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A locally commentative learning transformation by the entire cyclic cohomology theory for Banach algebras

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Abstract

We study the entire cyclic cohomology theory for Banach algebras by following a part of the noncommutative geometric theory invented by Connes.

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1 Introduction

Following Connes [3] we as beginners, outsiders, fools or not would like to make a personal locally commentative learning transformation by the entire cyclic cohomology theory for Banach algebras, as a short story specialized, with some considerable effort in time and space limited.

This is nothing but a review, added with some explicit computation or proofs, as a back to the past for a return to the future, after [14] and [15].

Sections presented by us are as follows.

The sections

Let's start with us and helpful $\star$ lines as hints added to explore the story, together with a pencil mightier than an apple, remembering the dream.

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2 Entire cyclic cohomology theory

Let A be a unital Banach algebra over $\mathbb{C}$ of complex numbers. Let us recall the construction of the (b, B) bicomplex of cyclic cohomology.

For a non-negative integer $n \in \mathbb{N}$, let $C^n(A, A^*) = C^n$ denote the space of continuous $(n+1)$-linear forms $\varphi : A^{n+1} \to A^*$ on A^{n+1}, as n times differentials.

Set $C^{-n}(A, A^*) = \{0\}$ for $n > 0$.

Define two differentials b and B as in the following.

The differential $b : C^n \to C^{n+1}$ is defined by

$$
\begin{aligned}
&(b\varphi)(a_0, \cdots, a_{n+1}) \quad a_0, \cdots, a_{n+1} \in A \\
&= \sum_{j=0}^{n} (-1)^j \varphi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, \cdots, a_n).
\end{aligned}
$$

In particular, $b : C^0 \to C^1$ is given by

$$
(b\varphi)(a_0, a_1) = \varphi(a_0 a_1) - \varphi(a_1 a_0).
$$

If $b\varphi = 0$, then $\varphi : A \to A^*$ is a trace map.

As well, $b : C^1 \to C^2$ is given by

$$
(b\varphi)(a_0, a_1, a_2) = \varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, a_1).
$$

The differential $B : C^n \to C^{n-1}$ is defined by $B = A_0 \circ B_0$, where

$$
\begin{aligned}
&(B_0\varphi)(a_0, \cdots, a_{n-1}) \\
&= \varphi(1, a_0, \cdots, a_{n-1}) - (-1)^n \varphi(a_0, \cdots, a_{n-1}, 1), \quad \varphi \in C^n, \\
&(A_0\psi)(a_0, \cdots, a_{n-1}) \\
&= \sum_{j=0}^{n-1} (-1)^{(n-1)j} \psi(a_j, a_{j+1}, \cdots, a_{j-1}), \quad \psi \in C^{n-1}.
\end{aligned}
$$

In particular, $B : C^1 \to C^0$ is given by

$$
\begin{aligned}
(B_0\varphi)(a_0) &= \varphi(1, a_0) + \varphi(a_0, 1), \\
(A_0\psi)(a_0) &= \psi(a_0)
\end{aligned}
$$

so that $B = B_0$ on C^1.

As well, $B : C^2 \to C^1$ is given by

$$
\begin{aligned}
(B_0\varphi)(a_0, a_1) &= \varphi(1, a_0, a_1) - \varphi(a_0, a_1, 1), \\
(A_0\psi)(a_0, a_1) &= \psi(a_0, a_1) - \psi(a_1, a_0).
\end{aligned}
$$

We have the differential property that

$$
b^2 = b \circ b = 0 = B^2 = B \circ B
$$

and $b \circ B = -B \circ b$ as a non-commutativity.

We then have the bicomplex $(C^{n,m}, d_1, d_2)$, where $C^{n,m}$ is defined by C^{n-m} for $n, m \in \mathbb{Z}$, and the local graded commutative square:

$$\begin{tikzcd} C^{n,m+1} \arrow[r, "d_1=(n-(m+1)+1)b"] & C^{n+1,m+1} \\ C^{n,m} \arrow[u, "d_2=\frac{1}{n-m}B"] \arrow[r, "d_1=(n-m+1)b"] & C^{n+1,m} \arrow[u, "d_2=\frac{1}{n+1-m}B"'] \end{tikzcd}$$

so that $d_2 \circ d_1 = B \circ b$ and $d_1 \circ d_2 = b \circ B$ as well.

$\star$ The cohomology of the complex ker$B$/imB by b is zero.

Proof. Consider the exact sequence of complexes of cochains

$$0 \to \mathrm{im}(B) \longrightarrow \ker(B) \longrightarrow \ker(B)/\mathrm{im}(B) \to 0$$

where the coboundary is given by Hochschild differential b. The first long map in the sequence above induces an isomorphism in cohomology. Then the cohomology of the quotient complex by b is zero (cf. [14]). $\square$

$\star$ The spectral sequence associated to the first filtration $F_pC = \sum_{n \geq p} C^{n,m}$ in the first variable n by b has the initial E_2 term equal to zero.

Proof. The initial term is given by $\ker(B)/\mathrm{im}(B)$.

Note that $F_{p+1}C$ is contained in F_pC. $\square$

The bicomplex $C^{n,m} = C^{n-m}$ has support in (n, m) with $n - m \geq 0$ (not $n + m$).

Thus $m \leq n$ on the (n, m) plane to make the lower triangle region.

The spectral sequence does not converge in general when we take cochains with finite support.

The cohomology of the bicomplex $C = C^{*,*} = C^{*-*}$, taken with supports finite, is nothing but the periodic cyclic cohomology $H^*(A)$.

Namely, $H^{2n}(C) = H^{\mathrm{ev}}(A)$ and $H^{2n-1}(C) = H^{\mathrm{od}}(A)$.

As with, $F^qC = \sum_{m \geq q} C^{n,m}$ the second filtration, then $H^p(F^qC) = cH^n(A)$ for $n = p - 2q$.

$\star$ Note that if p is even, then so is n, and if p is odd, then so is n. As well, when q varies, so does n.

Taking cochains with supports arbitrary, without controlling their growth, the corresponding cohomology is trivial.

Provided that we control the growth of the norm in cochains of even or odd degrees of the b and B bicomplex, we obtain the cohomology relevant to analyze infinite dimensional spaces and cycles.

Because of the periodicity

$$C^{n,m} = C^{n-m} = C^k \to C^{n+1,m+1} = C^{n-m} = C^k$$

in the bicomplex b and B, it is convenient to work with C^k.

Define

$$C^{\mathrm{ev}} = \{(\varphi_{2n})_{n\in\mathbb{N}} \,|\, \varphi \in C^{2n}, n \in \mathbb{N}\},$$
$$C^{\mathrm{od}} = \{(\varphi_{2n+1})_{n\in\mathbb{N}} \,|\, \varphi \in C^{2n+1}, n \in \mathbb{N}\}$$

as the even and odd spaces of sequences of even and odd cochains, respectively. And define the boundary operator $\partial = d_1 + d_2$ which maps C^{ev} to C^{od} and maps C^{od} to C^{ev}. Namely,

$$C^{\mathrm{ev}} \xrightarrow{\partial} C^{\mathrm{od}} \xrightarrow[\partial]{} C^{\mathrm{ev}}.$$

$\star$ Note that

$$\partial(C^k) = d_1(C^k) + d_2(C^k) \subset C^{k+1} \oplus C^{k-1}.$$

As well,

$$\partial^2 = \partial \circ \partial = d_1^2 + (d_1 \circ d_2) + (d_2 \circ d_1) + d_2^2 = 0$$

because of $b \circ B = -B \circ b$ so that ∂ is a derivation!

Definition 2.1. Cochain sequences $(\varphi_{2n}) \in C^{\mathrm{ev}}$ and $(\varphi_{2n+1}) \in C^{\mathrm{od}}$ of even and odd degrees are said to be entire if the radius(es) of convergence of the following series involving the supremum norm $\|\cdot\|$

$$\sum_{n\in\mathbb{N}} \frac{\|\varphi_{2n}\|}{n!} z^n \quad \text{and} \quad \sum_{n\in\mathbb{N}} \frac{\|\varphi_{2n+1}\|}{n!} z^n$$

in $\mathbb{C}$ are infinity, respectively.

$\star$ We may denote by r^{ev} and r^{od} theose respective radiuses.

By definition, for $z \in \mathbb{C}$ with $|z| < r^{\mathrm{ev}}$ and $|z| < r^{\mathrm{od}}$, the respective series converge absolutely, respectively, and for $z \in \mathbb{C}$ with $|z| > r^{\mathrm{ev}}$ and $|z| > r^{\mathrm{od}}$, the respective series diverge, respectively.

The ratio formula in power series known well implies that

$$r^{\mathrm{ev}} = \lim_{n\to\infty} \frac{\|\varphi_{2n}\|(n+1)}{\|\varphi_{2(n+1)}\|} \quad \text{and} \quad r^{\mathrm{od}} = \lim_{n\to\infty} \frac{\|\varphi_{2n+1}\|(n+1)}{\|\varphi_{2(n+1)+1}\|}$$

if they exist in $[0,\infty) \cup \{\infty\}$.

Entireness implies that the limits are infinity. Alternatively, the infinite limits are replaced as

$$\frac{1}{r^{\mathrm{ev}}} = \lim_{n\to\infty} \frac{\frac{\|\varphi_{2(n+1)}\|}{\|\varphi_{2n}\|}}{n+1} = 0 \quad \text{and} \quad \frac{1}{r^{\mathrm{od}}} = \lim_{n\to\infty} \frac{\frac{\|\varphi_{2(n+1)+1}\|}{\|\varphi_{2n+1}\|}}{n+1} = 0.$$

Namely, $\frac{\|\varphi_{2n}\|}{\|\varphi_{2(n-1)}\|} = o(n)$ $(n \to \infty)$ and $\frac{\|\varphi_{2n+1}\|}{\|\varphi_{2(n-1)+1}\|} = o(n)$ $(n \to \infty)$

In particular, if the limits of $\frac{\|\varphi_{2n}\|}{\|\varphi_{2(n+1)}\|}$ and $\frac{\|\varphi_{2n+1}\|}{\|\varphi_{2(n+1)+1}\|}$ are nonzero, then the respective cochain sequences are entire.

Moreover, the Cauchy-Hadamard formula implies that if

$$l^{\mathrm{ev}} = \limsup_{n\to\infty} \sqrt[n]{\frac{\|\varphi_{2n}\|}{n!}} \quad \text{and} \quad l^{\mathrm{od}} = \limsup_{n\to\infty} \sqrt[n]{\frac{\|\varphi_{2n+1}\|}{n!}}$$

exist in $[0,\infty)\cup\{\infty\}$, then we have $r^{\mathrm{ev}} = \frac{1}{l^{\mathrm{ev}}}$ and $r^{\mathrm{od}} = \frac{1}{l^{\mathrm{od}}}$.

The ratio formula existence is contained in the Cauchy-Hadamard formula

The norm $\|\varphi\|$ for a cochain $\varphi \in C^m = C(A^m, A^*)$ for any degree m as a continuous m-linear form on A to the dual A^* is defined to be the Banach space norm given by

$$\|\varphi\| = \sup\{|\varphi(a_0,\cdots,a_m)| \in \mathbb{R} \,|\, \|a_j\| \le 1, j \in \{0,\cdots,m\}\}.$$

⋆ Note that

$$|\varphi(a_0,\cdots,a_m)| = |\varphi(a_1,\cdots,a_m)(a_0)| \le \|\varphi(a_1,\cdots,a_m)\|\|a_0\|.$$

Therefore,

$$\|\varphi\| \le \sup_{\|a_j\|\le 1, j=1,\cdots,m} \|\varphi(a_1,\cdots,a_m)\|.$$

We also have

$$\|\varphi(a_1,\cdots,a_m)\| = \sup_{\|a_0\|\le 1} |\varphi(a_1,\cdots,a_m)(a_0)| \le \|\varphi\|.$$

It then follows that

$$\|\varphi\| = \sup_{\|a_j\|\le 1, j=1,\cdots,m} \|\varphi(a_1,\cdots,a_m)\|.$$

⋆ If $\|\varphi\| = 0$, then $|\varphi(a_0,\cdots,a_m)| = 0$ for $\|a_j\| \le 1$, $j = 0,\cdots,m$. Thus, for any nonzero $a_j \in A$, we have

$$|\varphi(a_0,\cdots,a_m)| = \|a_0\|\cdots\|a_m\|\|\varphi(\frac{a_0}{\|a_0\|},\cdots,\frac{a_m}{\|a_m\|})| = 0.$$

Also, $\varphi(0,a_1,\cdots,a_m) = 2\varphi(0,a_1,\cdots,a_m)$, hence $\varphi(0,a_1,\cdots,a_m) = 0$. It then follows that $\varphi = 0$. The converse also holds.

By definition, $\|\alpha\varphi\| = |\alpha|\|\varphi\|$ for $\alpha \in \mathbb{C}$ with absolute value $|\alpha|$.

By definition, $\|\varphi+\psi\| \le \|\varphi\| + \|\psi\|$ by triangle inequality of absolute value and by supremum.

For a Cauchy sequence (φ_k) of C^m by the norm, the completeness of $\mathbb{C}$ implies that there exists $\varphi = \lim \varphi_k$ at any $(a_0,a_1,\cdots,a_m)$. By completeness and linearity of A and A^*, we have $\varphi \in C^m$.

It follows in particular that any entire even cochain sequence $(\varphi_{2n}) \in C^{\mathrm{ev}}$ defines an entire function $f_\varphi = f_\varphi^{\mathrm{ev}}$ on the Banach space A given by

$$f_\varphi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{2n}(x,\cdots,x), \quad x \in A.$$

$\star$ Note that for any nonzero $x \in A$, with $0! = 1$,

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n!} \varphi_{2n}(x, \cdots, x) \right| = \sum_{n=0}^{\infty} \frac{1}{n!} \left| \varphi_{2n}(\frac{x}{\|x\|}, \cdots, \frac{x}{\|x\|}) \right| \|x\|^{2n+1}$$
$$\leq \|x\| \sum_{n=0}^{\infty} \frac{\|\varphi_{2n}\|}{n!} (\|x\|^2)^n,$$

which converges by entireness.

As well, for odd entire $(\varphi_{2n+1}) \in C^{\mathrm{od}}$, similarly we have

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n!} \varphi_{2n+1}(x, \cdots, x) \right| = \sum_{n=0}^{\infty} \frac{1}{n!} \left| \varphi_{2n+1}(\frac{x}{\|x\|}, \cdots, \frac{x}{\|x\|}) \right| \|x\|^{2n+2}$$
$$\leq \|x\|^2 \sum_{n=0}^{\infty} \frac{\|\varphi_{2n+1}\|}{n!} (\|x\|^2)^n,$$

which converges by entireness so that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{2n+1}(x, \cdots, x)$ is certainly defined to be $f_{\varphi}^{\mathrm{od}}(x)$.

Lemma 2.2. *If φ_* is an even or odd entire cochain sequence (φ_{2n}) or (φ_{2n+1}) respectively, then so is $\partial\varphi_* = (d_1 + d_2)\varphi_*$, where φ_* may be denoted as φ^{ev} or φ^{od} respectively.*

Proof. For $\varphi_m \in C^m$, we have $\|b\varphi_m\| \leq (m+2)\|\varphi_m\|$ and $\|B_0\varphi_m\| \leq 2\|\varphi_m\|$, and $\|A_0 B_0 \varphi_m\| \leq 2m\|\varphi_m\|$.

