebesgue measure on the measure space $\mathbb{R}$ is properly (un)normalized, the Fourier transform becomes a unitary operator on $L^2(\mathbb{R})$ with period 4, and with Hermite functions as eigen functions.

Note that the duality implies that

$$\begin{aligned}
&(m,k) \mapsto (\frac{1}{k}, \frac{1}{m}) \mapsto (m,k) \\
&(x,p) \mapsto (p,-x) \mapsto (-x,-p) \mapsto (-p,x) \mapsto (x,p) \\
&H \mapsto H \mapsto \cdots \mapsto H.
\end{aligned}$$

Also, for a differentiable function $\xi(x)$ in $L^2(\mathbb{R})$,

$$\begin{aligned}
[x,p]\xi(x) = (xp - px)\xi(x) &= -ix\frac{d}{dx}\xi(x) + i\frac{d}{dx}(x\xi)(x) \\
&= i(-x\xi' + \xi + x\xi')(x) = i\xi(x).
\end{aligned}$$

The operator $[x,p]$ is defined on a L^2-norm dense subspace of $L^2(\mathbb{R})$ of differentiable or smooth functions on $\mathbb{R}$ and is extended to $L^2(\mathbb{R})$ by continuity.

We may refer to [29]. The Fourier transform for integrable measurable functions f on $\mathbb{R}$ is defined to be

$$(\mathfrak{F}f)(\xi) = f^\wedge(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} f(x)dx, \quad \xi \in \mathbb{R}.$$

The inverse Fourier transform for such integrable functions g is defined by

$$(\mathfrak{F}^* g)(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} g(\xi)d\xi, \quad x \in \mathbb{R}.$$

Moreover, for square integrable measurable functions f, g in the Hilbert space $L^2(\mathbb{R})$, the inner product on L^2 is preserved by the Fourier $\mathfrak{F}$ and $\mathfrak{F}^*$ as

$$\langle f, g\rangle = \langle \mathfrak{F}f, \mathfrak{F}g\rangle = \langle \mathfrak{F}^* f, \mathfrak{F}^* g\rangle$$

so that $\mathfrak{F}^*\mathfrak{F}$ and $\mathfrak{F}\mathfrak{F}^*$ are the identity map on L^2. Namely, they are unitary.

Also, since $\mathfrak{F}^*\mathfrak{F}f(x) = f(x)$ for $f \in L^2$, we have

$$\mathfrak{F}^2 f(x) = \int_{\mathbb{R}} e^{-2\pi i x\xi}(\mathfrak{F}f)(\xi)d\xi = \int_{\mathbb{R}} e^{2\pi i(-x)\xi}(\mathfrak{F}f)(\xi)d\xi = f(-x).$$

Therefore, $\mathfrak{F}^4 f(x) = \mathfrak{F}^2 f(-x) = f(-(-x)) = f(x)$. That's it 4.

Suppose that $\mathfrak{F}f = \alpha f$ with some $\alpha \in \mathbb{C}$ and $f \neq 0$. Then $\mathfrak{F}^2 f = f(-x) = \alpha\mathfrak{F}f = \alpha^2 f$. On the other hand, we have

$$0 \neq \langle f, f\rangle = \langle \mathfrak{F}f, \mathfrak{F}f\rangle = \langle \alpha f, \alpha f\rangle = |\alpha|^2\|f\|_2$$

and thus $|\alpha| = 1$.

In particular, if f is even (and real), then $\alpha = 1$ or -1. If f is odd (and real), then $\alpha = \pm i$. The converse of both also holds.

As well, if f is even and real, then

$$\begin{aligned}\mathfrak{F}f(\xi) &= \int_{\mathbb{R}} e^{-2\pi i\xi x} f(-x)dx = \int_{\mathbb{R}} e^{2\pi i\xi x} f(x)dx \\ &= \int_{\mathbb{R}} \overline{e^{-2\pi i\xi x} f(x)}dx = \overline{\mathfrak{F}f}(\xi).\end{aligned}$$

T-Duality. T-duality is one of the important dualities in string theory, where T means target space or torus. This T-duality is viewed as an equivalence of string theories on two different space time manifolds. Namely, tori in a space X are replaced by dual tori in another dual space Y.

Example 2.3.4. Let V be a finite dimensional real vector space and L is a lattice in V. A torus is defined to be the quotient space $V/2\pi L$, and its dual torus is $V^*/2\pi\alpha' L^*$ for some real α', where V^* is the dual linear space of V and elements of $L^* \subset V^*$ take integral values on $L \subset V$.

The T-duality also involves changing in metric and B-field, known as the Buscher rules (cf. [11], [12], and [43]).

Example 2.3.5. Let Σ be a closed Riemannian 2-manifold and consider the following action for a map to a circle with radius r

$$S(\omega) = \frac{1}{4\pi\alpha'}\int_\Sigma \frac{r^2}{\alpha'}\omega \wedge *\omega$$

by integrating a 1-form ω on Σ with integral periods. We may add a parameter μ as a kind of Lagrange multiplier to the action as

$$S(\omega, \mu) = \frac{1}{4\pi\alpha'}\int_\Sigma \left(\frac{r^2}{\alpha'}\omega \wedge *\omega + 2\mu d\omega\right).$$

We need $d\omega = 0$ back to the original theory. Then taken is the variation in ω

$$\begin{aligned}\delta S &= \frac{r^2}{4\pi(\alpha')^2}\int_\Sigma \left(\delta\omega\wedge *\omega + \omega\wedge *\delta\omega + \frac{2\alpha'}{r^2}\mu d\delta\omega\right)\\ &= \frac{r^2}{4\pi(\alpha')^2}\int_\Sigma \delta\omega\wedge(2(*\omega) + \frac{2\alpha'}{r^2}d\mu).\end{aligned}$$