$\star$ Indeed, for $a_0, \cdots, a_{m+1} \in A$ with norm less than or equal to 1,

$$|(b\varphi_m)(a_0, \cdots, a_{m+1})| \leq$$
$$\sum_{j=0}^{m} |\varphi_m(a_0, \cdots, a_j a_{j+1}, \cdots, a_{m+1})| + |\varphi_m(a_{m+1}a_0, \cdots, a_m)|$$
$$\leq (m+2)\|\varphi_m\|.$$

As well,

$$|(B_0\varphi_m)(a_0, \cdots, a_{m-1})| \leq |\varphi_m(1, a_0, \cdots, a_{m-1})| + |\varphi_m(a_0, \cdots, a_{m-1}, 1)| \leq 2\|\varphi_m\|.$$

Moreover,

$$|(A_0\varphi_{m-1})(a_0, \cdots, a_{m-1})| \leq \sum_{j=0}^{m-1} |\varphi_{m-1}(a_j, a_{j+1}, \cdots, a_{j-1})| \leq m\|\varphi_{m-1}\|.$$

Note also that

$$(d_1 + d_2)\varphi_m = (m+1)b\varphi_m + \frac{1}{m} B\varphi_m.$$

Thus,

$$\begin{aligned}\|\partial\varphi_m\| &\le (m+1)\|b\varphi_m\| + \frac{1}{m}\|A_0B_0\varphi_m\| \\ &\le (m+1)(m+2)\|\varphi_m\| + \frac{1}{m}2m\|\varphi_m\|.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{\|\partial\varphi_{2n}\|}{n!}z^n \le \sum_{n=0}^{\infty}\frac{(2n+1)(2n+2)\|\varphi_{2n}\|}{n!}z^n + \sum_{n=0}^{\infty}\frac{2\|\varphi_{2n}\|}{n!}z^n,$$
$$\sum_{n=0}^{\infty}\frac{\|\partial\varphi_{2n+1}\|}{n!}z^n \le \sum_{n=0}^{\infty}\frac{(2n+2)(2n+3)\|\varphi_{2n+1}\|}{n!}z^n + \sum_{n=0}^{\infty}\frac{2\|\varphi_{2n+1}\|}{n!}z^n.$$

Entireness for $\partial\varphi_*$ would follow from the convergence of the right hand sides.

By the way, the convergence for the first terms may not follow from entireness for φ_* in general?

If either the ratio formula or the Cauchy-Hadamard formula imply the radius of convergence infinity, then there are no problem in convergence.

Anyhow, such a convergence may be involved in the definition for entireness from the first stage. Or assumed should be that infinite radius of convergence is given by the ratio formula or the Cauchy-Hadamard formula. □

Definition 2.3. Let A be a unital Banach algebra. The entire cyclic cohomology of A is defined to be the cohomology of the following short complex

$$C_{et}^{\mathrm{ev}}(A) \xrightarrow{\partial=\partial_{ev}} C_{et}^{\mathrm{od}}(A) \xrightarrow[\partial=\partial_{od}]{} C_{et}^{\mathrm{ev}}(A) \xrightarrow{\partial_{ev}} C_{et}^{\mathrm{od}}(A)$$

of entire cochain sequences of A with even and odd degrees respectively.

By definition, we have the two entire cyclic cohomology groups as

$$H_{et}^{\mathrm{ev}}(A) = \ker(\partial_{ev})/\mathrm{im}(\partial_{od}) \quad \text{and} \quad H_{et}^{\mathrm{od}}(A) = \ker(\partial_{od})/\mathrm{im}(\partial_{ev}).$$

There is an obvious map from $H(A)$ to $H_{et}(A)$, where $H(A) = H^{\mathrm{ev}}(A) \oplus H^{\mathrm{od}}(A)$ is the periodic cyclic cohomology of A, and $H_{et}(A) = H_{et}^{\mathrm{ev}}(A) \oplus H_{et}^{\mathrm{od}}(A)$.

$\star$ Certainly, an entire even or odd cochain sequence is an even or odd cochain sequence respectively. But the reason is that finite supportness implies entireness. Namely, constantness implies entireness.

There is a natural filtration of $H_{et}(A)$ by dimensions of cochains.

An even cochain sequence (φ_{2n}) is said to be of dimension $\le k$ if $\varphi_{2n} = 0$ for $2n > k$.

Unlike what happens for $H(A)$, that filtration does not exhaust all of $H_{et}(A)$ in general. Only the image of $H(A)$ in $H_{et}(A)$ is exhausted.

Example 2.4. Let $A = \mathbb{C}$ the trivial Banach algebra as the simplest case with dimension 1.

$\star$ By the way, the space $(\mathbb{C}^{n+1})^*$ of continuous linear forms ψ_n on $\mathbb{C}^{n+1}$ is given by $\mathbb{C}^{n+1}$ via the inner product as

$$\psi_n((z_j)) = \sum_{j=0}^{n} w_j z_j, \quad (z_j) \in \mathbb{C}^{n+1}$$

with ψ_n identified with $(w_j) \in \mathbb{C}^{n+1}$.

This ψ_n is certainly linear because

$$\begin{aligned} \psi_n((z_j) + (z_j')) &= \sum_{j=0}^{n} w_j(z_j + z_j') \quad (z_j), (z_j') \in \mathbb{C}^{n+1} \\ &= \sum_{j=0}^{n} w_j z_j + \sum_{j=0}^{n} w_j z_j' = \psi_n((z_j)) + \psi_n((z_j')). \end{aligned}$$

Then we have

$$|\psi_n((z_j))| \leq \sum_{j=0}^{n} |w_j z_j| \leq \sum_{j=0}^{n} |w_j|$$

for $|z_j| \leq 1$, $j = 0, \cdots, n$. Thus, $\|\psi_n\| \leq \sum_{j=0}^{n} |w_j|$. Conversely, with $z_j = \frac{\overline{w_j}}{|w_j|}$ for w_j nonzero, we have

$$\psi_n((z_j)) = \sum_{j=0}^{n} |w_j|.$$

Therefore, we obtain $\|\psi_n\| = \sum_{j=0}^{n} |w_j| \equiv \|(w_j)\|_1$.

Namely, the dual space $(\mathbb{C}^{n+1})^*$ is identified with the Banach space $\mathbb{C}^{n+1}$ with the 1-norm.

$\star$ On the other hand, the space $C^n(\mathbb{C}, \mathbb{C}^*)$ of continuous $(n+1)$-(multi-)linear forms φ_n on $\mathbb{C}$ is given by $\mathbb{C}$. Indeed,

$$\varphi_n(z_0, \cdots, z_n) = \lambda_n z_0 \cdots z_n, \quad z_0, \cdots, z_n \in \mathbb{C}$$

for some $\lambda_n \in \mathbb{C}$.

Then we have

$$|\varphi_n(z_0, \cdots, z_n)| = |\lambda_n||z_0| \cdots |z_n| \leq |\lambda_n|$$

for $z_0, \cdots, z_n \in \mathbb{C}$ with $|z_0| \leq 1, \cdots, |z_n| \leq 1$. Thus, $\|\varphi_n\| \leq |\lambda_n|$. Conversely, we have $\varphi_n(1, \cdots, 1) = \lambda_n$. Hence $|\lambda_n| \leq \|\varphi_n\|$. Therefore, $\|\varphi_n\| = |\lambda_n|$.

Note also that $\lambda_n = \varphi_n(1, \cdots, 1)$.

An element of $C_{et}^{\mathrm{ev}}(\mathbb{C})$ is given by an infinite sequence (λ_{2n}) with $\lambda_{2n} \in \mathbb{C}$ such that

$$\sum_{n=0}^{\infty} \frac{|\lambda_{2n}|}{n!} z^n < \infty, \quad z \in \mathbb{C}.$$

Similarly, an element of $C_{et}^{\mathrm{od}}(\mathbb{C})$ is given by replacing $2n$ with $2n+1$ in the power series.

The boundary $\partial = d_1 + d_2$ of (λ_{2n}) is zero since both b and B are zero on even cochains.

$\star$ Let $\varphi_0(z_0) = \lambda_0 z_0$ for some $\lambda_0 \in \mathbb{C}$.

$$\begin{aligned}
&(b\varphi_0)(z_0, z_1) = \varphi_0(z_0 z_1) - \varphi_0(z_1 z_0) = \lambda_0 z_0 z_1 - \lambda_0 z_1 z_0 = 0. \\
&B_0\varphi_0 = \varphi_0(1) - \varphi_0(1) = 0. \quad B\varphi_0 = A_0 B_0 \varphi = 0.
\end{aligned}$$

Let $\varphi_2(z_0, z_1, z_2) = \lambda_2 z_0 z_1 z_2$ for some $\lambda_2 \in \mathbb{C}$.

$$\begin{aligned}
&(b\varphi_2)(z_0, z_1, z_2, z_3) \\
&= \varphi_2(z_0 z_1, z_2, z_3) - \varphi_2(z_0, z_1 z_2, z_3) + \varphi_2(z_0, z_1, z_2 z_3) - \varphi_2(z_3 z_0, z_1, z_2) = 0. \\
&B_0\varphi_2(z_0, z_1) = \varphi_2(1, z_0, z_1) - \varphi_2(z_0, z_1, 1) = 0. \quad B\varphi_0 = A_0 B_0 \varphi = 0.
\end{aligned}$$

Namely, the image of the boundary map ∂_e is zero.

For m odd, let $\varphi(z_0, \cdots, z_m) = \lambda z_0 \cdots z_m$ with $\varphi = \varphi_m \in C^m(\mathbb{C})$, we have

$$\begin{aligned}
(b\varphi)(z_0, \cdots, z_{m+1}) &= \lambda z_0 \cdots z_{m+1}, \\
(B\varphi)(z_0, \cdots, z_{m-1}) &= 2m\lambda z_0 \cdots z_{m-1}.
\end{aligned}$$

$\star$ Let $\varphi_1(z_0, z_1) = \lambda z_0 z_1$. Then

$$\begin{aligned}
&(b\varphi_1)(z_0, z_1, z_2) = \varphi_1(z_0 z_1, z_2) - \varphi_1(z_0, z_1 z_2) + \varphi_1(z_2 z_0, z_1) = \lambda z_0 z_1 z_2, \\
&(B_0\varphi_1)(z_0) = \varphi_1(1, z_0) + \varphi_1(z_0, 1) = 2\lambda z_0. \\
&(B\varphi_1)(z_0) = (B_0\varphi)(z_0) = 2\lambda z_0.
\end{aligned}$$

It thus follows that

$$\begin{aligned}
(d_1\varphi)(z_0, \cdots, z_{m+1}) &= (m+1)\lambda z_0 \cdots z_{m+1}, \\
(d_2\varphi)(z_0, \cdots, z_{m-1}) &= 2\lambda z_0 \cdots z_{m-1}
\end{aligned}$$

since $d_1 = (m+1)b$ and $d_2 = \frac{1}{m}B$.

$\star$ Note that d_1 is essentially multiplication by degree $m+1$ and d_2 is also by only 2.

Therefore, the boundary $\partial((\varphi_{2n+1}))$ of an odd cochain sequence (φ_{2n+1}) at $2n$ is given by

$$d_1\varphi_{2n-1} + d_2\varphi_{2n+1} = (2n)\varphi_{2n-1} + 2\varphi_{2n+1} = \partial((\varphi_{2n+1}))_{2n}.$$

If the boundary $\partial((\varphi_{2n+1}))$ is zero, then

$$\varphi_{2n+1} = -n\varphi_{2n-1} = \cdots = (-1)^n n! \varphi_1$$

and $d_2\varphi_1 = 0$. Hence, if so, the sequence is zero. This is the same for $C_{et}^{\mathrm{od}}(\mathbb{C})$. Namely, the kernel of the boundary map ∂_{od} is zero.

It then follows that

$$H_{et}^{\mathrm{od}}(\mathbb{C})) = \ker(\partial_{od})/\mathrm{im}(\partial_{ev}) = \ker(\partial_{od}) = \{0\}.$$

Since the kernel $\ker(\partial_{od})$ is zero, then the map ∂_{od} is injective.

Since the image $\mathrm{im}(\partial_{ev})$ is zero, then the kernel $\ker(\partial_{ev})$ is $C_{et}^{\mathrm{ev}}(\mathbb{C})$.

Moreover, for $(\varphi_{2n}) \in C_{et}^{\mathrm{ev}}(\mathbb{C})$, in particular, the series

$$\sigma((\varphi_{2n})) = \sum_{n=0}^{\infty}(-1)^n \frac{\lambda_{2n}}{n!}$$

is absolutely convergent and convergent.

There is a linear map h from $C_{et}^{ev}(\mathbb{C})$ to the space of holomorphic functions defined on $\mathbb{C}$, denoted as $\mathfrak{H}_{et}(\mathbb{C})$, defined as

$$h((\varphi_{2n}))(z) = \sum_{n=0}^{\infty}(-1)^n \frac{\lambda_{2n}}{n!} z^n, \quad z \in \mathbb{C}.$$

There is also a linear quotient map from $C_{et}^{ev}(\mathbb{C})$ to $\mathbb{C}$ defined by $\mathrm{ev}_1 \circ h$, where the linear map ev_1 on $\mathfrak{H}_{et}(\mathbb{C})$ means the evaluation map at $1 \in \mathbb{C}$.

Note that $\mathrm{ev}_1 \circ h = \sigma$. Namely, we have the following commutative diagram:

$$\begin{tikzcd}
C_{et}^{\mathrm{ev}} \arrow[r, "\sigma"] \arrow[d, "h"'] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
\mathfrak{H}_{et}(\mathbb{C}) \arrow[r, "\mathrm{ev}_1"] & \mathbb{C} \arrow[r] & 0.
\end{tikzcd}$$

Note also that the map h is injective, but not surjective.

Indeed, if the function $h((\varphi_{2n}))$ is zero, then in particular, $h(\varphi_{2n})(0) = \lambda_0 = 0$. As well, differentiating the function term-wise and evaluating the derivative at zero implies $\lambda_{2n} = 0$. Continuing this process inductively implies that the sequence (φ_{2n}) is zero.

Furthermore, $\sigma((\varphi_{2n}))$ is zero if and only if (φ_{2n}) is in the boundary of $C_{et}^{\mathrm{od}}(\mathbb{C})$.

$\star$ If (φ_{2n}) is in the boundary, then $\varphi_{2n} = 2n\varphi_{2n-1} + 2\varphi_{2n+1}$. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n \frac{\lambda_{2n}}{n!} &= \sum_{n=1}^{\infty}(-1)^n \frac{2\lambda_{2n-1}}{(n-1)!} + \sum_{n=0}^{\infty}(-1)^n \frac{2\lambda_{2n+1}}{n!} \quad (k = n-1) \\
&= \sum_{k=0}^{\infty}(-1)^{k+1} \frac{2\lambda_{2k+1}}{k!} + \sum_{n=0}^{\infty}(-1)^n \frac{2\lambda_{2n+1}}{n!} = 0.
\end{aligned}$$

Conversely, if the series $\sigma((\varphi_{2n}))$ is zero, then we can define $\lambda_1 = \frac{1}{2}\lambda_0$, $\lambda_3 = \frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_0$, $\lambda_5 = \frac{1}{2}\lambda_4 - \frac{(3+1)}{2}\lambda_3$, and inductively, thus

$$\lambda_{2n+1} = \frac{1}{2}\lambda_{2n} - \frac{(2n-1)+1)}{2}\lambda_{2n-1} = \frac{1}{2}\lambda_{2n} - n\lambda_{2n-1}.$$

By the construction, we obtain $\partial((\lambda_{2n+1})) = (\lambda_{2n})$. As well,

$$\sum_{n=0}^{\infty} \frac{\lambda_{2n+1}}{n!} z^n = \frac{1}{2}\lambda_0 + (\frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_0)z + \frac{1}{2!}(\frac{1}{2}\lambda_4 - 2\lambda_3)z^3 + \cdots$$
$$+ \frac{1}{n!}(\frac{1}{2}\lambda_{2n} - n\lambda_{2n-1})z^n + \cdots$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{\lambda_{2n}}{n!} z^n - \sum_{n=1}^{\infty} \frac{\lambda_{2n-1}}{(n-1)!} z^{n-1} z,$$

Therefore,

$$(1+z)\sum_{n=0}^{\infty} \frac{\lambda_{2n+1}}{n!} z^n = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\lambda_{2n}}{n!} z^n.$$

Hence $(\lambda_{2n+1}) \in C_{et}^{\mathrm{od}}(\mathbb{C})$.

It then follows that

$$H_{et}^{\mathrm{ev}}(\mathbb{C})) = \ker(\partial_{ev})/\mathrm{im}(\partial_{od}) = C_{et}^{\mathrm{ev}}(\mathbb{C})/\ker(\sigma) \cong \mathbb{C}.$$

Proposition 2.5. *We have $H_{et}^{\mathrm{od}}(\mathbb{C}) = \{0\}$ and $H_{et}^{\mathrm{ev}}(\mathbb{C}) = \mathbb{C}$, with isomorphism given by*

$$\sigma((\varphi_{2n})) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{2n}(1, \cdots, 1) \in \mathbb{C}.$$

Definition 2.6. A cocycle sequence (φ_{2n}) or (φ_{2n+1}) is said to be normalized if we have

$$B_0\varphi_m = \frac{1}{m} A_0 B_0 \varphi_m = \frac{1}{m} B\varphi_m$$

for any respective order m.