If $\delta S = 0$, then $*\omega = -\frac{\alpha'}{r^2}d\mu$. Then $\omega = \frac{\alpha'}{r^2} * d\mu$. Substituting $\eta = *d\mu$ into $S(\omega,\mu)$ implies that (with $d\omega = 0$,)

$$\begin{aligned}S^{\sim}(\eta) &= \frac{1}{4\pi\alpha'}\int_\Sigma \left(\frac{r^2}{\alpha'}(\frac{\alpha'}{r^2})^2\eta\wedge *\eta + 2\frac{\alpha'}{r^2}\mu d * d\mu\right)\\ &= \frac{1}{4\pi\alpha'}\int_\Sigma \frac{\alpha'}{r^2}\eta\wedge *\eta.\end{aligned}$$

This is the original like action with replacing of 1-form ω with η and of radius r with $\frac{\alpha'}{r}$.

Example 2.3.6. The t-duality is related to the classical theory of theta (θ)-functions. We consider a theory where $\Sigma = S^1$ the circle and $X = \mathbb{R}/2\pi r\mathbb{Z}$ also a torus. The spaces may be the space-like directions and there may be another time direction like $\mathbb{R}$. The harmonic maps are given by sending $x + \mathbb{Z} \in \mathbb{R}/\mathbb{Z} = S^1$ to $2\pi nrx + 2\pi r\mathbb{Z} \in X$. The action for this map is given by

$$\frac{1}{4\pi\alpha}\int_0^1 |\frac{d}{dx}(2\pi nrx)|^2 dx = \frac{1}{4\pi\alpha}\int_0^1 4\pi^2 n^2 r^2 dx = \frac{\pi n^2 r^2}{\alpha}.$$

The partition function is given by

$$z_r = \sum_{n=-\infty}^{\infty} e^{-\frac{1}{\alpha}\pi n^2 r^2}$$

which is a classical θ-function. The Poisson summation formula is now that for f a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions on $\mathbb{R}$ with Fourier transform $f^\wedge$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} f^\wedge(n).$$

This formula also holds for $f(x)$a continuous function on $\mathbb{R}$ such that the series $\sum_{m=-\infty}^{\infty} f(x+m)$ on the interval $[0,1]$ converges absolutely and uniformly and that the right hand side series above converges absolutely.

Note that

$$\begin{aligned}\mathfrak{F}(e^{-\pi x^2})(\xi) &= \int_{\mathbb{R}} e^{-2\pi i\xi x}e^{-\pi x^2}dx = \int_{\mathbb{R}} e^{i(-2\pi\xi)x-\pi x^2}dx\\ &= e^{-\frac{(-2\pi\xi)^2}{4\pi}} = e^{-\pi\xi^2}\end{aligned}$$

where for showing the equality we use the Taylor expansion

$$e^{i(-2\pi\xi)x} = \sum_{k=0}^{\infty} \frac{1}{k!}(i(-2\pi\xi)x)^k$$

and term-wise integration and differentiation in integration with respect to even and odd degree cases.

Therefore, with $\alpha' > 0$,

$$\begin{aligned}\mathfrak{F}(e^{-\frac{1}{\alpha}\pi x^2 r^2})(\xi) &= \int_{\mathbb{R}} e^{-2\pi i \xi x} e^{-\frac{1}{\alpha}\pi x^2 r^2} dx \quad (s = \frac{1}{\sqrt{\alpha}} xr) \\ &= \int_{\mathbb{R}} e^{-2\pi i (\xi \frac{\sqrt{\alpha}}{r}) s} e^{-\pi s^2} \frac{\sqrt{\alpha}}{r} ds = \frac{\sqrt{\alpha}}{r} e^{-\frac{\pi \xi^2 \alpha}{r^2}}.\end{aligned}$$

The Poisson summation formula implies that

$$z_r = \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2 r^2}{\alpha}} = \frac{\sqrt{\alpha}}{r} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2 \alpha}{r^2}} = \frac{\sqrt{\alpha}}{r} z_{\frac{\alpha}{r}}$$

and equivalently, $\sqrt{r} z_r = \sqrt{\frac{\alpha}{r}} z_{\frac{\alpha}{r}}$. This is the basic form of T-duality, which is used for proving the functional equation in the Riemann zeta (ζ)-function.

Example 2.3.7. We may refer to [36]. The theta transformation formulae are given as

$$\begin{aligned}\theta^{\sim}(\frac{1}{x}) &= \sqrt{x}\theta^{\sim}(x), \\ \theta(\frac{1}{x}) &= \sqrt{x}(\frac{1}{2} + \theta(x)) - \frac{1}{2},\end{aligned}$$

where the theta $\theta(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ and the extended theta $\theta^{\sim}(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$.

Also, the symmetric functional equation for the gamma-zeta function $\Gamma\zeta(s) = \frac{1}{\pi^{\frac{s}{2}}}\Gamma(\frac{s}{2})\zeta(s)$ is given as $(\Gamma\zeta)(s) = (\Gamma\zeta)(1-s)$ by using the θ-transformation above in the integration formula for $\Gamma\zeta$ by θ-function.