In other words, the cochain $B_0\varphi_m$ is cyclic. Namely, $B_0\varphi_m \in C_c^{m-1}$. Then $\frac{1}{m}A_0(B_0\varphi_m) = B_0\varphi_m$.

Only the normalized cocycle sequences have a natural interpretation in terms of the universal differential algebra ΩA.

Lemma 2.7. *For any entire cocycle sequence, there is a normalized cohomologous entire cycle sequence.*

Refer to [2]. Also refer to [8].

Remark 2.8. The entire cyclic cohomology defined above and its pairing with K-theory given below is adapted to arbitrary locally convex algebras A over $\mathbb{C}$ as well as in the following.

A cochain sequence (φ_{2n}) (or (φ_{2n+1})) on A is said to be entire if for any bounded subset $B \subset A$, there exists a constant C depending on B such that

$$|\varphi_{2n}(a_0, \cdots, a_{2n})| \leq Cn!, \quad a_j \in B, n \in \mathbb{N}.$$

⋆ In particular, it then follows that if we take B as the unit ball of A (by the norm if any), then

$$\|\varphi_{2n}\| \leq Cn!.$$

Therefore,

$$\lim_{n\to\infty} \sqrt[n]{\frac{\|\varphi_{2n}\|}{n!}} \leq 1.$$

Hence, given is the spectral radius as

$$\frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{\|\varphi_{2n}\|}{n!}}} \geq 1$$

for the series $\sum_{n=0}^{\infty} \frac{\|\varphi_{2n}\|}{n!} z^n$. Is this radius infinite? Or so.

But, replacing B by $\lambda^{-1}B$ for $\lambda > 0$, we have that for any bounded subset B of A and $\lambda > 0$, there exists a constant C depending on B and λ such that

$$|\varphi_{2n}(a_0, \cdots, a_{2n})| \leq C\lambda^{2n} n!, \quad a_j \in B, n \in \mathbb{N}.$$

⋆ In the first definition above, replacing B by $\lambda^{-1}B$ for $\lambda > 0$ implies that

$$|\varphi_{2n}(\lambda^{-1}a_0, \cdots, \lambda^{-1}a_{2n})| \leq Cn!, \quad \lambda^{-1}a_j \in \lambda^{-1}B, n \in \mathbb{N}$$

with the left hand side equal to $\lambda^{-(2n+1)}|\varphi_{2n}(a_0, \cdots, a_{2n})|$. Hence, the power $2n$ of the multiple λ^{2n} should be replaced with $2n+1$.

It then follows that

$$\lim_{n\to\infty} \sqrt[n]{\frac{\|\varphi_{2n}\|}{n!}} \leq \lambda^2.$$

Taking $\lambda > 0$ to zero implies that the limit is zero. Infinite obtained is the radius of convergence!

Let A be an algebra over $\mathbb{C}$. Then A is a locally convex algebra with the finest locally convex topology (by some semi-norms or norms). Its entire cyclic cohomology theory is defined well as shown above.

Bounded subsets of A are given by convex hulls of finite subsets F.

A cochain sequence (φ_{2n}) (or (φ_{2n+1})) of A is defined to be entire if for any finite subset F of A, there exists a constant C such that

$$|\varphi_{2n}(a_0, \cdots, a_{2n})| \leq Cn!, \quad a_j \in F, n \in \mathbb{N}.$$

In the (d_1, d_2) bicomplex with $d_1 = (n+1)b$ at C^n and $d_2 = \frac{1}{n}B$ at C^n, we have

$$d_1 d_2^{-1} = S = n(n+1)bB^{-1}$$

at C^{n-1}.

⋆ If $\varphi = d_2\psi = \frac{1}{n}B\psi$ at C^{n-1}, then $\psi = d_2^{-1}\varphi = nB^{-1}\varphi$.

The pairing of the K-theory group $K_0(A)$ with $H_{et}^{\mathrm{ev}}(A)$ is given by the function f_φ by functional inserting of projections of matrix algebras over A up to

K-theory class via the canonical trace of matrix algebras over $\mathbb{C}$. Its existence comes from the growth condition such as entireness.

⋆ We certainly have the pairing given as

$$\begin{aligned}\langle [p], [\varphi] \rangle &= f_\varphi(p^\sim) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{2n}(\mathrm{tr}(p')p^\sim, \cdots, \mathrm{tr}(p')p^\sim) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{tr}(p')^{2n+1}}{n!} \varphi_{2n}(p^\sim, \cdots, p^\sim)\end{aligned}$$

for $[p] \in K_0(A)$ and $[\varphi] = [(\varphi_{2n})] \in H^{\mathrm{ev}}_{et}(A)$ with $p \in M_k(A)$ for some $k \geq$ identified with $p' \otimes p^\sim \in M_n(\mathbb{C}) \otimes A$.

⋆ We may consider $\varphi_{2n} = (n!)^2 \psi_{2n}$ degree wise. Since $\|\varphi_{2n}\| \leq C\lambda^{2n+1} n!$ with C depending on λ, then we have $\|\psi_{2n}\| \leq \frac{C\lambda^{2n+1}}{n!}$.

3 Cycles of dimension infinite

The notion of a cycle of dimension n as a starting point of cyclic cohomology theory is given by a graded differential algebra (Ω, d) with $\Omega = \oplus_{j=0}^{n} \Omega^j$, $d : \Omega_j \to \Omega_{j+1}$ of degree 1, and $d^2 = 0$ and a homogeneous, linear, closed, graded trace form $\int : \Omega^n \to \mathbb{C}$ of degree n such that

$$\int \omega_1 \omega_2 = (-1)^{k_1 k_2} \int \omega_2 \omega_1, \quad \omega_j \in \Omega^{k_j}, j = 1, 2$$

with (differential forms) $\omega_j \omega_l = \omega_j \otimes \omega_l \in \Omega^{k_j} \otimes \Omega^{k_l}$ of dimension or degree $k_j + k_l = n$, and $\int d\omega = 0$ for any $\omega \in \Omega^{n-1}$.

In order to handle the infinite dimensional case, the conditions above is replaced by the inhomogeneous condition as

$$\int (\omega_1 \omega_2 - (-1)^{k_1 k_2} \omega_2 \omega_1) = (-1)^{k_1} \int d\omega_1 d\omega_2$$

on forms of degrees $k_1 + k_2$ and $(k_1 + 1) + (k_2 + 1)$ respectively, which may not be zero.

A linear form μ on a differential graded algebra (Ω, d) with $\Omega = \oplus_{j=0}^{\infty} \Omega_j$ is said to be even (or odd) if $\mu(\omega) = 0$ for any $\omega \in \Omega^k$ with degree k odd (or even, respectively).

Proposition 3.1. ([2]). *Let A be an algebra over $\mathbb{C}$, (Ω, d) a graded differential algebra such that $A = \Omega_0$, and μ an even linear, closed cycle form on Ω (over A) satisfying the inhomogeneous condition given above.*

Then a normalized cocycle sequence (φ_{2n}) in the (d_1, d_2) bicomplex $C^{n,m} = C^{n-m}(A, A^*)$ is defined by

$$\varphi_{2n}(a_0, a_1, \cdots, a_{2n}) = (-1)^n (2n-1)!! \mu(a_0 da_1 \cdots da_{2n})$$

for $a_j \in A$, $0 \le j \le 2n$, with $(2n-1)!! = \Pi_{j=1}^n (2n+1-2j)$.

If μ is odd, then a normalized cocycle sequence (φ_{2n+1}) in the (d_1, d_2) bicomplex is defined by

$$\varphi_{2n+1}(a_0, a_1, \cdots, a_{2n+1}) = (-1)^n (2n)!! \mu(a_0 da_1 \cdots da_{2n+1})$$

for $a_j \in A$, $0 \le j \le 2n+1$, with $(2n)!! = \Pi_{j=1}^n (2n+2-2j)$.

Conversely, for a normalized cocycle sequence (φ_{2n}) and (φ_{2n+1}) in the (d_1, d_2) bicomplex, even and odd, linear closed, inhomogeneous cycle forms μ on the universal differential algebra $\Omega^* A$ are defined respectively by

$$\begin{aligned}
&\mu((a_0 + \lambda 1) da_1 \cdots da_{2n}) \\
&= \frac{(-1)^n}{(2n-1)!!} \{\varphi_{2n}(a_0, \cdots, a_{2n}) + \lambda (B_0 \varphi_{2n})(a_1, \cdots, a_{2n})\}, \\
&\mu((a_0 + \lambda 1) da_1 \cdots da_{2n+1}) \\
&= \frac{(-1)^n}{(2n)!!} \{\varphi_{2n+1}(a_0, \cdots, a_{2n+1}) + \lambda (B_0 \varphi_{2n+1})(a_1, \cdots, a_{2n+1})\}
\end{aligned}$$

respectively, with $\lambda \in \mathbb{C}$.

Proof. $\star$ As for the first half, we have

$$\begin{aligned}
&(B_0 \varphi_{2n})(a_0, \cdots, a_{2n-1}) \\
&= \varphi_{2n}(1, a_0, \cdots, a_{2n-1}) - \varphi_{2n}(a_0, \cdots, a_{2n-1}, 1) \\
&= (-1)^n (2n-1)!! \{\mu(da_0 \cdots da_{2n-1}) - \mu(a_0 da_1 \cdots da_{2n-1} d1)\} \\
&= (-1)^n (2n-1)!! \mu(da_0 \cdots da_{2n-1}) = \varphi_{2n}(1, a_0, \cdots, a_{2n-1}).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&A_0 (B_0 \varphi_{2n})(a_0, \cdots, a_{2n-1}) \\
&= \sum_{j=0}^{2n-1} (-1)^{(2n-2)j} \varphi_{2n}(1, a_j, a_{j+1}, \cdots, a_{j-1}) \\
&= \sum_{j=0}^{2n-1} (-1)^{(2n-2)j} (-1)^{(2n)j} \varphi_{2n}(1, a_0, \cdots, a_{2n-1}) \\
&= 2n \varphi_{2n}(1, a_0, \cdots, a_{2n-1}) = 2n (B_0 \varphi_{2n})(a_0, \cdots, a_{2n-1}).
\end{aligned}$$

This means that $B_0 \varphi_{2n}$ is cyclic, so that (φ_{2n}) is normalized by definition.

As for the converse, we check the even case. Let $\psi_{2n} = \frac{(-1)^n}{(2n-1)!!} \varphi_{2n}$. Then $B_0 \psi_{2n}$ is cyclic and we have $B_0 \psi_{2n} = b \psi_{2n-2}$ (why?) for any n.

$\star$ Since (φ_{2n}) is normalized, $B_0 \varphi_{2n}$ is cyclic. Then so is $B_0 \psi_{2n}$.

$$
\begin{aligned}
&(B_0\varphi_{2n})(a_0,\cdots,a_{2n-1})\\
&=\varphi_{2n}(1,a_0,\cdots,a_{2n-1})-\varphi_{2n}(a_0,\cdots,a_{2n-1},1).\\
&(b\varphi_{2n-2})(a_0,\cdots,a_{2n-1})\\
&=\sum_{j=0}^{2n-2}(-1)^j\varphi_{2n-2}(a_0,\cdots,a_ja_{j+1},\cdots,a_{2n-1})\\
&\quad+(-1)^{2n-1}\varphi_{2n-2}(a_{2n-1}a_0,\cdots,a_{2n-2}).
\end{aligned}
$$

We have $B_0\psi_{2n}=b\psi_{2n-2}$ for any n.

$\star$ The reason certainly comes from $b(B_0\psi_{2n})=0$ and cohomology triviality as checked before. Or it may be included in the definition from the first.

It is shown that da for any $a\in A$ belongs to the centralizer of the functional μ defined so above.

It follows from the cyclic of $B_0\psi_{2n}$ that

$$\mu(da(da_1\cdots da_{2n-1}))=(-1)^{2n-1}\mu((da_1\cdots da_{2n-1})da).$$

$\star$ Note that in this case, we have $a_0=0$ so that $\varphi_{2n}(0,a,a_1,\cdots,a_{2n-1})=0$.

Since we have $B_0\psi_{2n}=b\psi_{2n-2}$, then $bB_0\psi_{2n}=0$. Also, $B_0b\psi_{2n}=0$ since $b\psi_{2n}$ is cyclic.

$\star$ We have $0=bB_0\psi_{2n}=bB\psi_{2n}=-Bb\psi_{2n}=-B_0b\psi_{2n}$.

Let $D=B_0b+b'B_0$. It then follows that

$$
\begin{aligned}
&\psi_{2n}(a_0,\cdots,a_{2n-1},a)-(-1)^{2n}\psi_{2n}(a,a_0,\cdots,a_{2n-1})\\
&\quad+(-1)^{2n}B_0\psi_{2n}(aa_0,a_1,\cdots,a_{2n-1})=0.
\end{aligned}
$$

That is

$$\mu(da(a_0da_1\cdots da_{2n-1}))=(-1)^{2n-1}\mu((a_0da_1\cdots da_{2n-1})da).$$

$\star$ Note that since $bB_0\psi_{2n}=0$, we have

$$b'B_0\psi_{2n}(a_0,\cdots,a_{2n-1},a)=(-1)^{2n-1}(B_0\psi_{2n})(aa_0,\cdots,a_{2n-1}).$$

Also,

$$
\begin{aligned}
&\mu((a_0da_1\cdots da_{2n-1})da)=\frac{(-1)^n}{(2n-1)!!}\varphi_{2n}(a_0,a_1,\cdots,a_{2n-1},a)\\
&=\psi_{2n}(a_0,a_1,\cdots,a_{2n-1},a).
\end{aligned}
$$

As well, $d(aa_0)=da(a_0)+ada_0$. We then obtain

$$
\begin{aligned}
&\mu(d(aa_0)da_1\cdots da_{2n-1})=(B_0\psi_{2n})(aa_0,a_1,\cdots,a_{2n})\\
&=\mu(da(a_0)da_1\cdots da_{2n-1})+\psi_{2n}(a,a_0,\cdots,a_{2n-1}).
\end{aligned}
$$

Namely, da commutes with $a_0da_1\cdots ad_{2n-1}$ with respect to μ up to sign $(-1)^{2n-1}$.

It follows that $d\omega$ such as $db_1db_2\cdots db_k$ belongs to the centralizer with respect to μ, inductively.

It is shown that $\mu(a\omega-\omega a)=\mu(dad\omega)$ for any $a\in A$.

With $\omega=a_0da_1\cdots da_{2n}$, we have

$$\begin{aligned}&\mu(\omega a)=\mu(a_0(da_1\cdots da_{2n})a)\\&=\psi_{2n}(a_0,a_1,\cdots,a_{2n-1},a_{2n}a)-\psi_{2n}(a_0,a_1,\cdots,a_{2n-1}a_{2n},a)+\cdots\\&+(-1)^j\psi_{2n}(a_0,\cdots,a_{2n-j}a_{2n-j+1},\cdots,a)+\cdots+(-1)^{2n}\psi_{2n}(a_0a_1,\cdots,a).\end{aligned}$$

Thus,

$$\begin{aligned}&\mu(\omega a-a\omega)=\mu(\omega a)-\mu(a\omega)=b\psi_{2n}(a_0,a_1,\cdots,a_{2n},a)\\&=B_0\psi_{2n+2}(a_0,\cdots,a_{2n},a)=\mu(d\omega da)=-\mu(dad\omega).\end{aligned}$$

$\star$ Note that $d(a_{2n}a)=(da_{2n})a+a_{2n}da$. It then follows that

$$\begin{aligned}&\mu(a_0(da_1\cdots da_{2n})a)=\mu((a_0da_1\cdots da_{2n-1})(da_{2n})a)\\&=\mu(a_0da_1\cdots da_{2n-1}d(a_{2n}a))-\mu(a_0da_1\cdots(da_{2n-1})a_{2n}da)\\&=\psi_{2n}(a_0,a_1,\cdots,a_{2n-1},a_{2n}a)-\mu(a_0da_1\cdots(da_{2n-1})a_{2n}da).\end{aligned}$$

Next consider that $d(a_{2n-1}a_{2n})=(da_{2n-1})a_{2n}+a_{2n-1}da_{2n}$. Hence

$$\begin{aligned}&\mu(a_0da_1\cdots(da_{2n-1})a_{2n}da)\\&=\mu(a_0da_1\cdots d(a_{2n-1}a_{2n})da)-\mu(a_0da_1\cdots a_{2n-1}(da_{2n})da)\\&=\psi_{2n}(a_0,a_1,\cdots,a_{2n-1}a_{2n},a)-\mu(a_0da_1\cdots a_{2n-1}(da_{2n})da).\end{aligned}$$

Inductively, we need to consider the derivation equations to obtain the equality for $\mu(\omega a)$.