Proof. There is the Poisson sum formula says that for a real valued function $f(x)$ with $f^\wedge(\xi)$ Fourier transform satisfying $f(x) = O(\frac{1}{|x|^{1+\delta}})$ $(|x| \to \infty)$ and $f^\wedge(\xi) = O(\frac{1}{|\xi|^{1+\varepsilon}})$ $(|\xi| \to \infty)$ for some $\delta > 0$ and $\varepsilon > 0$ respectively, and f even and so $f^\wedge$, as essential conditions, it then holds that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} f^\wedge(m).$$

We then apply this Poisson formula to a function $f(n) = e^{-\pi\frac{(n+\alpha)^2}{x}}$ for reals α and $x > 0$ and integers n. Then deduced from the formula at $\alpha = 0$ involving computing Fourier transform $f^\wedge(m)$ as integral with change of variables is the theta transformation. □

S-Duality. Another duality in string theory is S-duality for strong and weak. We may say it as sw-duality. This is an outgrowth of the classical electric and magnetic duality by Maxwell, via Dirac that charge of a magnetic monopole not yet or never observed should be a multiple of charge of an electron by $\frac{1}{2}\hbar c$. Charge of an electron is small and observable. So charge of a magnetic monopole should be large enough to be not observed. The sw-duality interchanges strong coupling limit of a string theory with weak coupling limit of another string one. We may find some in [54] or [16].

The S-duality and the T-duality are closely linked as [25]. The S-duality involves the duality between a compact Lie group G and its Langlands dual $G^\vee$. For a Cartan subalgebra $\mathfrak{h}$ in the Lie algebra $\mathfrak{g}$ of G, the dual space $\mathfrak{h}^*$ can be identified with a Cartan subalgebra in $\mathfrak{g}^\vee$. We then obtain $\mathfrak{h}/\Lambda$ and $\mathfrak{h}^*/\Lambda^*$ as a pair of dual tori, where Λ is the coweight lattice for $\mathfrak{g}$ and is the weight lattice for $\mathfrak{g}^\vee$, while Λ^* is the coweight lattice for $\mathfrak{g}^\vee$ and is the weight lattice for $\mathfrak{g}$. The T-duality between the pair reproduces the S-duality. The S and T-dulities mixing gives a family of dualities, called (unified) U-duality [28].

Perturbation theory is one of the calculational tools in some quantum field theories and string theories. That is expanding in a power series in some parameter and computing coefficients, with parameter small to reasonable convergence of the series.

String coupling constant is the important parameter in string theory with dimension less. That measures intensity of interactions between strings. With the constant small, we may consider perturbation expansions. However, the constant is not fixed. It can be expressed as the exponential of the expectation valued of a scalar-valued field called dilaton.

The S-duality exchanges dramatically one string theory with the constant small with another with the constant large, by reversng the sign of the dilaton field. On the other hand, the T-duality shifts the dilaton field as a translation, so that effect on the constant is less and not dramatic.

The consequence of S-duality is not amenable to perturbation theory by changing the constant from small to large.

AdS or CFT duality. AdS stands for anti-de Sitter space as a space-time manifold of constant curvature. CFT stands for conformal field theory. This duality is discovered by Juan Maldacena [37]. That is an equivalence between gauge theories in dimension d like 4 and string theories in a space-times of dimension $d+1$.

For Yang-Mills theory for $U(n)$-bundles, taking the limit as $n \to \infty$ gives the discovery duality AdS or CFT by observation of behaving in similar to string theory. It is conjectured that the limit of super-symmetric Yang-Mills theory on 4-dimensional Minkowski space is dual to type IIB string theory on the product space of the anti-de Sitter as a 5-dimensional Lorentz monifold of constant curvature with the 5-sphere S^5. This is the homogeneous space $SO(4,2)/SO(4,1)$ up to coverings.

String theory is connected to Yang-Mills theory. D-branes carry bundles and gauge fields, $U(n)$-bundles in type IIB theory. Open string massless states in type IIB theory have an effective Lagrangian that looks like that of Yang-Mills theory. This makes it possible to construct duality correspondence from string theory to gauge theory in the reverse direction.

In that correspondence, string coupling corresponds to a coupling constant in Yang-Mills theory (YM). S like duality gives a match between the two theories of weakly coupled and strongly coupled.

We may refer to [39] or [1].

String theory with duality

$$\begin{array}{ccccc} \text{Type I} & & \text{Type IIA} & \xrightarrow{T} & \text{Type IIB} \\ {\scriptstyle S}\big\downarrow & & & & \big\downarrow{\scriptstyle S} \\ \text{Heterotic } SO(32) & \xleftarrow{T} & \text{Heterotic } E_8 \times E_8 & & \text{Type IIB} \end{array}$$

Super-string theories are required to be 10-dimensional to eliminate certain anomalies. Their dualities involve an 11-dimensional theory, that is called M-theory. That is reduced to 11-dimensional super-gravity in the low energy limit from a 12-dimensional theory, that is called F-theory. Namely,

$$\begin{array}{ccccc} \text{String theory} & \longleftarrow & \text{M-theory} & \longleftarrow & \text{F-theory} \\ {\scriptstyle \text{duality}}\big\downarrow{\scriptstyle \text{U}} & & \| & & \| \\ \text{String theory} & \longleftarrow & \text{M-theory} & \longleftarrow & \text{F-theory} \end{array}$$

We may refer to [55].

Example 2.3.8. A closed string looks like a rubber band (or circle). An open string looks like a cut off rubber band (or line).

S duality is like expanding such a band from small to large. We may call it L duality. T duality is like changing a rubber band to a doughnut by magic.

Right handed is like a right hand. Left handed is like a left hand.

Super symmetry is like putting together both left and right hands.

Two hands have 10 fingers!

One has two hands.

3 K-theory and more

3.1 D-brane charges

Charges in electromagnetism. Electric charge is quantized as a discrete invariant as in the following. Observed charges are integral multiples of the charge of an electron. This is confirmed experimentally by Millikan as oil dropping experiment in early 1900. Charges of quarks if allowed should be integral multiples of the one-third charge of an electron. In space-time of dimension 4, the electro-magnetic field can be identified with a two-form F locally given as dA for A the potential combined of electro-static potential and magnetism vector potential of dimension 3. This is not globally since the integration of F over S^2 as the worldline of an electron is not zero by Gauss law. It is then suggested by Dirac [16] that A should be viewed as a connection on a line bundle with base space the complement of the world-lines of charged particles. Namely, A is a $U(1)$ gauge field and F is the field strength curvature. Note that $H^2(\mathbb{R}^4 \setminus \mathbb{R}, \mathbb{Z}) \cong \mathbb{Z}$, and the class $[F]$ is Chern class of the line bundle, up to constant, by Chern-Weil. So quantized is charge.