Note also that

$$\begin{aligned}&B_0\psi_{2n+2}(a_0,\cdots,a_{2n},a)=0+1B_0\psi_{2n+2}(a_0,\cdots,a_{2n},a)\\&=\psi_{2n}(0,a_0,\cdots,a_{2n},a)+1B_0\psi_{2n+2}(a_0,\cdots,a_{2n},a)\\&=\mu(da_0\cdots da_{2n}da)=\mu(d\omega da).\end{aligned}$$

That's it!

Finally, we need to check the following for $\omega_1=ad\omega$ of degree k_1 with $a\in A$ and ω_2 of degree k_2.

Since $d\omega$ belongs to the centralizer with respect to μ, we have

$$\begin{aligned}&\mu(\omega_1\omega_2-(-1)^{k_1k_2}\omega_2\omega_1)=\mu(a(d\omega)\omega_2-(-1)^{k_1k_2}\omega_2ad\omega)\\&=\mu(a(d\omega)\omega_2-(-1)^{k_1k_2^2(k_1+k_2-1)}a(d\omega)\omega_2)\\&=\mu(a(d\omega)\omega_2-(-1)^{k_1(k_2-1)}a(d\omega)\omega_2)\\&=\mu(a(d\omega)\omega_2-(-1)^{k_1(k_2-1)(k_1+k_2)}(d\omega)\omega_2)a)\\&=\mu(a(d\omega)\omega_2-(-1)^{k_2+k_1-k_1k_2}(d\omega)\omega_2)a).\end{aligned}$$

Is this correct?

It follows that

$$\mu(\omega_1\omega_2 - (-1)^{k_1k_2}\omega_2\omega_1) = \mu(dad(d\omega\omega_2)) = (-1)^{k_1}\mu(d\omega_1 d\omega_2).$$

$\star$ Note that

$$\mu(a(d\omega)\omega_2 - (d\omega)\omega_2)a) = \mu(dad((d\omega)\omega_2))$$

with

$$\mu(dad((d\omega)\omega_2)) = \mu(dad\omega d\omega_2) = \mu(d\omega_1 d\omega_2).$$

Note as well that $k_1 + k_2$ as well as k_1k_2 may be even because μ is even. It seems that the factor $(-1)^{k_1}$ may be removed from the formula in the even case. The factor may represent the odd case degree if involved.

Conversely, as for the first half, the proof above implies that any functional μ on $\Omega^* A$ of even or add, satisfying the inhomogeneous condition given like above defines even or odd, normalized cochain sequences (ψ_{2n}) and (ψ_{2n+1}) such that $b\psi_m = B_0\psi_{m+2}$ for any m, given by

$$\psi_m(a_0, \cdots, a_m) = \mu(a_0 da_1 \cdots da_m).$$

By universality of the differential graded algebra $\Omega^* A$ over A we obtain the first half. $\square$

Let A be a Banach algebra. Consider the norms for the universal differential algebra $\Omega^* A$ defined by, with $r > 0$,

$$\| \oplus_{k=0}^{\infty} \omega_k \|_r = \sum_{k=0}^{\infty} r^k \|\omega_k\|_{pr}$$

where $\|\omega_k\|_{pr}$ is the projective tensor product norm on

$$\Omega^k = \Omega^k(A) = \otimes_A^k (A^\sim \otimes_{\mathbb{C}} A) \cong A^\sim \otimes (\otimes^n A)$$

with $A^\sim = A \oplus \mathbb{C}1$ the unitization of A by 1 (cf. [1]).

Theorem 3.2. *There is a canonical bijection between normalized entire cocycle sequences on A and linear forms on $\Omega^* A$, of even and odd respectively, satisfying the inhomogeneous condition, given above, and continuous for all the norms $\| \cdot \|_r$.*

The natural topology on $\Omega^* A$ provided by the statement above is not the projective limit $\varprojlim(\Omega^* A, \| \cdot \|_r)$ of the normed spaces $(\Omega^* A, \| \cdot \|_r)$ given by the normes $\| \cdot \|_r$ as $r \to \infty$.

That is the inductive limit for $r \to 0$. Namely,

$$\Omega^* A = \varinjlim(\Omega^* A, \| \cdot \|_r).$$

$\star$ For $0 < r_1 < r_2$, we have $\|\omega\|_{r_1} \leq \|\omega\|_{r_2}$ for $\omega \in \Omega^* A$. There is a continuous identity map from the normed space $(\Omega^* A, \|\cdot\|_{r_2})$ to $(\Omega^* A, \|\cdot\|_{r_1})$. Namely,

$$\varprojlim(\Omega^* A, \|\cdot\|_r) \xrightarrow{q_{r_2}} (\Omega^* A, \|\cdot\|_{r_2}) \to (\Omega^* A, \|\cdot\|_{r_1}) \xrightarrow{i_{r_1}} \varinjlim(\Omega^* A, \|\cdot\|_r)$$

with q_{r_2} the quotient map by projectiveness and i_{r_1} the injective map by inductiveness.

For each $r > 0$, the completion of $\Omega^* A$ by the norm $\|\cdot\|_r$ is a Banach algebra, denoted by $\Omega_r(A)$

There is a natural homomorphism from $\Omega_r(A)$ to $\Omega_{r'}(A)$ for $0 < r' < r$, which is the identity on $\Omega^* A$ and is norm decreasing.

Let $\Omega_\varepsilon(A) = \varinjlim \Omega_r(A)$ for $r > 0$.

Then $\Omega_\varepsilon(A)$ is a locally convex algebra with the continuous homomorphism from $\Omega_\varepsilon(A)$ to $A^\sim$ given by the augmentation as sending $\oplus_{k=0}^\infty \omega_k$ to ω_0 of $\Omega^* A$.

Proposition 3.3. *A linear form μ on $\Omega^* A$ is continuous for all the norms $\|\cdot\|_r$ for $r > 0$ if and only if that on $\Omega_\varepsilon(A)$ is continuous.*

There is the following short exact sequence of Banach algebras

$$0 \to J = \ker(\mathrm{aug}) \longrightarrow \Omega_\varepsilon(A) \xrightarrow{\mathrm{aug}} A^\sim \to 0$$

by augmentation aug. *Then any element $\omega \in J$ the kernel is quasi-nilpotent, i.e. $\lambda 1 - \omega$ is invertible in $\Omega_\varepsilon(A)$ for any λ nonzero.*

Proof. $\star$ Note that

$$\begin{tikzcd} \Omega_r \arrow[r, "\mu"] \arrow[d, "i_r"'] & \mathbb{C} \arrow[d, equal] \\ \Omega_\varepsilon(A) \arrow[r, "\mu"] & \mathbb{C}. \end{tikzcd}$$

Let $\omega \in \Omega_r$ for some $r > 0$. If $\omega \in J$, then $\omega = \sum_{k=1}^\infty \omega_k$ with $\omega_k \in \Omega^k A$ and $\|\omega\|_r = \sum_{n=1}^\infty r^n \|\omega_n\|_{pr} < \infty$.

Replacing r by a smaller $0 < r' < r$ implies that $\|\lambda^{-1}\omega\|_{r'} < 1$. It then follows that $1 - \lambda^{-1}\omega$ is invertible in $\Omega_{r'}$.

$\star$ Let A be a unital Banach algebra and $a \in A$ with $\|a\| < 1$. Then $1 - a$ is invertible in A with inverse given by $\sum_{n=0}^\infty a^n$ (cf. [13]).

Indeed, we have

$$\|\sum_{n=0}^\infty a^n\| \leq \sum_{n=0}^\infty \|a^n\| \leq \sum_{n=0}^\infty \|a\|^n = \frac{1}{1 - \|a\|}.$$

We also have

$$(1-a)\sum_{k=0}^n a^k = \sum_{k=0}^n a^k(1-a) = 1 - a^{n+1}$$

Taking the limit as $n \to \infty$ implies the statement ($\star$) above. $\square$

⋆ Certainly, it looks like that those inductive limits are just $A^{\sim}$. Then J is trivial zero. Is this right?

$\Omega_\varepsilon(A)$ is a $\mathbb{Z}_2$-graded differential algebra, since the differential d of $\Omega^* A$ is continuous for all the norms $\|\cdot\|_r$. The range of d is contained in the ideal J.

4 Traces

Lemma 4.1. ([4], cf. [9]). *Let (Ω, d) be a differential $\mathbb{Z}_2$-graded algebra and $\lambda \in \mathbb{C}$. An associative bilinear product on $\Omega = \Omega^{\mathrm{ev}} \oplus \Omega^{\mathrm{od}}$ is defined as*

$$\begin{aligned}\omega_1 \cdot_\lambda \omega_2 &= \omega_1\omega_2 + \lambda\omega_1 d\omega_2, \quad \omega_1 \in \Omega^{\mathrm{od}}, \omega_2 \in \Omega,\\ \omega_1 \cdot_\lambda \omega_2 &= \omega_1\omega_2, \quad \omega_1 \in \Omega^{\mathrm{ev}}, \omega_2 \in \Omega.\end{aligned}$$

Proof. For $\omega_1, \omega_2 \in \Omega^{\mathrm{od}}$ and $\omega_3 \in \Omega$, we compute

$$\begin{aligned}\omega_1 \cdot_\lambda (\omega_2 \cdot_\lambda \omega_3) &= \omega_1 \cdot_\lambda (\omega_2\omega_3 + \lambda\omega_2 d\omega_3)\\ &= \omega_1(\omega_2\omega_3) + \lambda\omega_1\omega_2 d\omega_3 + \lambda\omega_1 d(\omega_2\omega_3) + \lambda^2\omega_1 d(\omega_2 d\omega_3)\\ (\omega_1 \cdot_\lambda \omega_2) \cdot_\lambda \omega_3 &= (\omega_1\omega_2 + \lambda\omega_1 d\omega_2) \cdot_\lambda \omega_3\\ &= (\omega_1\omega_2)\omega_3 + \lambda\omega_1 d\omega_2(\omega_3) + \lambda(\omega_1\omega_2 + \lambda\omega_1 d\omega_2)d\omega_3.\end{aligned}$$

But it seems to have both lines above not equal in general. □

The algebra corresponding to $\lambda = 0$ is Ω.

The algebras for $\lambda \neq 0$ are independent.

Possibly, we may consider non-associative algebras like.

The $\mathbb{Z}_2$-grading of Ω is given by the involutive automorphism defined as $\sigma_0(\omega) = (-1)^{\deg\omega}\omega$. The grading is extended to a $\mathbb{Z}_2$-grading of the deformed algebra given by

$$\sigma_\lambda(\omega) = (-1)^{\deg\omega}(\omega - \lambda d\omega)$$

which is an involutive automorphism of the deformed product.

Proof. We have

$$\sigma_0^2(\omega) = (-1)^{\deg((-1)^{\deg\omega}\omega)}(-1)^{\deg\omega}\omega = \omega.$$

We also have

$$\begin{aligned}\sigma_\lambda^2(\omega) &= \sigma_\lambda((-1)^{\deg\omega}(\omega - \lambda d\omega))\\ &= \omega - \lambda d\omega + \sigma_\lambda((-1)^{\deg d\omega}\lambda d\omega)\\ &= \omega - \lambda d\omega + \lambda d\omega - \lambda^2 d^2\omega = \omega!\end{aligned}$$

□

Lemma 4.2. ([2]). *Let $(\Omega, d, \cdot_\lambda)$ be as above with the $\mathbb{Z}_2$-grading σ_λ.*

Any odd linear form τ on (Ω, d) corresponds to $\tau^\sim$ as its restriction to Ω^{od} extended to 0 on Ω^{ev}.

There is a canonical bijection between odd traces on $(\Omega, d, \cdot_\lambda, \sigma_\lambda)$ as $\tau^\sim$ and odd linear forms τ on (Ω, d) such that

$$\tau^\sim(\omega_1\omega_2 - (-1)^{k_1k_2}\omega_2\omega_1) = \frac{1}{2}\lambda^2(-1)^{k_1}\tau^\sim(d\omega_1 d\omega_2)$$

for $\omega_j \in \Omega^{k_j}$.

Proof. Let τ be an odd trace on $(\Omega, \cdot_\lambda)$, corresponding to the linear form on Ω. Assume that $\omega_1 \in \Omega^{\mathrm{od}}$, $\omega_2 \in \Omega^{\mathrm{ev}}$. Then we have $\tau(\omega_1 \cdot_\lambda \omega_2) = \tau(\omega_2 \cdot_\lambda \omega_1)$. For $\omega = \omega_1 d\omega_2$ even, we have

$$\tau(\omega) = \frac{1}{2}\tau(\omega - \sigma_\lambda(\omega)).$$

Indeed,

$$\begin{aligned}
&\tau(\omega_1 d\omega_2 - \sigma_\lambda(\omega_1 d\omega_2)) \\
&= \tau(\omega_1 d\omega_2 - \omega_1 d\omega_2 + \lambda d(\omega_1 d\omega_2)) \\
&= \lambda\tau(d\omega)
\end{aligned}$$

because the degree of $d(\omega_1 d\omega_2)$ is odd. It seems that the formula above is wrong and is corrected so.

It then follows that

$$\begin{aligned}
&\tau(\omega_1 \cdot_\lambda \omega_2 - \lambda\omega) - \tau(\omega_2 \cdot_\lambda \omega_1) \\
&= \tau(\omega_1\omega_2 + \lambda\omega_1 d\omega_2 - \lambda\omega_1 d\omega_2) - \tau(\omega_2\omega_1) \\
&= \tau(\omega_1\omega_2 - (-1)^{k_1k_2}\omega_2\omega_1)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\tau(\omega_1 \cdot_\lambda \omega_2 - \lambda\omega) - \tau(\omega_2 \cdot_\lambda \omega_1) \\
&= -\lambda\tau(\omega) = -\lambda\tau(\lambda d\omega + \sigma_\lambda(\omega)) \\
&= -\lambda\tau(\lambda d\omega + \omega - \lambda d\omega) = -\lambda\tau(\omega).
\end{aligned}$$

Hence the condition in the statement may be corrected so.

We may skip the second half left. $\square$

There is an analogue in the even case.

Let $E_\lambda = (\Omega, \cdot_\lambda) \rtimes_{\sigma_\lambda} \mathbb{Z}_2$ denote the crossed product of $(\Omega, \cdot_\lambda)$ by $\mathbb{Z}_2$ by the $\mathbb{Z}_2$-grading autumorphism σ_λ.

There is the dual $\mathbb{Z}_2$-grading $\hat{\sigma}_\lambda$ of E_λ defined by $\hat{\sigma}_\lambda(\omega) = \omega$ for $\omega \in \Omega$ and $\hat{\sigma}_\lambda(F) = -F$, where F is the element of E_λ associated to the generator of $\mathbb{Z}_2$ such that $F^2 = 1$ the identity.

Lemma 4.3. ([2]). *Any odd linear form τ on E_λ corresponds to $\tau^\sim$ as the restriction of $\tau(F\omega)$ to $\omega \in \Omega^{\mathrm{ev}}$ extended by 0 on Ω^{od}.*

There is a canonical bijection between odd traces τ on E_λ and even linear forms $\tau^\sim$ on (Ω, d) satisfying the same condition in the lemma above.

Let $Q_\varepsilon(A)$ denote the $\mathbb{Z}_2$-graded algebra obtained as $(\Omega_\varepsilon(A), d, \cdot_\lambda)$

Let $Q_\varepsilon^\wedge(A)$ denote the crossed product of $Q_\varepsilon(A)$ by the $\mathbb{Z}_2$-grading σ such that $\sigma^2 = 1$ the identity.

Both $Q_\varepsilon(A)$ and $Q_\varepsilon^\wedge(A)$ are locally convex algebras.

There is a canonical bijection between continuous odd traces on $Q_\varepsilon(A)$ (and $Q_\varepsilon^\wedge(A)$ respectively) and normalized odd (and even) entire cocycle sequences on the Banach algebra A.

Let QA denote the $\mathbb{Z}_2$-graded algebra obtained as $(\Omega A, d, \cdot_{\sqrt{2}})$.

Let $Q^\wedge A$ denote the crossed product of QA by the $\mathbb{Z}_2$-grading σ.

Proposition 4.4. ([4]). *Let A be an algebra over $\mathbb{C}$.*

(a) The pair (ρ_1, ρ_2) of homomorphisms from $A^\sim$ to QA are defined as

$$\rho_1(a) = a \in \Omega^0 A \quad \textit{and} \quad \rho_2(a) = a - \sqrt{2}da \in \Omega^0 A \oplus \Omega^1 A$$

for $a \in A^\sim$, giving an isomorphism of the free product $A^\sim *_{\mathbb{C}} A^\sim$ with QA. The $\mathbb{Z}_2$-grading σ of QA is the automorphism exchanging $\rho_1(a)$ with $\rho_2(a)$ for $a \in A$.

(b) The pair $(\rho_1^\wedge, \rho_2^\wedge)$ of homomorphisms from $A^\sim$ and $\mathbb{Z}_2$ to $Q^\wedge A$ are defined as

$$\rho_1^\wedge(a) = a \in \Omega^0 A \quad \textit{and} \quad \rho_2^\wedge(n) = F^n$$

for $a \in A^\sim$ and $n \in \mathbb{Z}_2$, giving an isomorphism of $A^\sim *_{\mathbb{C}} \mathbb{C}[\mathbb{Z}_2]$ with $Q^\wedge(A)$. The $\mathbb{Z}_2$-grading $\sigma^\wedge$ of $Q^\wedge A$ satisfies $\sigma^\wedge \circ \rho_1^\wedge = \rho_1^\wedge$ and $\sigma^\wedge(F) = -F$.

Proof. For $a, b \in A^\sim$,

$$\begin{aligned}
\rho_2(ab) &= ab - \sqrt{2}d(ab) \\
\rho_2(a) *_{\sqrt{2}} \rho_2(b) &= \rho_2(a)\rho_2(b) + \sqrt{2}\rho_2(a)d\rho_2(b) \\
&= (a - \sqrt{2}da)(b - \sqrt{2}db) + \sqrt{2}(a - \sqrt{2}da)d(b - \sqrt{2}db) \\
&= ab - \sqrt{2}(adb + (da)b) + 2dadb + \sqrt{2}(adb - \sqrt{2}dadb) \\
&= ab - \sqrt{2}(da)b.
\end{aligned}$$

It seems that ρ_2 is not homo. □

For $a \in A^\sim$, the difference qa is defined by

$$qa = \rho_1(a) - \rho_2(a) = \sqrt{2}da \in QA.$$

We may identify $A^\sim$ with $\rho_1(A^\sim)$ in QA.

Proposition 4.5. *Let τ be an odd trace on QA such that $\tau(qa) = \sqrt{2}\tau(da) = 0$ for any $a \in A$. Then a normalized odd cocycle sequece (φ_{2n+1}) in the (d_1, d_2) bicomplex is defined as*

$$\begin{aligned}\varphi(a_0, \cdots, a_{2n+1}) &= (-1)^n n! \tau(a_0 q a_1 \cdots q a_{2n+1}) \quad a_j \in A \\ &= (-1)^n n! 2^n \sqrt{2} \tau(a_0 da_1 \cdots da_{2n+1}).\end{aligned}$$

Let τ be an odd trace on $Q^\wedge(A)$. Then a normalized even cocycle sequence (φ_{2n}) in the (d_1, d_2) bicomplex is defined as

$$\varphi_{2n}(a_0, \cdots, a_{2n}) = \Gamma(n + \frac{1}{2})\tau(Fa_0[F, a_1] \cdots [F, a_{2n}]).$$

Proof. Note that

$$\begin{aligned}Fa_0[F, a_1] &= Fa_0(Fa_1 - a_1F) = Fa_0Fa_1 - Fa_0a_1F \\ &= (Fa_0F)a_1 - Fa_0a_1F\end{aligned}$$

which is viewed as belonging to A.

Also $[F, a_2] = Fa_2 - a_2F$ is in $A \rtimes \mathbb{Z}_2$. □

The crossed product $Q^\wedge(A) \rtimes_{\sigma'} \mathbb{Z}_2$ of $Q^\wedge(A)$ by its $\mathbb{Z}_2$-grading σ' such that $\sigma'(F) = -F$ and σ' is the identity on $Q^\wedge(A)$ is defined so and is then isomorphic to $Q(A) \otimes M_2(\mathbb{C}) \cong M_2(Q(A))$, denoted as $(Q^\wedge)^\wedge(A)$.

This is generated by a copy of $A^\sim$ and a pair (F, γ) of elements such that

$$[\gamma, a] = \gamma a - a\gamma = 0, \quad a \in A, \quad \gamma F = -F\gamma, \quad \text{and} \quad \gamma^2 = F^2 = 1.$$

$\star$ Note that $\sigma'(F) = \gamma F \gamma = -F$. Also, $\gamma a \gamma = a$. As well, $\sigma(a) = FaF$, which may not be a.

Those relations represents $(Q^\wedge)^\wedge(A)$.

Corollary 4.6. *Let τ be a trace on $(Q^\wedge)^\wedge(A)$ such that $\tau(\gamma a) = 0$ for $a \in A$. A normalized cocycle sequence in the (d_1, d_2) bicomplex is defined by*

$$\varphi_{2n+1}(a_0, \cdots, a_{2n+1}) = n!\tau(\gamma F a_0[F, a_1] \cdots [F, a_{2n+1}]), \quad a_j \in A.$$

Such an explicit construction of cocycle sequences of ([4]) by the traces is viewed as the translation of the triviality of the first spectral sequence of the (b, B) bicomplex.

Let A be a Banach algebra. The algebra QA defined as above has the locally convex topology inherited from the inductive limit topology of ΩA.

Let $Q_\varepsilon(A)$ correspond in the same way to $\Omega_\varepsilon(A)$.

Then QA is a subalgebra of $Q_\varepsilon A$, and its topology is the restriction of that of $Q_\varepsilon A$.

There is a bijective correspondence between continuous odd traces on both algebras QA and $Q_\varepsilon(A)$ by restriction and extension by continuity.

As established in the proposition and the corollary above, we obtain

Theorem 4.7. *Let A be a Banach algebra. There is a canonical bijection between continuous odd traces on $Q_\varepsilon^\wedge(A)$ and on $(Q_\varepsilon^\wedge)^\wedge(A)$ vanishing on γA and entire normalized even and odd cocycle sequences on A, respectively.*

As well, QA or $Q_\varepsilon A$ and their dual crossed products can be used instead.

The structure of the locally convex algebra $Q_\varepsilon A$ is similar to that of $\Omega_\varepsilon A$ as described by continuous linear forms with respect to the norms and the kernel of the augmentation.

The augmentation morphism $\varepsilon : Q_\varepsilon A \to A^\sim$ is given as the morphism $\varepsilon : \Omega_\varepsilon A \to A^\sim$. This homomorphism is insensitive to the deformation product.

We have $\varepsilon \circ d = 0$. Also, $\varepsilon \circ q = 0$.

Proposition 4.8. *Let $J = \ker(\varepsilon)$, where $\varepsilon : Q_\varepsilon A \to A^\sim$. Any element x of J is quasi-nilpotent in the sense that $\lambda 1 - x$ is invertible in $Q_\varepsilon A$ for any nonzero $\lambda \in \mathbb{C}$.*

Namely, $Q_\varepsilon A$ is a quasi-nilpotent extension by $A^\sim$.

5 Pairing with K-theory groups

Lemma 5.1. *Let (φ_{2n}) be a normalized entire cocycle sequence even on a Banach algebra A. If the sequence belongs to the image under the boundary ∂ contained in $C_{\rm et}^{\rm ev}$, then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{2n}(p, \cdots, p) = 0$$

for any idempotent $p \in A$.

Proof. Let $(\psi_{2n+1}) \in C_{\rm et}^{\rm od}$ such that $\partial(\psi_{2n+1}) = (\varphi_{2n})$. For each n, we have

$$\varphi_{2n} = 2nb\psi_{2n-1} + \frac{1}{2n+1} B\psi_{2n+1}.$$

$\star$ Recall that $\partial = d_1 + d_2$ so that

$$\partial(\psi_{2n-1}, \psi_{2n+1}) = ((2n-1)+1)b\psi_{2n-1} + \frac{1}{2n+1} B\psi_{2n+1}.$$

Since (φ_{2n}) is normalized, we have $B_0\varphi_{2n} \in cC^{2n}$ is cyclic, so that $B_0\varphi_{2n} = B\varphi_{2n}$, and

$$B_0 b\psi_{2n-1} = \frac{1}{2n} B_0 \varphi_{2n}$$

is cyclic for any n.

$\star$ This follows from multiplying the equation above with B_0 not $B = A_0 B_0$ from the left, if $B_0 \circ B = 0$. It seems that the last equation is not equivalent to $B^2 = 0$. That follows from multiplying with B. The reason is that

$$Bb\psi_{2n-1} = -bB\psi_{2n-1} = -bB_0\psi_{2n-1}$$

but which should be equal to $B_0 b\psi_{2n-1}$. Or just $Bb\psi_{2n-1} = B_0 b\psi_{2n-1}$ if normalization is preserved by the boundary b.

$\star$ Since (ψ_{2n+1}) is also normalized, we let

$$\alpha_n = (B_0\psi_{2n+1})(p, \cdots, p) = \frac{1}{2n+1} B\psi_{2n+1}(p, \cdots, p).$$

Since $p^2 = p$, we have

$$\begin{aligned}\alpha_n &= (b' B_0 \psi_{2n+1})(p, \cdots, p) \\ &= ((D - B_0 b)\psi_{2n+1})(p, \cdots, p) \\ &= (D\psi_{2n+1})(p, \cdots, p) = 2\psi_{2n+1}(p, \cdots, p).\end{aligned}$$

$\star$ Note that for the $((2n+1)+1)$-tuple $(p, \cdots, p)$,

$$(b' B_0\psi_{2n+1})(p, \cdots, p) = \sum_{j=0}^{2n} (-1)^j (B_0\psi_{2n+1})(p, \cdots, p^2, \cdots, p)$$
$$= (B_0\psi_{2n+1})(p, \cdots, p, \cdots, p).$$

Also, $D = B_0 b + b' B_0$. As well, multiplying that equation with b implies that

$$0 = 0 + \frac{1}{2n+1} bB\psi_{2n+1}$$

and hence $Bb\psi_{2n+1} = 0$, so that $B_0 b\psi_{2n+1} = 0$ by the reason mentioned above. Since ψ_{2n+1} is cyclic and the signature of odds is -1, we obtain

$$D\psi_{2n+1} = \psi_{2n+1} - (-1)A_0\psi_{2n+1} = 2\psi_{2n+1}.$$

Also.

$$(b\psi_{2n+1})(p, \cdots, p) = \psi_{2n+1}(p, \cdots, p) = \frac{1}{2}\alpha_n.$$

$\star$ Note that for the $((2n+2)+1)$-tuple $(p, \cdots, p)$,

$$(b\psi_{2n+1})(p, \cdots, p) = \sum_{j=0}^{2n+1} (-1)^j \psi_{2n+1}(p, \cdots, p^2, \cdots, p)$$
$$+ (-1)^{2n+2}\psi_{2n+1}(p^2, p, \cdots, p) = \psi_{2n+1}(p, \cdots, p).$$

Thus,

$$\varphi_{2n}(p, \cdots, p) = 2n(b\psi_{2n-1})(p, \cdots, p) + \frac{1}{2n+1}(B\psi_{2n+1})(p, \cdots, p)$$
$$= 2n\frac{1}{2}\alpha_{n-1} + \alpha_n.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{2n}(p, \cdots, p) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (n\alpha_{n-1} + \alpha_n)$$
$$= -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \alpha_{n-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha_n = 0.$$

$\star$ Note that $\|p\| = \|p^2\| \leq \|p\|^2$ so that $1 \leq \|p\|$ if $p \neq 0$.
Also, the spectral radius of p is computed by

$$r(p) = \lim_{n\to\infty} \|p^n\|^{\frac{1}{n}} = 1.$$

The Gelfand representation is only norm-decreasing (cf. [13]).
If A is a C^*-algeba, then $\|p\| = \|p^2\| = \|p\|^2$, so that $\|p\| = 1$ if $p \neq 0$.
In such a case, we have

$$|\alpha_n| = 2|\psi_{2n+1}(p, \cdots, p)| \leq 2\|\psi_{2n+1}\|.$$

Then entireness implies that the series $\sum_{n=0}^{\infty} \frac{\|\psi_{2n+1}\| z^n}{n!}$ converges at any $z \in \mathbb{C}$. In particular, the series converges at $z = -1$. $\square$

We next let $A_q = M_q(\mathbb{C}) \otimes A = M_q(A)$ be the Banach algebra of $q \times q$ matrices over a Banach algebra A, for $q \in \mathbb{N}$.

For any $\varphi \in C^m(A)$ as a multi-linear functional on A^{m+1}, we denote by φ^q the natural multi-linear extension of φ to $M_q(\mathbb{C}) \otimes A$ defined as $\varphi^q = \mathrm{tr}\#\varphi$. Namely,

$$\varphi^q(\mu_0 \otimes a_0, \cdots, \mu_m \otimes a_m) = \mathrm{tr}(\mu_0 \cdots \mu_m)\varphi(a_0, \cdots, a_m)$$

for $\mu_j \in M_q(\mathbb{C})$ and $a_j \in A$.

Lemma 5.2. *For any entire even and odd cochain sequence (φ_{2n}) and (φ_{2n+1}) on A, the extended even and odd cochain sequences (φ_{2n}^q) and (φ_{2n+1}^q) on $M_q(A)$ are also entire, respectively.*

The map sending entire cochain sequences φ on A to φ^q on $M_q(A)$ is a morphism of the complexes of entire cochain sequences.

Proof. For $\varphi \in C^m(A)$, we have $\|\varphi^q\| \leq q^m \|\varphi\|$.
$\star$ It seems that we in fact have

$$\|\varphi^q\| \leq q\|\varphi\|$$

since $\mathrm{tr}(A) = \mathrm{tr}(P^{-1}AP)$ on $M_q(\mathbb{C})$, and upper trianglization as $P^{-1}AP$ by an invertible P holds for any matrix A in $M_q(\mathbb{C})$, as $A = \mu_0 \cdots \mu_m$, and the spectral radius $r(P^{-1}AP) = r(A) \leq \|A\| \leq 1$, where the norm of φ^q may be defined only on such simple tensors $\mu_j \otimes a_j$ with their norms less than or equal to 1 coordinatewise.

It then follows that for instance,

$$\sum_{n=0}^{\infty} \frac{\|\varphi_{2n}^q\||z|^n}{n!} \leq q \sum_{n=0}^{\infty} \frac{\|\varphi_{2n}\||z|^n}{n!} \quad z \in \mathbb{C}$$

as desired.

We have the following even morphism

$$\Phi : C_{et}^{\mathrm{ev}}(A) \to C_{et}^{\mathrm{ev}}(M_q(\mathbb{C}) \otimes A)$$

where $\Phi((\varphi_{2n})) = (\mathrm{tr}_{M_q(\mathbb{C})} \# \varphi_{2n})$. The odd case is defined similarly. $\square$

Theorem 5.3. *Let $\varphi = (\varphi_{2n})$ be an entire normalized cocycle sequence on a Banach algebra A. Define*

$$f_\varphi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{2n}^q(x, \cdots, x), \quad x \in M_q(\mathbb{C}) \otimes A$$

as the corresponding entire function on $M_\infty(A) = \cup_{q \in \mathbb{N}} M_q(A)$. The the restriction of f_φ to the idempotents $p = p^2$, $p \in M_\infty(A)$ defines an additive map $f_\varphi : K_0(A) \to \mathbb{C}$ defined as up to K-theory equivalence. The value $\langle \varphi, [p] \rangle$ as $f_\varphi(p)$ only depends on the entire cohomology class of φ in $H_{et}^{\mathrm{ev}}(A)$.

Proof. We may replace A with $A^\sim$ and do φ_{2n} with $\varphi_{2n}^\sim$.

We have that for $x_j + \lambda_j 1 \in A^\sim$,

$$\begin{aligned} &\varphi_{2n}^\sim(x_0 + \lambda_0 1, \cdots, x_{2n} + \lambda_{2n} 1) \\ &= \varphi_{2n}(x_0, \cdots, x_{2n}) + \lambda_0 B_0 \varphi_{2n}(x_1, \cdots, x_{2n}). \end{aligned}$$

$\star$ Note that

$$\begin{aligned} &(B_0 \varphi_{2n})(x_1, \cdots, x_{2n}) \\ &= \varphi_{2n}(1, x_1, \cdots, x_{2n}) - \varphi_{2n}(x_1, \cdots, x_{2n}, 1). \end{aligned}$$

This is zero if φ_{2n} is cyclic. Being cyclic may be assumed from the beginning. Then the second term in the formula for $\varphi_{2n}^\sim$ above is zero.