Lemma 3.1.1. *Let X be a compact space and E a vector bundle over X of rank n. Then there is a compact space Y and a map f from Y to X such that f^* is injective on cohomology and that $f^*(E)$ splits into a direct sum of line bundles as $L_1 \oplus \cdots \oplus L_n$.*

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ Y & \xrightarrow{\ f\ } & X \end{array}$$

where the pull back in the diagram is defined as

$$f^*(E) = \{(y, e) \in Y \times E \,|\, f(y) = p(e) \in X\}.$$

Definition 3.1.2. Define the Chern class

$$c(f^*(E)) = \Pi_{j=1}^n (1 + c_1(L_j)) = 1 + \sum_{j=1}^n c_j(f^*(E))$$

where $c_1(L_j)$ means the class in H^2 of the line bundle L_j, and $c_j(f^*(E)) \in H^{2j}(X, \mathbb{Z})$ is the j-th elementary symmetric function of $c_1(L_j)$ as variables. Then define the Chern class $c(E)$ (by composite duality) so that $f^*(c(E)) = c(f^*(E))$, independent of the choice of space Y and map f.

Connections and more. A connection (operator) ∇ on a vector bundle $E \to X$ is defined by

$$\nabla_Y(s) \in \Gamma^\infty(X, E), \quad Y \in \Gamma^\infty(X, TX), \quad s \in \Gamma^\infty(X, E)$$

where TX is the tangent bundle over X and Γ^∞ stands for smooth sections of the bundles over X, such that

$$\nabla_Y(fs) = (Yf)s + f\nabla_Y(s), \quad f \in \Gamma^\infty(X).$$

As well, the connection ∇ is an operator from sections of $\Gamma^\infty(X, E)$ to E-valued 1-forms of $\Gamma^\infty(X, T^*X \otimes E)$. Then $\nabla^2 = \nabla \circ \nabla$ is an operator from sections of E to E-valued 2-forms.

The curvature of the connection is given by an E-valued 2-form defined as

$$\Theta(Y, W) = \nabla_Y \nabla_W - \nabla_W \nabla_Y - \nabla_{[Y,W]}.$$

As the theorem of Chern-Weil theory, we have

Theorem 3.1.3. *The de Rham classes of the coefficients of the characteristic polynomial of* $-(2\pi i)^{-1}\Theta$ *are independent of the choice of a connection and live in the image from* $H^{\mathrm{ev}}(X,\mathbb{Z})$ *of even degree to even* $H^{\mathrm{ev}}(X,\mathbb{R})$*.*

These classes are the images of the Chern $c_j(E)$ in $H^{\mathrm{ev}}(X,\mathbb{R})$. We may refer to [18].

Chan-Paton bundles.

D-branes are submanifolds of the manifold X of space and time. In superstring theory, X is assumed to be a 10-dimensional Lorentz manifold, or to be the product of a Riemann manifold with $\mathbb{R}$ of time. D_p or p-branes are $(p+1)$-dimensional, where p represents the dimension of space part of the branes.

D-branes carry Chan-Paton (CP) bundles. There is a local $U(n)$ gauge symmetry for n different D-branes coincided as mapped to the same submanifold of X by the principle of quantum mechanics so that there should be mix states supported on the different branes by mixing matrices of $U(n)$ which vary along the brane pointwise. Namely, the brane carries a $U(n)$ gauge field locally. Thus, the brane carries such a bundle of dimension n over the brane globally. That's it. The gauge field is a connection on the CP bundle, whose field strength is given by the curvature of the connection. Branes and their CP bundles are allowed to be coalesced or split apart, so that the rank n of the bundle can vary at the moment.

There are anti-branes. Bundles on the branes should have dimension negative as viewed.

***D*-brane charges.** D-branes carry topological charges given by the (nontrivial) Chan-Paton bundles (over $\mathbb{R}$) as in the case of electro-magnetism (EM). In the EM case, X is a space time with world lines of electrons, and admits a line bundle, equivalent to the charges. The string case is analogous, but its gauge theory is non-abelian, involving vector bundles of rank higher. Then charges should be classified by (topological) K-theory of space time by physicists. We may refer to [42], [56], [57], [38], and [45].

The Minasian-Moore formula. We may deal with D_9-branes filling a space time X of dimension 10. Then the K-theory charge is nothing but the class $[E]$ of the Chan-Paton bundle E over $\mathbb{R}$. The charge of an anti-D_9-brane is given by $-[E]$ ([42]). As well, for branes W that are proper submanifolds of a space time X with an embedding $f : W \to X$, if both W and X have spin structure defining spinors to involve fermion theory, the K-thoery charge should be identified with $f_!([E])$, where $f_!$ is the Gysin map in K-theory, or a wrong way (or cohomological) map defined by Atiyah-Singer [4].

Note that $f_! : K_*(X) \to K_*(W)_{*+1}$ of degree 1?

Example 3.1.4. We may refer to [13]. Let $f : X \to Y$ be a K-oriented (spin structured) smooth map of manifolds. The Gysin wrong-way functoriality map $f! : K(X) \to K(Y)$ can be described as an element of the Kasparov group $KK(X,Y)$ or $E(X,Y)$.