It then follows that each φ_{2n} vanishes if some x_j, $j \geq 0$ is equal to 1.

$\star$ Namely, by definition,

$$\varphi_{2n}^\sim(x_0, \cdots, 0 + 1, \cdots, x_{2n}) = \varphi_{2n}(x_0, \cdots, 0, \cdots, x_{2n}) = 0.$$

We need to show that the value $f_\varphi(p)$ for p a projection of $M_q(A)$ only depends upon the connected component of p in the space $P_q(A)$ of projections of $M_q(A)$.

Since the map from φ to φ^q is a morphism of complexes, we may assume that $q = 1$.

Let $p(t)$ be a C^1-class map from the interval $[0, 1]$ to the space $P(A)$ of projections of A.

It is shown that

$$\frac{d}{dt} f_\varphi(p(t)) = 0.$$

We have

$$\frac{d}{dt} p(t) = [a(t), p(t)], \quad a(t) = (1 - 2p(t))\frac{d}{dt} p(t).$$

⋆ Note that

$$\frac{d}{dt} p(t) = \frac{d}{dt} p(t)^2 = 2p(t)p'(t).$$

We then have $(1-2p(t))p'(t) = 0$. Multiplying $p(t)$ from the left to the equation implies that $-p(t)p'(t) = 0$. Therefore, we obtain $p'(t) = 0 = a(t)$. By the way, projection valued functions may not be differentiable at some points, in general, as in the case of real valued functions such as characteristic functions.

We need to compute $\frac{d}{dt} f_\varphi(p(t))$ at $t = 0$.

Let $p = p(0)$ and $a = a(0) = 0$.

⋆ We then compute in a way different from the original text that

$$\frac{d}{dt}\varphi_{2n}(p(t), \cdots, p(t))|_{t=0} = \sum_{j=0}^{2n} \varphi_{2n}(p(t), \cdots, p'(t), \cdots, p(t))|_{t=0}$$
$$= \sum_{j=0}^{2n} \varphi_{2n}(p, \cdots, 0, \cdots, p) = 0.$$

Being zero of the derivative implies that the value $f_\varphi(p(t))$ is a constant. Hence it does depend only on the (C^1-)connected component of p.

Does it hold that $f_\varphi(p+q) = f_\varphi(p) + f_\varphi(q)$ as an additive sense?

But φ_{2n} is multi-linear, and then

$$\varphi_{2n}(p+q, \cdots, p+q) = \varphi_{2n}(p, \cdots, p) + \varphi_{2n}(q, \cdots, q)?$$

Let $n = 1$.

$$\begin{aligned}
&\varphi_2(p+q, p+q, p+q) = \varphi_2(p, p+q, p+q) + \varphi_2(q, p+q, p+q) \\
&= \varphi_2(p,p,p) + \varphi_2(p,p,q) + \varphi_2(p,q,p) + \varphi_2(p,q,q) \\
&\quad + \varphi_2(q,p,p) + \varphi_2(q,p,q) + \varphi_2(q,q,p) + \varphi_2(q,q,q).
\end{aligned}$$

But cycling implies

$$\begin{aligned}
\varphi_2(p,p,q) = \varphi_2(q,p,p) = \varphi_2(p,q,p), \\
\varphi_2(p,q,q) = \varphi_2(q,p,q) = \varphi_2(q,q,p).
\end{aligned}$$

□

Theorem 5.4. *Let τ be a continuous odd trace on $Q_\varepsilon^\wedge A$. Then the map f_φ of $K_0(A)$ to $\mathbb{C}$ given above by the entire even cocycle sequence $\varphi = (\varphi_{2n})$ associated to τ is obtained by the formula*

$$f_\varphi(p) = \tau(Fp(1-(qp)^2)^{-\frac{1}{2}}), \quad p \in P(A).$$

Proof. The entire cocycle sequence φ associated to τ has components given by

$$\varphi_{2n}(a_0, \cdots, a_{2n}) = \frac{(-1)^n(2n-1)!!}{2^n}\tau(Fa_0q(a_1)\cdots q(a_{2n}))$$

up to an overall normalization constant, where $qa = \sqrt{2}da$.

$\star$ If so, then we have

$$\varphi_{2n}(p, \cdots, p) = (-1)^n(2n-1)!!\tau(Fpdp\cdots dp)$$

Then we have

$$\begin{aligned} f_\varphi(p) &= \sum_{n=0}^\infty \frac{(-1)^n}{n!}\varphi_{2n}(p, \cdots, p) \\ &= \sum_{n=0}^\infty \frac{(2n-1)!!}{n!}\tau(Fpdp\cdots dp). \end{aligned}$$

Also, we have

$$dp = d(p^2) = (dp)p + pdp.$$

Thus, $p(dp)p = 0$. Also, $pdp = (dp)(1-p)$.

On the other hand, we have $1-(qp)^2 = 1-2p$. This is positive. Thus there exists the positive $\sqrt{1-2p}$ such that $(\sqrt{1-2p})^2 = 1-2p$. But $\sqrt{1-2p}$ may not be invertible.

There may be more reasons for the formula attained. □

Remark 5.5. The normalization condition for the cocycle sequence can be removed by the following minor modification

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!}\varphi_{2n}(p-\frac{1}{2}, p, \cdots, p).$$

to be zero (cf. [11]).

When A is a C^*-algebra then $E_\lambda A$ has a natural C^*-algebra norm (but not complete) which defines a stronger topology than that used above. There are continuous traces on $E_\lambda A$ for the C^*-norm (cf. [5]).

The pairing above is applied to the case of arbitrary algebras over $\mathbb{C}$ by using Remark given above to define entire cyclic cohomology in a generality.

6 The entire cyclic cohomology for the circle algebra

The periodic cyclic cohomology $H^*(A)$ of an algebra A with Hochschild dimension finite n is given by the image of the cyclic cohomology groups $cH^q(A)$ with $q \leq n$ in $H^*(A)$ described by the diagram of $I \circ S \circ B$ maps. In order to obtain such a result for entire cyclic cohomology assuming entireness for cochains or that A is a Banach algebra, we need to construct a homotopy σ_k for $k > n$ of the bar resolution with controlling the size of σ_k for k large (cf. [12]). See also [6] and [7].

Let us recall that the standard bar resolution of a unital algebra A as the bimodule A over $B = A \otimes A^{\mathrm{op}}$ is given by the acyclic chain complex (M_k, b) defined as $M_k = B \otimes (\otimes^k A)$ and the B-module map $b_k : M_k \to M_{k-1}$,

$$b_k((1 \otimes 1) \otimes a_1 \otimes \cdots \otimes a_k) = (a_1 \otimes 1) \otimes (a_2 \otimes \cdots \otimes a_k) +$$
$$\sum_{j=1}^{k-1} (-1)^j (1 \otimes 1) \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k + (-1)^k (1 \otimes a_k^\circ) \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

$\star$ The map $b_1 : M_1 \to M_0 = B$ is defined by

$$b_1((1 \otimes 1) \otimes a) = (a \otimes 1) - (1 \otimes a^\circ) \in B, \quad a \in A, a^\circ \in A^{\mathrm{op}}.$$

Also, we check the differentiability at M_2 as $b_1 \circ b_2 = 0$ as the following:

$$\begin{aligned}
&(b_1 \circ b_2)((1 \otimes 1) \otimes a_1 \otimes a_2) = b_1((a_1 \otimes 1) \otimes a_2) \\
&- b_1((1 \otimes 1) \otimes a_1 a_2) + b_1((1 \otimes a_2^\circ) \otimes a_1) \\
&= (a_1 a_2 \otimes 1) - (a_1 \otimes a_2^\circ) \\
&- (a_1 a_2 \otimes 1) + (1 \otimes (a_1 a_2)^{\mathrm{op}}) + (a_1 \otimes a_2^\circ) - 1 \otimes (a_2^\circ a_1^\circ) = 0
\end{aligned}$$

where the bimodule structure for B is given by right multiplication by elements of A on each tensor factor A and A^{op}.

In the topological context, the above tensor products are (projective or) π-tensor products of locally convex vector spaces, with topology given by some continuous semi-norms on A or by continuous product $A \times A \to A$.

We now consider an algebra A of complex-valued-functions in one complex variable $z \in \mathbb{C}$.

Since A is commutative, we may assume that $A = A^{\mathrm{op}}$ so that $B = A \otimes A^{\mathrm{op}} = A \otimes A$.

Any element $f \in M_k = B \otimes (\otimes^k A)$ is viewed as a function $f(z, z_0, z_1, \cdots, z_k)$ of complex $k+2$ variables.

The domain of the complex variables or the regularity of functions f may not be specified.

In particular, we consider the case of $A = \mathbb{C}[z, z^{-1}]$ of Laurent polynomials in what follows. The algebra A with generators as variables z and $z^{-1} = \overline{z}$ with

$|z| = 1$ (or nonzero) may be called as the circle (or torus or annulus) algebra by us.

Define a B-module map $\sigma_n : M_n \to M_{n+1}$ for $n \geq 1$ by

$$(\sigma_n f)(z, z_0, z_1, \cdots, z_{n+1}) = (-1)^{n+1} f(z, z_0, z_1, \cdots, z_n)$$
$$+ (-1)^n \frac{z_{n+1} - z_0}{z_n - z_0}(f(z, z_0, z_1, \cdots, z_n) - f(z, z_0, z_1, \cdots, z_{n-1}, z_0)).$$

Lemma 6.1. *We have $b_{n+1}\sigma_n + \sigma_{n-1}b_n = \mathrm{id}_n$ on M_n for $n \geq 2$.*

$\star$ Namely, it holds as in the diagram.

$$\begin{tikzcd}
M_{n-1} \arrow[r, "\sigma_{n-1}"] & M_n \arrow[r, "\sigma_n"] \arrow[d, equal] & M_{n+1} \arrow[d, "\mathrm{id}_{n+1}"] \\
M_{n-1} \arrow[u, "\mathrm{id}_{n-1}"] & M_n \arrow[l, "b_n"'] & M_{n+1}. \arrow[l, "b_{n+1}"']
\end{tikzcd}$$

Proof. For $g \in M_n$ we have

$$(b_n g)(z, z_0, z_1, \cdots, z_{n-1}) = g(z, z_0, z, z_1, \cdots, z_{n-1})$$
$$+ \sum_{j=1}^{n-1} (-1)^j g(z, z_0, z_1, \cdots, z_j, z_j, \cdots, z_{n-1}) + (-1)^n g(z, z_0, z_1, \cdots, z_{n-1}, z_0).$$

$\star$ Assume that b_n is defined so. We then compute

$$(b_3\sigma_2 f)(z, z_0, z_1, z_2) = (\sigma_2 f)(z, z_0, z, z_1, z_2)$$
$$- (\sigma_2 f)(z, z_0, z_1, z_1, z_2) + (\sigma_2 f)(z, z_0, z_1, z_2, z_0)$$
$$= -f(z, z_0, z, z_1) + \frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z, z_1) - f(z, z_0, z, z_0))$$
$$+ f(z, z_0, z_1, z_1) - \frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z_1, z_1) - f(z, z_0, z_1, z_0))$$
$$- f(z, z_0, z_1, z_2) + \frac{z_0 - z_0}{z_2 - z_0}(f(z, z_0, z_1, z_2) - f(z, z_0, z_1, z_0))$$
$$= -f(z, z_0, z, z_1) + \frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z, z_1) - f(z, z_0, z, z_0))$$
$$+ f(z, z_0, z_1, z_1) - \frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z_1, z_1) - f(z, z_0, z_1, z_0)) - f(z, z_0, z_1, z_2)$$

Also,

$$
\begin{aligned}
&(\sigma_1 b_2 f)(z, z_0, z_1, z_2) = (b_2 f)(z, z_0, z_1) \\
&+ \frac{z_2 - z_0}{z_1 - z_0}((b_2 f)(z, z_0, z_1) - (b_2 f)(z, z_0, z_0)) \\
&= f(z, z_0, z, z_1) - f(z, z_0, z_1, z_1) + f(z, z_0, z_1, z_0) \\
&+ \frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z, z_1) - f(z, z_0, z_1, z_1) + f(z, z_0, z_1, z_0)) \\
&- \frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z, z_0) - f(z, z_0, z_0, z_0) + f(z, z_0, z_0, z_0)) \\
&= f(z, z_0, z, z_1) - f(z, z_0, z_1, z_1) + f(z, z_0, z_1, z_0) \\
&+ \frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z, z_1) - f(z, z_0, z_1, z_1) + f(z, z_0, z_1, z_0) - f(z, z_0, z, z_0))
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&(b_3 \sigma_2 + \sigma_1 b_2) f(z, z_0, z_1, z_2) = -f(z, z_0, z_1, z_2) + f(z, z_0, z_1, z_0) + \\
&2\frac{z_2 - z_0}{z_1 - z_0}(f(z, z_0, z, z_1) - f(z, z_0, z_1, z_1) + f(z, z_0, z_1, z_0) - f(z, z_0, z, z_0)).
\end{aligned}
$$

On the other hand we obtain

$$
\begin{aligned}
&(-b_3 \sigma_2 + \sigma_1 b_2) f(z, z_0, z_1, z_2) = 2(f(z, z_0, z, z_1) - f(z, z_0, z_1, z_1)) \\
&+ f(z, z_0, z_1, z_2) + f(z, z_0, z_1, z_0).
\end{aligned}
$$

Possibly, something may be wrong in a way along. □

Theorem 6.2. *Let $A = \mathbb{C}[z, z^{-1}]$ be the circle algebra of Laurent polynomials. Then its entire cyclic cohomology is given by $H_{et}^{\mathrm{ev}}(A) = \mathbb{C}$ and $H_{et}^{\mathrm{od}}(A) = \mathbb{C}$. Their generators are given respectively by the cyclic cocycles*

$$
\tau_0(f) = \int f(z)dz \quad \textit{and} \quad \tau_1(f_0, f_1) = \int f_0 df_1.
$$

Specializing the general homotopy σ_k given above to the bimodule A^* over A yields linear maps

$$
\alpha_n : C^{n+1}(A, A^*) \to C^n(A, A^*)
$$

such that $\alpha_n b + b\alpha_{n-1} = \mathrm{id}$ on $C^n(A, A^*)$ for $n \geq 2$.

⋆ Namely,

$$
\begin{array}{ccc}
C^n(A, A^*) & \xrightarrow{\alpha_{n-1}} & C^{n-1}(A, A^*) \\
b \downarrow & & \downarrow b \\
C^{n+1}(A, A^*) & \xrightarrow{\alpha_n} & C^n(A, A^*).
\end{array}
$$

The transposed map $\alpha_n^t : \otimes^n A \to \otimes^{n+1} A$ is given by

$$(\alpha_n^t f)(z_0, \cdots, z_{n+1}) = (-1)^{n+1} f(z_0, \cdots, z_n)$$
$$+ (-1)^n \frac{z_{n+1} - z_0}{z_n - z_0} (f(z_0, \cdots, z_{n-1}, z_n) - f(z_0, \cdots, z_{n-1}, z_0)).$$

$\star$ Note that

$$M_n \xrightarrow{\sigma_n} M_{n+1} \xrightarrow[bilinear]{\varphi = f} A^* \text{ or } \mathbb{C}.$$

Define $\alpha_n(\varphi) = \varphi \circ \sigma_n$. For $f \in M_n$, define as $\alpha_n^t f = \sigma_n f$.

Given an odd cocycle sequence $\varphi = (\varphi_{2k+1})$ in the (b, B) bicomplex, produced is a cohomologous cocycle sequence (φ'_{2k+1}) with $\varphi'_{2k+1} = 0$ for any $k \geq 1$ by adding to φ the coboundary of the cochain sequence (ψ_{2k}) whose components are given by, using the homotopy $\alpha = (\alpha_k)$

$$\psi_{2k} = \sum_{m=0}^{\infty} \alpha (B\alpha)^m \varphi_{2m+2k+1}.$$

In particular, $\psi_2 = \alpha\varphi_3 + \alpha B \alpha \varphi_5 + \cdots$.

$\star$ In the last case, we have $\alpha\varphi_3 = \alpha_2 \varphi_3$. Also, $\alpha B \alpha \varphi_5 = \alpha_2 B \alpha_4 \varphi_5$. More precisely, in the general case, we have

$$\alpha (B\alpha)^m \varphi_{2m+2k+1} =$$
$$\alpha_{2k} (B\alpha_{2k+2}) \cdots (B\alpha_{2k+2m}) \varphi_{2k+2m+1}.$$

The formulae above are given as standard homotopy formulae for cocycles with support finite in any bicomplex.

Only the difficulty we carry is to show that the formulae continue to make sense for entire cocycle sequences with support infinite or arbitrary.

The growth condition on cochain sequeneces is given by that for any finite subset Σ of A, there exists $C = C_\Sigma$ such that

$$|\varphi_{2k+1}(a_0, \cdots, a_{2k+1})| \leq \frac{C}{k!}, \quad a_j \in \Sigma$$

in the (b, B) bicomplex instead of the equivalent (d_1, d_2) bicomplex.

$\star$ Note that

$$\frac{(2k+1)!}{k!} = (2k+1)(2k) \cdots (k+1)$$
$$= k!(2 + \frac{1}{k})(\frac{2k}{k-1}) \cdots \frac{k+1+l}{l} \cdots \frac{k+2}{1}(k+1)$$

whose factors except $k!$ may be involved to the constant C as another constant.

Given a finite subset Σ of the algebra A of Laurent polynomials, the maximal degree of elements of Σ is denoted as $d = d(\Sigma)$, so that any $f \in \Sigma$ can be written as $\sum_{j=-d}^{d} f_j z^j$.

Lemma 6.3. *Let $f \in \otimes^{n+1} A$ be a Laurent polynomial of degree at most d in each variable z_j for $0 \leq j \leq n$. Then $\alpha_n^t B^t f \in \otimes^{n+1} A$ has the same degree property.*

Let $\| \cdot \|_1$ denote the l^1-norm on $\otimes^{n+1} A$ for any n, so that

$$\|f\|_1 = \| \sum \lambda_{i_0 \cdots i_n} z_0^{i_0} \cdots z_n^{i_n} \|_1 = \sum_{i_0, \cdots, i_n} |\lambda_{i_0 \cdots i_n}|.$$

If f has degree less than d, then we have

$$\|\alpha_n^t B_0^t f\|_1 \leq (2d+2)\|f\|_1.$$

Proof. Cyclic permutations A_0^t for variables z_j do not change the degree for f.

It is enough to prove the first statement for $\alpha_n^t B_0^t f$, where $B^t = A_0^t B_0^t$.

We have

$$(B_0^t f)(z_0, \cdots, z_{n+1}) = f(z_1, \cdots, z_{n+1}) - (-1)^n f(z_0, \cdots, z_n).$$

Thus,

$$\begin{aligned}(\alpha_{n+1}^t B_0^t f)(z_0, \cdots, z_{n+1}, z_{n+2}) &= (-1)^{n+2} (B_0^t f)(z_0, \cdots, z_{n+1}) \\ &+ (-1)^{n+1} \frac{z_{n+2} - z_0}{z_{n+1} - z_0} ((B_0^t f)(z_0, \cdots, z_{n+1}) - (B_0^t f)(z_0, \cdots, z_n, z_0))\end{aligned}$$

and

$$\begin{aligned}&(B_0^t f)(z_0, \cdots, z_{n+1}) - (B_0^t f)(z_0, \cdots, z_n, z_0) \\ &= f(z_1, \cdots, z_{n+1}) - (-1)^n f(z_0, \cdots, z_n) - f(z_1, \cdots, z_n, z_0) + (-1)^n f(z_0, \cdots, z_n) \\ &= f(z_1, \cdots, z_{n+1}) - f(z_1, \cdots, z_n, z_0) \quad \text{(cancelled)}.\end{aligned}$$

It is true that $B_0^t f$ has the same (or less as zero) degree as f.

We need to deal with the other term. We may assume that f has the form

$$f(z_0, \cdots, z_n) = h(z_0, \cdots, z_{n-1}) z_n^q, \quad |q| \leq d.$$

Then we evaluate a part of the Laurent polynomial (terms) such that

$$\frac{z_{n+2} - z_0}{z_{n+1} - z_0} (z_{n+1}^q - z_0^q).$$

If q is positive, then the part is divided as to be

$$(z_{n+2} - z_0)(z_{n+1}^{q-1} + z_{n+1}^{q-2} z_0 + \cdots + z_0^{q-1}).$$

If q is negative, with $p = -q > 0$, we have

$$z_{n+1}^q - z_0^q = \frac{1}{z_{n+1}^p} - \frac{1}{z_0^p} = \frac{z_0^p - z_{n+1}^p}{z_{n+1}^p z_0^p}$$

so that the part is divided as to be

$$(z_{n+2}-z_0)z_{n+1}^q z_0^q(z_0^{p-1}+z_0^{p-2}z_{n+1}+\cdots+z_{n+1}^{q-1})$$
$$=(z_{n+2}-z_0)(z_{n+1}^q z_0^{-1}+z_{n+1}^{q+1}z_0^{-2}+\cdots+z_{n+1}^{-1}z_0^q).$$

In both cases, checked is that the degree (with respect to each variable) is less than (or equal to) d.

Also checked is that the l^1-norm satisfies the inequality in the statement.

$\star$ If cyclic permutations not involved, then the l^1-norm of the left hand side is estimated by

$$2\|f\|_1+2|q|\|h\|_1=2(|q|+1)\|f\|_1\le 2(d+1)\|f\|_1$$

where $q\neq 0$ and the factorization by z_n^q is assumed, but both may not be assumed in general.

□

Proof. (For the theorem above). The formula

$$\psi_{2k}=\sum_{m=0}^{\infty}\alpha(B\alpha)^m\varphi_{2m+2k+1}$$

is convergent.

Indeed, given a finite subset Σ of A, there exists a growth constant $C=C_\Sigma$ such that

$$|\varphi_{2n+1}(a_0,\cdots,a_{2n+1})|\le C\frac{1}{n!},\quad a_j\in\Sigma.$$

Thus, taking the monomials $\frac{1}{\lambda}z^q$, $|q|\le d$ as for Σ, it then follows that for any Laurent polynomials f_j of degree less than (or equal to) d (with respect to each variable) we have

$$|\varphi_{2n+1}(f_0,\cdots,f_{2n+1})|\le C_{\lambda,d}\frac{\lambda^{2n+2}}{n!}\Pi_{j=0}^{2n+1}\|f_j\|_1.$$

$\star$ Let $f_j=\sum_{s_j=-d}^{d}c_{s_j}z^{s_j}$ with $c_{s_j}\in\mathbb{C}$. Then

$$|\varphi_{2n+1}(f_0,\cdots,f_{2n+1})|=|\varphi_{2n+1}(\sum_{s_0=-d}^{d}c_{s_0}z^{s_0},\cdots,\sum_{s_{2n+1}=-d}^{d}c_{s_{2n+1}}z^{s_{2n+1}})|$$
$$=|\sum_{s_0=-d}^{d}c_{s_0}\cdots\sum_{s_{2n+1}=-d}^{d}c_{s_{2n+1}}\varphi_{2n+1}(z^{s_0},\cdots,z^{s_{2n+1}})|$$
$$=|\sum_{s_0=-d}^{d}c_{s_0}\cdots\sum_{s_{2n+1}=-d}^{d}c_{s_{2n+1}}\lambda^{2n+2}\varphi_{2n+1}(\lambda^{-1}z^{s_0},\cdots,\lambda^{-1}z^{s_{2n+1}})|$$
$$\le\sum_{s_0=-d}^{d}|c_{s_0}|\cdots\sum_{s_{2n+1}=-d}^{d}|c_{s_{2n+1}}|\lambda^{2n+2}C_{\lambda,d}\frac{1}{n!}$$
$$=C_{\lambda,d}\frac{\lambda^{2n+2}}{n!}\|f_0\|_1\cdots\|f_{2n+1}\|_1.$$

Using the lemma above with the equality $B = A_0 B_0$ we then obtain

$$
\begin{aligned}
&|(B\alpha)^m \varphi_{2m+2k+1}(f_0, \cdots, f_{2k+1})| \\
&\leq C_{\lambda,d} \frac{\lambda^{2m+2k+2}}{(m+k)!}(2d+2)^m \Pi_{j=1}^m (2k+2j) \Pi_{j=0}^{2k+1} \|f_j\|_1
\end{aligned}
$$

for any $f_j \in A$ of degree less than (or equal to) d.

$\star$ Note that for $m = 1$,

$$
(B\alpha)\varphi_{2+2k+1}(f_0, \cdots, f_{2k+1}) = A_0(B_0\alpha)\varphi_{2+2k+1}(f_0, \cdots, f_{2k+1})
$$

with

$$
(B_0\alpha)\varphi_{2+2k+1}(f_0, \cdots, f_{2k+1}) = \varphi_{2+2k+1}(\alpha^t B_0^t(f_0 \otimes \cdots \otimes f_{2k+1})).
$$

Do this seem to make sense? Then

$$
\begin{aligned}
&|(B\alpha)\varphi_{2+2k+1}(f_0, \cdots, f_{2k+1})| \\
&\leq (2k+2) C_{\lambda,d} \frac{\lambda^{2k+4}}{(k+1)!}(2d+2) \Pi_{j=0}^{2k+1} \|f_j\|_1.
\end{aligned}
$$

That's it for $m = 1$!

Taking λ small enough implies that there is a constant C_d such that

$$
|(B\alpha)\varphi_{2+2k+1}(f_0, \cdots, f_{2k+1})| \leq C_d \frac{1}{2^m k!} \Pi_{j=0}^{2k+1} \|f_j\|_1
$$

for any $f_j \in A$ of degree less than (or equal to) d.

$\star$ Note that it seems that the behavior of $C_{\lambda,d}$ for λ small enough is not so clear to obtain such an estimate.

It then follows that the series converge as desired so that (ψ_{2k}) is an entire cochain sequence.

$\star$ A possible solution for this is to have that

$$
\lim_{k \to \infty} \sqrt[k]{\frac{\sum_{m=0}^{\infty} \|\alpha(B\alpha)^m \varphi_{2m+2k+1}\|}{k!}} = 0.
$$

We have

$$
\|\psi_{2k}\| \leq \sum_{m=0}^{\infty} \|\alpha(B\alpha)^m \varphi_{2m+2k+1}\| \leq \sum_{m=0}^{\infty} \|\alpha_{2k}\| \|(B\alpha)^m \varphi_{2m+2k+1}\|
$$

with

$$
\|(B\alpha)^m \varphi_{2m+2k+1}\| \leq C_d \frac{1}{2^m k!}
$$

if the last estimate is correct. It seems that the behavior of $\|\alpha_{2k}\|$ as k large enough is not clear. As well, what is the norm? Masaka, infinity?

On the other hand, we have

$$\log \sqrt[k]{k!} = \frac{1}{k} \log k!$$

For any $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $k > n$ such that

$$k! = n!(n+1) \cdots k \geq n! n \cdots n = n! n^{k-n}$$

so that

$$\sqrt[k]{k!} \geq \sqrt[k]{n!} n^{1-\frac{n}{k}} = \sqrt[k]{\frac{n!}{n^n}} n.$$

It then follows that $\lim_{k \to \infty} \log \sqrt[k]{k!} \geq n$. Therefore, we obtain $\lim_{k \to \infty} \log \sqrt[k]{k!} = \infty$.

□

Corollary 6.4. *Let A be a locally convex algebra. The pairing between the K-theory $K_1(A)$ and the odd entire cohomology theory $H_{et}^{\mathrm{od}}(A)$ is defined by*

$$\langle u, \varphi \rangle = \frac{1}{\sqrt{2\pi i}} \sum_{m=0}^{\infty} (-1)^m m! \varphi_{2m+1}^q (u^{-1}, u, u^{-1}, \cdots, u, u^{-1}, u)$$

for $u \in GL_q(A^\sim) \subset M_q(A^\sim) = M_q(\mathbb{C}) \otimes A^\sim$ and any normalized cocycle sequence $\varphi = (\varphi_{2m+1})$ in the (b, B) bicomplex, where

$$\varphi_{2m+1}^q (u_0 \otimes a_0, \cdots, u_{2m+1} \otimes a_{2m+1}) = \mathrm{tr}(u_0 \cdots u_{2m+1}) \varphi_{2m+1}(a_0, \cdots, a_{2m+1}).$$

Proof. Any invertible element $u \in GL_n(A)$ of a locally convex algebra A unital determines a homomorphism ρ_u of the unital algebra L of Laurent polynomials of z, z^{-1} to $M_n(A)$.

$\star$ We may define as seen by us that $\rho_u(z) = u$ and $\rho_u(z^{-1}) = u^{-1}$. This is extended to such a homomorphism from L to $M_n(A)$.

The pull-back $\rho_u^* \varphi$ of any odd entire cocycle sequence φ on A is cohomologous to a multiple $\lambda \tau_1$ of τ_1.

$\star$ Note that $\tau_1(f_0, f_1) = \int f_0 df_1 \in \mathbb{C}$. Also, φ_{2m+1}^n is defined on $\oplus_{j=0}^{2m+1} M_n(A)$. Then $\rho_u^* \varphi_{2m+1}^n$ with u fixed is defined on $\oplus_{j=0}^{2m+1} L$. In particular, $\rho_u^* \varphi_1^n$ is defined on $L \oplus L$ to $\mathbb{C}$. It seems to be possible to find such a multiple λ, but how to?

The pairing $\langle u, \varphi \rangle$ is defined to be λ.

The explicit formula for λ follows from the proof above (cf. [6] and also [10]).

$\star$ As for $m = 0$, the corresponding term is $\varphi_1^q(u^{-1}, u)$. Does differentiation look like multiplication by such an invertible? □

Given an entire odd cocycle sequence (φ_{2n+1}) on $A = \mathbb{C}[\mathbb{Z}]$ as in the theorem above, we further compute exlicitly the entire even cochain $\psi = (\psi_{2n})$ such that

$$\varphi_{2n+1} = b\psi_{2n} + B\psi_{2n+2}, \quad n \geq 1.$$

The formula for ψ_{2k} to be simplified is that

$$\psi_{2k} = \sum_{m=0}^{\infty} \alpha(B\alpha)^m \psi_{2k+2m+1}.$$

In order to simplify the computation we may assume that

$$\varphi_{2n+1}(f_0, \cdots, f_{2n+1}) = 0$$

for some $f_j = 1$ for some $j \geq 1$. This is a normalization condition weaken.

At the level of chains as elements of $\otimes^{n+1} A$, that means that any function $f(z_0, \cdots, z_n)$ which is independent of some z_j for some $j \geq 1$ can be ignored.

It then follows that the formula for the map $\alpha_n^t B_0^t$ given above is converted (?) as to

$$(\alpha_n^t B_0^t f)(z_0, \cdots, z_{n+1}) = (-1)^n (z_{n+1} - z_0) \frac{f(z_1, \cdots, z_n) - f(z_1, \cdots, z_{n-1}, z_0)}{z_n - z_0}$$

for $f \in \otimes^{n+1} A$ (with the factor $z_{n+1} - z_0$ changed from only z_{n+1}).

Therefore, $\alpha_n^t B_0^t$ is essentially a divided difference. As well, $(\alpha^t B^t)^n$ is viewed as iterated divided differences which satisfy remarkable identities.

The computation may become straightforward and as well the result is formulated in terms of the algebra $(Q^\wedge)^\wedge(A)$.

The algebra $(Q^\wedge)^\wedge(A)$ is generated by $A^\sim$ and two elements F, γ such that $F^2 = \gamma^2 = 1$, $\gamma a = a\gamma$ for any $a \in A$, and $\gamma F = -F\gamma$.

$\star$ Recall that

$$(Q^\wedge)^\wedge(A) = Q^\wedge(A) \rtimes_\gamma \mathbb{Z}_2 = QA \rtimes_F \mathbb{Z}_2 \rtimes_\gamma \mathbb{Z}_2 \cong QA \otimes M_2(\mathbb{C}).$$

This is a version of the Takai duality for crossed products of C^*-algebras by actions of abelian groups.

The weak normalization condition above implies that the distinction between A and $A^\sim$ is not necessary. Thus, the unit of A is that of $(Q^\wedge)^\wedge(A)$.