K-homology. K-homology (K_*) for spaces is the homology theory for spaces dual to K-theory (K^*) for spaces as cohomology theory for spaces. In this theory, maps of spaces imply the right way maps in the theory. K-theory charges (class) in $K_*(W)$ is mapped forwardly to $K_*(X)$ under f_* for $f : W \to X$.

A geometric realization of K-homology theory is given as follows. We may refer to [6], [7] Let X be a compact space. Any K-homology class on X is defined as a (half) cycle consisting of a compact spin manifold W with a map f from W to X and a vector bundle E over W.

$$\begin{array}{ccc} E & & \\ \big\downarrow & & \\ W & \xrightarrow{\ f\ } & X \end{array}$$

Cycles make an abelian semigroup by disjoint union such that homologus cycles define the same K-homology class in the sense of equivalence relation generated by spin bordism that if M is a compact spin manifold with boundary W, if E is a vector bundle over M, and if $f : M \to X$, then the class $(W, E|_W, f|_W)$ on W is zero,

$$\begin{array}{ccccc} E|_W \subset E & \longrightarrow & M & = & M \\ \big\downarrow & & \big\downarrow & & \big\downarrow f \\ W & = & W = \partial M & \xrightarrow{\ f|_W\ } & X \end{array}$$

and the additive relation is given by

$$(W, E_1, f) + (W, E_2, f) = (W, E_1 \oplus E_2, f)$$

with vector bundle modification as a way of building in Bott periodicity.

Analytical K-homology. The Kasparov K-homology is based on generalized elliptic operators or Fredholm modules. This is a special case of the Kasparov KK-theory for (extensions of) C^*-algebras. Namely,

$$K_*(X) = KK_*(C(X), \mathbb{C})$$

where X is a compact space and $C(X)$ is the C^*-algebra of continuous complex-valued functions on X.

An even-dimensional K-homology cycle on X is given by a triple of a $\mathbb{Z}_2$ graded Hilbert space $H = H_0 \oplus H_1$, a representation of $C(X)$ on H as bounded operators, and an odd bounded self-adjoint operator T such that $T^2 - 1$ and $[T, f] = Tf - fT$ for $f \in C(X)$ are compact operators on H. Namely, T is a self-adjoint unitary or symmetry mod compact and T and $C(X)$ commute mod compact to make a circle around $C(X)$.

A typical example is given by a compact manifold X, the operator $T = D(1+D^2)^{-\frac{1}{2}}$ for some self-adjoint elliptic first-order partial differential operator D like the Dirac operator, and H the space of L^2-sections of a vector bundle on X on which D acts.

$$T^2 = D(1 + D^2)^{-1}D = (1 + D^2 - 1)(1 + D^2)^{-1} = 1 - (1 + D^2)^{-1}$$

Is $(1+D^2)^{-1}$ compact in general? How to check this? Spectrum to be discrete may be checked or assumed from the first.

In particular, if $T = (-1)1$, then $T^2 = 1$ and $[T, C(X)] = \{0\}$.

The (K)K-homology equivalence relation is generated by homotopy, block (matrix) addition, and the relation such that if T can be changed by a compact operator so that $T^2 = 1$ and $[T, C(X)] = \{0\}$, then the class is trivial.

Odd-dimensional cycles in the KK-homology are defined similarly dropping the grading on H and requiring T to be odd from the first.

If an even-dimensional K-homology cycle forgets its grading, then it becomes an odd-dimensional trivial cycle.

Example 3.1.5. Let W be a closed spin manifold admitting a Dirac operator D. For E a vector bundle over W and $f : W \to X$ a map, there is D_E the Dirac with coefficients in E. Then defined is a class in the Kasparov $K_*(W)$ with dimension by that of W mod 2. For a compact space X, the class of (W, E, f) in the Baum-Douglas (BD)K-homology corresponds to the $f_*([D_E])$ in the KK-homology. Namely, the half cycle diagram

$$\begin{array}{ccc} E & & \\ \downarrow & & \\ W & \xrightarrow{\ f\ } & X \end{array}$$

does correspond to

$$K_*(W) \xrightarrow{\ f_*\ } K_*(X), \quad [D_E] \mapsto f_*([D_E]).$$

To obtain an isomorphism, we need to assume that X is homotopically finite as a technical reason. The reason is that the BDK-homology is like singular homology, while the KK-homology is like Steenrod homology. These homology theories do agree on finite CW complexes, but do not on general compact spaces. For usual physical applications we need not to case about this difference.

Example 3.1.6. Let W be a D-brane in X space time, which has Chan-Paton bundle E. Then, for the inclusion map $f : W \to X$, the triple (W, E, f) gives a class in BD $K_*(X)$, provided that W is spinning. This crucial condition but to be modified is usually needed for anomaly cancellation. We may refer to [19].

The identification of D-brane charges with K-homology classes is Poincaré dual to the identification of them with K-theory classes, under such conditions on X. Namely, points of view on X are equivalent in this sense.

Example 3.1.7. The Eilenberg singular homology theory is given as in the following. We may refer to [40].

The integral singular homology group of a topological space X of q dimension is defined to be the quotient group $H_q(X) = Z_q(X)/B_q(X)$ for . Let $S_q(X)$ be the free abelian group with basis of singular q simplexes, and with zero (map).

A singular q simplex is a continuous map σ from Δ^q to X, where

$$\Delta^q = \{(t_0, \cdots, t_q) \in \mathbb{R}^{q+1} \,|\, t_j \geq 0, \sum_{j=0}^{q} t_j = 1\}$$

is a q-dimensional standard simplex, that is convex in the Euclidean space $\mathbb{R}^{q+1}$. In particular, Δ^0 is a point $\{1\}$ in $\mathbb{R}$, Δ^1 is a line segment (closed interval) in $\mathbb{R}^2$, and Δ^2 is a plane segment (triangle face) in $\mathbb{R}^3$. Define edging (or face making) maps for $j = 0, \cdots, q)$,

$$\varepsilon_j : \Delta^{q-1} \to \Delta^q, \quad \varepsilon_j(t_0, \cdots, t_{q-1}) = (t_0, \cdots, t_{j-1}, 0, t_j, \cdots, t_{q-1}).$$

In particular, $\varepsilon_0(1) = (0, 1)$, $\varepsilon_1(1) = (1, 0)$. Also, $\varepsilon_0(t_0, t_1) = (0, t_0, t_1)$, $\varepsilon_1(t_0, t_1) = (t_0, 0, t_1)$ and $\varepsilon_2(t_0, t_1) = (t_0, t_1, 0)$. There is the boundary map defined to be a homomorphism by

$$\partial_q : S_q(X) \to S_{q-1}(X), \quad \partial_q \sigma = \sum_{j=0}^{q} (-1)^q \sigma \circ \varepsilon_j.$$

It then holds that the composition $\partial_{q-1} \circ \partial_q$ is zero map. The kernel $\ker(\partial_q)$ is $Z_q(X)$ of singular cycles. The image $\mathrm{im}(\partial_{q-1})$ is $B_q(X)$ of boundary cycles. Two singular cycles with the same dimension are homologous if their difference is a boundary cycle.

Let $S_{-1}(X) = \{0\}$. We have $\partial_0 \sigma = \sigma \circ \varepsilon_0 = 0$, where the second term is formal. We also have

$$(\partial_1 \sigma)(1) = (\sigma \circ \varepsilon_0)(1) - (\sigma \circ \varepsilon_1)(1) = \sigma(0, 1) - \sigma(1, 0).$$

In particular, if X is a one point set, then ∂_1 is zero map(?), that is the zero map. Then there is only one map from $\{1\}$ to X the point. Thus, $H_0(X)$ is $\mathbb{Z}$. As well, there is only one map from Δ^q to X the point.

Example 3.1.8. For a pair (X, A) of a topogical space X and a subspace A, there correspond to an abelian groups $H_q(X, A)$ for q integers by H a homology theory. For a continuous map f from (X, A) to (Y, B) and q integers, there correspond to homomorphisms $f_q : H_q(X, A) \to H_q(Y, B)$. For (X, A) and q, there correspond to $\partial_* : H_q(X, A) \to H_{q-1}(A)$, which is induced by boundary maps ∂ in homology long exact sequence of a short exact sequence of chain complexes of a pair of a topological space and a subspace. With inclusion maps $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$, there is the following exact sequence:

$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_*} H_{q-1}(A) \to \cdots$$

The relative singular homology $H_q(X, A)$ is the homology theory given by the quotient complex $\{S_q(X)/S_q(A), \partial_q\}$ of $\{S_q(X)\}$ by the subcomplex $\{S_q(A)\}$, where

$$\partial_q : S_q(X)/S_q(A) \to S_{q-1}(X)/S_{q-1}(A).$$

The Eilenberg-Streenrod axioms are given as in the following.

(I) Identity functor. id_* is the identity homology map for the identity map id on X or (X, A) as in the following.

(II) Categorical functor from topological pairs to abelian groups. For continuous maps $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ with $f(A) \subset B$ and $g(B) \subset C$, we have $(g \circ f)_* = g_* \circ f_*$. Namely, the following diagram commutes.

$$\begin{array}{ccccc} H_q(X,A) & \xrightarrow{f_*} & H_q(Y,B) & \xrightarrow{g_*} & H_q(Z,C) \\ {\scriptstyle (g\circ f)_*}\downarrow & & & & \| \\ H_q(Z,C) & = & H_q(Z,C) & = & H_q(Z,C) \end{array}$$

(III) Homotopy. For homotopic continuous maps $f, f' : (X, A) \to (Y, B)$, we have $f_* = (f')_*$ on H_q.

(IV) Homology long exact sequence. Namely,

$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial_*} H_{q-1}(A) \to \cdots$$

(V) The following diagram commutes.

$$\begin{array}{ccc} H_q(X,A) & \xrightarrow{\partial_*} & H_{q-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f|_A)_*} \\ H_q(Y,B) & \xrightarrow{\partial_*} & H_{q-1}(B). \end{array}$$

(VI) Excision. For an open subset U of X with $\overline{U} \subset \mathrm{int}(A)$ and the inclusion map $i : (X \setminus U, A \setminus U) \to (X, A)$, we have the following isomorphism

$$i_* : H_q(X \setminus U, A \setminus U) \xrightarrow{\cong} H_*(X, A).$$

(VII) Dimension. We have $H_q(\{x\}) = 0$ for q nonzero and x a point of X.

$H_0(\{x\})$ is called the homology coefficient group.

For finite CW pairs of subcomplexes $X_1, X_2 \subset X$ a finite CW complex, we have (VI) replaced with the following, with the inclusion map $i : (X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$,

$$i_* : H_q(X_1, X_1 \cap X_2) \xrightarrow{\cong} H_q(X_1 \cup X_2, X_2).$$

4 Twisted K-theory and more

4.1 Twisted K-theory

String theory. Brane charges are classified by K-theory when the H-flux is trivial, but not in general.

It is shown by [19] in type II string theory that W_3 of a stable D-brane Y, that is a characteristic class of Y taking values in the 2-torsion subgroup of

$H^3(Y, \mathbb{Z})$ does match the restriction of the H-flux. If this is non-zero, then such D-branes do not have spin structure and thus do not define classes in topological K-homology. However, they define classes in twisted K-homology. A duality is obtained by Poincaré duality, since the space time manifold is a spin manifold.

In general type II string theory, the brane charges (not changes) take values in $K^*(X, H)$. The Ramond-Ramond charges in the even group in type IIB and in the odd group in type IIA. We may refer to [56].

Because of differences in sign, we may use $K^*(X, -H)$. The groups $K^*(X, \pm H)$ are isomorphic as groups, though.

Twisted K-theory to be defined. If an n-manifold has a spin structure, then it satisfies Poincaré duality in K-theory. Namely, it is the following isomorphism.

$$K^j(M) \cong K_{n-j}(M).$$

If there are no spin structures, we need take twisted K-theory. As a way to define this, given a class $h \in H^3(M, \mathbb{Z})$, we take the K-theory of a noncommutative C^*-algebra such as a stable continuous trace C^*-algebra $\mathfrak{A}_{tr}(M, h)$ of M with h its Dixmier-Douady class. We then define

$$K^{\pm j}(M, h) = K_j(\mathfrak{A}_{tr}(M, h)).$$

The idea of using continuous trace C^*-algebras is due to Rosenberg [50], [51]. The other idea of using Azumaya algebra is due to Donovan and Karoubi [17]. For more treatments in K-theory twisting, we may see [2], [3], and [33].

Note that K-theory groups for spaces or C^*-algebras have degree only zero or one. The Chern character map sends isometrically K-theory groups with 0 or 1 degree for spaces to homology theory groups with degree even or odd degree, respectively, but by killing torsion parts by tensoring with $\mathbb{Q}$ of rational numbers. Degree 3 may be beyond its K-theory, like.

Example 4.1.1. Let $M = SU(3)/SO(3)$, where elements of $SO(3)$ are viewed as real spacial unitary matrices over reals. The Lie group $SO(3)$ has $SU(2)$ as a double cover, and that is $P^3(\mathbb{R})$ as a topological space, that is a arcwise connected compact Hausdorff space. There is a long exact homotopy sequence as

$$\cdots \xrightarrow{\partial} \pi_n(SO(3)) \longrightarrow \pi_n(SU(3)) \longrightarrow \pi_n(M) \xrightarrow{\partial} \cdots$$

The $SU(3)$ is 2-connected and $\pi_1(SO(3)) \cong \mathbb{Z}_2$, and $\pi_2(SO(3)) = 0$. Thus, $\pi_1(M) = 0$ and $\pi_2(M) \cong \mathbb{Z}_2$.

We have

$$\begin{array}{lllll} \pi_2(SO(3)) = 0 & \longrightarrow \pi_2(SU(3)) = 0 & \longrightarrow \pi_2(M) \cong \mathbb{Z}_2 & \xrightarrow{\partial} \\ \pi_1(SO(3)) \cong \mathbb{Z}_2 & \longrightarrow \pi_1(SU(3)) = 0 & \longrightarrow \pi_1(M) = 0 & \xrightarrow{\partial} \\ \pi_0(SO(3)) = 0 & \longrightarrow \pi_0(SU(3)) = 0 & \longrightarrow \pi_0(M) = 0. & \end{array}$$

Example 4.1.2. We have group isomorphisms $SU(2) \cong S^3$ and $SO(2) \cong S^1$.
We also have $O(n)/SO(n) \cong S^0 = \{\pm 1\}$ and $U(n)/SU(n) \cong S^1$.
The $SO(n)$ and $SU(n)$ are arcwise connected.
We have $\pi_1(P^1(\mathbb{R})) \cong \mathbb{Z}$ and $\pi_1(P^n(\mathbb{R})) \cong \mathbb{Z}_2$ for $n \geq 2$.
We also have $\pi_1(SU(n)) \cong 0$ for $n \geq 1$.

Example 4.1.3. Let $X = S^3$ with $h = k \neq 0$, where $H^3(S^3)$ is identified with $\mathbb{Z}$. In this case, the Steenrod operation $sq^3 = 0$, but the differential as the sum of sq^3 and cup product with h as

$$d_3 : H^0(X, \mathbb{Z}) \cong \mathbb{Z} \to H^3(X, \mathbb{Z}) \cong \mathbb{Z}$$

is multiplication by k. It then follows that $K^0(S^3, h) = 0$ and $K^1(S^3, h) \cong \mathbb{Z}_k$.

4.2 Brauer groups and C^*-algebras of continuous trace

Theorem 4.2.1. (Green). *Let X be a second-countable, locally compact Hausdorff space. Then the Dixmier-Douady class defines an isomorphism between the Brauer group $Br(X)$ and $H^3(X, \mathbb{Z})$. If X is a finite CW-complex, then the Grothendieck-Serre Brauer group is embedded as a torsion subgroup.*

Proof. Stable C^*-algebras of continuous trace over the space X are classified by their Dixmier-Douady (DD) invariants. Every class in $H^3(X, \mathbb{Z})$ are obtained by this way. For separable C^*-algebras, stable isomorphism equivalence coincide with Morita equivalence. This is also true for separable C^*-algebras over X. Therefore, $H^3(X, \mathbb{Z})$ classifies C^*-algebra of continuous trace over X, up to $C_0(X)$-linear Morita equivalence.

We need to check that group operation in the Brauer group corresponds to addition of DD invariants. As well, $\mathfrak{A}_{tr}(X, h)^\circ \cong \mathfrak{A}_{tr}(X, -h)$ over X.

The C^*-algebra $\mathfrak{A}_{tr}(X, h)$ is viewed as a continuous section algebra $\Gamma_0(X, \mathcal{A})$, where $\mathcal{A}$ is a bundle of algebras with fibers $\mathbb{K}$. Let $\mathfrak{A}_{tr}(X, h') = \Gamma_0(X, \mathcal{A}')$ as well. Then we have

$$\mathfrak{A}_{tr}(X, h) \otimes_X \mathfrak{A}_{tr}(X, h') = \Gamma_0(X, \mathcal{A} \otimes \mathcal{A}'),$$

where the last $\otimes = \otimes_f$ is the fiberwise tensor product of bundles of algebras Transition functions for $\mathcal{A} \otimes \mathcal{A}'$ are obtained by tensoring those for $\mathcal{A}$ and $\mathcal{A}'$, so that the DD class for $\mathcal{A} \otimes \mathcal{A}'$ is the sum of those for $\mathcal{A}, \mathcal{A}'$. □

For a connected finite CW-complex X, the Azumaya algebras over $C(X)$ are unital C^*-algebras of continuous trace with center $C(X)$. Let X be a locally compact Hausdorff space and consider all C^*-algebras of continuous trace with spectrum of irreducible representation classes, homeomorphic to X, viewed as algebras over $C_0(X)$ of continuous functions on X vanishing at infinity. The Brauer group $Br(X)$ is obtained by those C^*-algebras up to $C_0(X)$-linear Morita equivalence, with group operation given by $\otimes_{C_0(X)}$ the topological tensor product over $C_0(X)$, and inversion given by opposite algebra class. For more details, we may refer to [49].

Theorem 4.2.2. (Grothendieck-Serre). *Let X be a connected finite CW-complex. Then the Brauer group of $C(X)$ is identified with the torsion subgroup of $H^3(X, \mathbb{Z})$. The Azumaya algebras over $C(X)$ are obtained as $\Gamma(X, \mathcal{A})$, where $\mathcal{A}$ is a locally trivial bundle of algebras over X with fibers matrix algebra $M_n(\mathbb{C})$ and structure group $Aut(M_n(\mathbb{C})) \cong PGL_n(\mathbb{C})$.*

For the proof, we may see [24] or [14].

Let R be a commutative ring. The Brauer group $Br(R)$ of R is obtained by R-Morita equivalence classes $[A]$ of A Azumaya algebras or central separable algebras such that A are R-algebras with R center and as well A are finitely generated projective modules over $A \otimes_R A^\circ$, with group operation as tensor product over R and $[A]^{-1} = [A^\circ]$. We may refer to [5].

Let F be a field in mathematics, not in physics. This is a commutative ring. The Wedderburn theorem says that every finite dimensional central algebra over F (or simple with center F) has the form of $M_n(D)$ a matrix algebra over D a division algebra with center F. Such two algebras are F-Morita equivalent if and only if division algebras D are F-isomorphic.

The Brauer group $Br(F)$ of F is defined to be an abelian group of Morita equivalence classes $[A]$ of central simple algebras A over F with tensor product operation $\otimes_F$ and with $[F]$ identity element and $[A^\circ]$ inverse of $[A]$. The point is that

$$A \otimes_F A^\circ \cong \mathrm{End}_F(A),$$

which is isomorphic to a matrix algebra over F and thus is Morita equivalent to F.

The group $Br(F)$ can be shown to be isomorphic to $H^2(\mathrm{Gal}(F^s/F), (F^s)^*)$, where F^s is the separable closure of F. The group is also viewed as $H^2_{et}(\mathrm{Sp}(F), G_m)$. The point is that étale open subsets of $\mathrm{Sp}(F)$ correspond to finite Galois coverings L of F, and G_m on which takes value L^*, viewed as a module over $\mathrm{Gal}(L/F)$. We may see [41].

Theorem 4.2.3. (Dixmier-Douady). *Let X be a second-countable locally compact Hausdorff space, and let A be a separable C^*-algebra of continuous trace with spectrum X. Suppose either that A is stable, i.e., $A \cong A \otimes \mathbb{K}$, or that X is finite dimensional and every irreducible representation of A has dimension of the representation space infinity. Then A is isomorphic to the algebra $\Gamma_0(X, \mathcal{A})$ of continuous sections on X vanishing at infinity of a locally trivial bundle $\mathcal{A}$ over X with fibers $\mathbb{K}$.*

It is shown that A has a characteristic class $\delta(A) \in H^3(X, \mathbb{Z})$ on whether or not A is locally trivial in the sense above. This is the Dixmier-Douady class, which does not change by replacing A with $A \otimes \mathbb{K}$.

It is certainly known in personal knowledge some quite long time ago.

Suppose that A is locally trivial in that sense. This bundle has fibers $\mathbb{K}$ on a Hilbert space H with dimension infinity and structure group

$$Aut(\mathbb{K}) \cong pU(H) = U(H)/\mathbb{T}$$

the projective unitary group on H. The unitary group $U(H)$ of unitaries is contractible since H has dimension infinity. Thus, $pU(H)$ has homotopy type $B\mathbb{T}$, which is a $K(\mathbb{Z}, 2)$ space such that the space has homotopy type of a CW complex with non-zero homotopy group of only one degree, namely, π_2 is $\mathbb{Z}$, and is unique up to homotoy equivalence. Moreover, principal pU-bundles over X are classified by

$$[X, BpU] = [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z}).$$

It then follows that every stable C^*-algebra of continuous trace defines a class in H^3 and also every class in H^3 comes from such a C^*-algebra.

There is another theory of gerbes to approach the theory of C^*-algebras of continuous trace by the Dixmier-Douady class. We may refer to [46], [9], or [26] and as well [20].

Roughly speaking, a line bundle is a geometric object giving rise to a class in H^2 such as the Chern class c_1, or the de Rham class of the curvature form of a connection on the bundle in terms of Chern-Weil theory. A gerbe is a geometric object giving rise to a class in H^3.

More to be continued if we like it in the next time.

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Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Senbaru 1, Nishihara, Nakagami, Okinawa 903-0213, Japan.
Email: sudo@math.u-ryukyu.ac.jp
Visit: www.math.u-ryukyu.ac.jp