Lemma 6.5. *Let τ be a trace on $(Q^\wedge)^\wedge(A)$ vanishing on γA, and $\varphi_{2n+1})$ the cocycle sequence in the (b, B) bicomplex given by*

$$\varphi_{2n+1}(a_0, \cdots, a_{2n+1}) = t_n \tau(\gamma F a_0 [F, a_1] \cdots [F, a_{2n+1}]), \quad a_j \in A$$

with $t_n^{-1} = 2^n (2n+1)!!$. With u as the generator of $A = \mathbb{C}[\mathbb{Z}]$, the cochain sequence $\psi = (\psi_{2n})$ such that $\varphi_{2n+1} = b\psi_{2n} + B\psi_{2n+2}$ for $n \geq 2$ is then given by $\psi_{2n} = \alpha(\varphi_{2n+1} - A_0 \theta_{2n+1})$, where

$$\theta_{2n+1}(f_0, \cdots, f_{2n+1}) =$$

$$= \int_0^{\frac{1}{2}} \tau(F \frac{\partial}{\partial \lambda} f_0(u + \lambda[F, u])[\gamma F, f_1(u + \lambda[F, u])] \cdots [\gamma F, f_{2n+1}(u + \lambda[F, u])]) d\mu_n(\lambda),$$

where $d\mu_n(\lambda) = t_n (1 - 4\lambda^2)^{n + \frac{1}{2}} d\lambda$.

Proof. There is the possibility of applying the Laurent polynomials $f_j \in A$ to any invertible element of an algebra, and in particular to $u + \lambda[F, u]$ which is invertible in $(Q^\wedge)^\wedge_\varepsilon(A)$.

$\star$ We have $u = z$ with $u^{-1} = \frac{1}{z}$. We may assume that λ is small enough to have that $u + \lambda[F, u]$ close to u is invertible.

That formula does fit with the quantized calculus where the quantum differential is given by the graded commutator operator $[F, \cdot]$ by F. Or equivalently it is done by $[\gamma F, \cdot]$.

$\star$ Note that

$$[\gamma F, u] = \gamma F u - u\gamma F = \gamma[F, u].$$

Thus, $u + \lambda[F, u]$ plays the role of $u + \lambda du$.

The following formula is needed in this proof:

$$\gamma[\gamma F, f(u + \lambda[F, u])] = \frac{1}{2\lambda}(f(u + \lambda[F, u]) - f(u - \lambda[F, u])).$$

This relates the quantum differential of $f(u + \lambda du)$ to the difference slope:

$$\frac{1}{2\lambda}(f(u + \lambda du) - f(u - \lambda du)).$$

$\star$ Note that

$$\begin{aligned}
&\gamma[\gamma F, u + \lambda[F, u]] = \gamma[\gamma F, u] + \gamma[\gamma F, \lambda(Fu - uF)] \\
&= [F, u] + \lambda(F^2 u - FuF) - \lambda\gamma(Fu - uF)\gamma F \\
&= [F, u] + \lambda u - \lambda FuF + \lambda FuF - \lambda u = [F, u] \\
&= \frac{1}{2\lambda}(u - u + \lambda[F, u] - (-\lambda[F, u])) \\
&= \frac{1}{2\lambda}(u + \lambda[F, u] - (u - \lambda[F, u])).
\end{aligned}$$

That's it!

We may check the case of multiples of $u + \lambda[F, u]$ as f similarly. But not. See the computation given below as a partial part.

The proof of that formula is straightforward for Laurent polynomials, or for $f(u) = \frac{1}{u-z}$, as computed as above. $\square$

The formula of the lemma above is interpreted using a natural deformation of the algebra $(Q^\wedge)^\wedge(A)$ to an exterior algebra over A.

Define an endomorphism σ_λ of $(Q^\wedge)^\wedge(A)$ for $\lambda \in [0, \frac{1}{2})$ by

$$\begin{cases}
\sigma_\lambda(u) = u + \lambda[F, u], \\
\sigma_\lambda(F) = F, \\
\sigma_\lambda(\gamma) = \frac{1}{\sqrt{1-4\lambda^2}}\gamma(1 - 2\lambda F).
\end{cases}$$

⋆ Certainly, σ_λ is linear at u, F, and γ. Check that

$$
\begin{aligned}
&\sigma_\lambda(u)^2 = (u + \lambda[F,u])^2 \\
&= u^2 + \lambda u(Fu - uF) + \lambda(Fu - uF)u + \lambda^2(Fu - uF)^2 \\
&= u^2 - \lambda u^2 F + \lambda F u^2 + \lambda^2(FuFu - Fu^2F - u^2 + uFuF) \\
&= u^2 + \lambda[F,u^2] + \lambda^2(FuFu - Fu^2F - u^2 + uFuF) \\
&= \sigma_\lambda(u^2) + \lambda^2(FuFu - Fu^2F - u^2 + uFuF).
\end{aligned}
$$

The second term may not vanish but it does if u commutes with F. Note that $Fu^2F = (FuF)^2$. Also, $FuFu = F(uFuF)F$. So we should have that $Fu^2F = -u^2$ and $F(uFuF)F = -uFuF$ as a possible choice. This seems to involve the definition of F given as $FuF = \pm iu$ so that $Fu^2F = -u^2$ but

$$F(uFuF)F = \pm iu^2 \neq \mp iu^2 = -uFuF.$$

It should be a canonical case. In this case, we do not have $\sigma_\lambda(u)^2 = \sigma_\lambda(u^2)$. Also, as another case, if we take $FuF = -u$, then $(FuF) = u^2$.

We have $\sigma_\lambda(\gamma)^2 = 1$.

⋆ Check that

$$
\begin{aligned}
\sigma_\lambda(\gamma)^2 &= \frac{1}{1-4\lambda^2}\gamma(1-2\lambda F)\gamma(1-2\lambda F) \\
&= \frac{1}{1-4\lambda^2}(\gamma^2 - 2\lambda\gamma^2 F - 2\lambda\gamma F\gamma + 4\lambda^2\gamma F\gamma F) \\
&= \frac{1}{1-4\lambda^2}(1 - 2\lambda F + 2\lambda F - 4\lambda^2 F\gamma^2 F) = 1.
\end{aligned}
$$

We have that $\sigma_\lambda(\gamma)$ commutes with $\sigma_\lambda(u)$ and anti-commutes with F so that $[\sigma_\lambda(\gamma), \sigma_\lambda(u)] = 0$ and $\sigma_\lambda(\gamma)F + F\sigma_\lambda(\gamma) = 0$.

⋆ Check that zero is

$$
\begin{aligned}
&[\sigma_\lambda(\gamma), \sigma_\lambda(u)] = \frac{1}{\sqrt{1-4\lambda^2}}\{\gamma(1-2\lambda F)(u + \lambda[F,u]) - (u + \lambda[F,u])\gamma(1-2\lambda F)\} \\
&= \frac{1}{\sqrt{1-4\lambda^2}} \\
&\quad \{\gamma u - u\gamma + \lambda(\gamma[F,u] - [F,u]\gamma) - 2\lambda(\gamma F u - u\gamma F) - 2\lambda^2(\gamma F[F,u] - [F,u]\gamma F)\}
\end{aligned}
$$

with the first term and the second plus third term to be zero as

$$
\begin{aligned}
&\gamma u - u\gamma = 0, \\
&\gamma[F,u] - [F,u]\gamma = \gamma F u - \gamma u F - F u\gamma + uF\gamma \\
&= -F\gamma u - u\gamma F - F\gamma u - u\gamma F = -2(F\gamma u + u\gamma F), \\
&\gamma F u - u\gamma F = -F\gamma u - u\gamma F = -(F\gamma u + u\gamma F),
\end{aligned}
$$

and the forth zero as

$$
\begin{aligned}
&\gamma F[F,u] - [F,u]\gamma F = \gamma u - \gamma F u F - F u \gamma F + u F \gamma F \\
&= \gamma u - \gamma F u F + \gamma F u F - u\gamma = 0.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\sigma_\lambda(\gamma)F + F\sigma_\lambda(\gamma) \\
&= \frac{1}{\sqrt{1-4\lambda^2}}\{\gamma(1-2\lambda F)F + F\gamma(1-2\lambda F)\} \\
&= \frac{1}{\sqrt{1-4\lambda^2}}\{\gamma F - 2\lambda\gamma + F\gamma - 2\lambda F\gamma F\} \\
&= \frac{1}{\sqrt{1-4\lambda^2}}\{\gamma F - 2\lambda\gamma - \gamma F + 2\lambda\gamma\} = 0.
\end{aligned}
$$

We have $\sigma_\lambda \circ \sigma_{\lambda'} = \sigma_{\lambda''}$ with $2\lambda'' = \frac{2\lambda+2\lambda'}{1+4\lambda\lambda'}$. Thus, we have a semi-group of σ_λ for $\lambda \in [0, \frac{1}{2})$.

$\star$ Note that as a possible computation,

$$
\begin{aligned}
&(\sigma_\lambda \circ \sigma_{\lambda'})(u) = \sigma_\lambda(u + \lambda'[F,u]) \\
&= \sigma_\lambda(u) + \lambda' F\sigma_\lambda(u) - \lambda'\sigma_\lambda(u)F \\
&= u + \lambda[F,u] + \lambda' F(u + \lambda[F,u]) - \lambda'(u + \lambda[F,u])F \\
&= u + (\lambda+\lambda')[F,u] + \lambda\lambda'\{F(Fu - uF) - (Fu - uF)F\} \\
&= u + (\lambda+\lambda')[F,u] + \lambda\lambda'(u - FuF - FuF + u) \\
&= (1 + 2\lambda\lambda')u + (\lambda+\lambda')[F,u] - 2\lambda\lambda' FuF.
\end{aligned}
$$

For any $f \in A$, we have

$$
[\gamma F, f(u + \lambda[F,u])] = \frac{1}{\sqrt{1-4\lambda^2}}\sigma_\lambda(\gamma[F, f(u)]).
$$

$\star$ Possibly, the multiple should be changed to $\sqrt{1-4\lambda^2}$? Note that

$$
\begin{aligned}
&\sigma_\lambda(\gamma[F,u]) = \sigma_\lambda(\gamma F u - \gamma u F) \\
&= \frac{1}{\sqrt{1-4\lambda^2}}\gamma(1-2\lambda F)F(u + \lambda[F,u]) - \frac{1}{\sqrt{1-4\lambda^2}}\gamma(1-2\lambda F)(u + \lambda[F,u])F \\
&= \frac{1}{\sqrt{1-4\lambda^2}}\gamma F(1-2\lambda F)(u + \lambda[F,u]) - \frac{1}{\sqrt{1-4\lambda^2}}(1+2\lambda F)\gamma(u + \lambda[F,u])F.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\lambda F(u + \lambda[F,u]) = \lambda F u + \lambda^2(u - FuF). \\
&\gamma(u + \lambda[F,u]) = u\gamma + \lambda(-Fu\gamma + uF\gamma) = (u - \lambda[F,u])\gamma. \\
&\lambda F(u - \lambda[F,u]) = \lambda F u - \lambda^2(u - FuF).
\end{aligned}
$$

Therefore,

$$\sigma_\lambda(\gamma[F,u]) = \frac{1}{\sqrt{1-4\lambda^2}}[\gamma F, u+\lambda[F,u]]$$
$$+\frac{1}{\sqrt{1-4\lambda^2}}(-2\lambda\gamma u - 2\lambda^2(\gamma F u - \gamma u F) - 2\lambda F u\gamma F + 2\lambda^2(u\gamma F - FuF\gamma F)).$$

The second term is converted to

$$\frac{1}{\sqrt{1-4\lambda^2}}(-2\lambda\gamma(u-FuF) - 2\lambda^2(\gamma F u - 2\gamma u F + Fu\gamma))$$
$$= \frac{1}{\sqrt{1-4\lambda^2}}(-2\lambda\gamma(u-FuF) + 4\lambda^2\gamma u F).$$

It seems to be necessary to have that the second term vanishes.

The formula for θ_{2n+1} in the lemma above is simplified to

$$\theta_{2n+1}(f_0, \cdots, f_{2n+1})$$
$$= t_n \int_0^{\frac{1}{2}} \tau(\frac{\partial}{\partial\lambda}\sigma_\lambda(f_0)\sigma_\lambda([F,f_1]\cdots[F,f_{2n+1}]\gamma F))d\lambda.$$

$\star$ Note that for $f_0 = f_0(u) \in A$, we have

$$\sigma_\lambda(f_0(u)) = f_0(u+\lambda[F,u]),$$

but as a possible or probable computation. Also, as a possible or probable computation,

$$[\gamma F, f_1(u+\lambda[F,u])]\cdots[\gamma F, f_{2n+1}(u+\lambda[F,u])]$$
$$= \frac{1}{\sqrt{1-4\lambda^2}}\sigma_\lambda(\gamma[F,f_1(u)])\cdots\frac{1}{\sqrt{1-4\lambda^2}}\sigma_\lambda(\gamma[F,f_{2n+1}(u)])$$
$$= \frac{1}{(1-4\lambda^2)^{n+\frac{1}{2}}}\sigma_\lambda(\gamma[F,f_1(u)]\cdots\gamma[F,f_{2n+1}(u)]\gamma^2).$$

Moreover,

$$\gamma[F,f_1(u)] = \gamma[F,f_1(u)]\gamma^2 = \gamma(Ff_1(u) - f_1(u)F)\gamma^2$$
$$= (-F\gamma f_1(u) + f_1(u)F\gamma)\gamma^2 = (-1)[F,f_1(u)]\gamma.$$

As well,

$$\gamma[F,f_1(u)]\gamma[F,f_2(u)]\gamma[F,f_3(u)]$$
$$= (-1)[F,f_1(u)]\gamma\gamma[F,f_2(u)](-1)[F,f_3(u)]\gamma$$
$$= [F,f_1(u)][F,f_2(u)][F,f_3(u)]\gamma.$$

Furthermore,

$$\gamma[F,f_1(u)]\gamma[F,f_2(u)]\gamma[F,f_3(u)]\gamma[F,f_4(u)]\gamma[F,f_5(u)]$$
$$= (-1)[F,f_1(u)]\gamma\gamma[F,f_2(u)](-1)[F,f_3(u)]\gamma\gamma[F,f_4(u)](-1)[F,f_5(u)]\gamma$$
$$= (-1)[F,f_1(u)][F,f_2(u)][F,f_3(u)][F,f_4(u)][F,f_5(u)]\gamma.$$

Therefore, the factor $(-1)^n$ as a multiple should be attached to the simplified formula. Since τ is a trace, then $\tau(F(\cdots)) = \tau((\cdots)F)$.

Since τ is a trace, we can replace the endomorphisms σ_λ by α_s for $s \in \mathbb{R}$ as their inner conjugates, defined as

$$\begin{cases} \alpha_s(u) = \frac{1}{2}(u + FuF) + \frac{e^{-s}}{2}(u - FuF), \\ \alpha_s(\gamma) = \gamma, \quad \alpha_s(F) = F. \end{cases}$$

Indeed, we have

$$\begin{aligned} &\alpha_s(x) = z_s^{-1}\sigma_\lambda(x)z_s, \quad x \in (Q_\varepsilon^\wedge)^\wedge(A), \\ &z_s = z_{-\frac{1}{2}\log(1-4\lambda^2)} = \cosh\frac{t}{2} + F\sinh\frac{t}{2}, \quad 2\lambda = \tanh t. \end{aligned}$$

$\star$ Since $s = -\frac{1}{2}\log(1-4\lambda^2)$ with $\lambda \in [0, \frac{1}{2})$, we have non-negative $s \in [0, \infty) \subset \mathbb{R}$. We also have $2\lambda \in [0, 1)$ so that $t \in [0, \infty)$. If we take $\pm s$, then real $\pm s \in \mathbb{R}$.

We compute to check that

$$\begin{aligned} &(\cosh\frac{t}{2} + F\sinh\frac{t}{2})(\cosh\frac{t}{2} - F\sinh\frac{t}{2}) \\ &= \cosh^2\frac{t}{2} - F^2\sinh^2\frac{t}{2} = 1. \end{aligned}$$

Hence, $\cosh\frac{t}{2} + F\sinh\frac{t}{2}$ is invertible with inverse $\cosh\frac{t}{2} - F\sinh\frac{t}{2}$. In particular,

$$\begin{aligned} &(\cosh\frac{t}{2} - F\sinh\frac{t}{2})\sigma_\lambda(u)(\cosh\frac{t}{2} + F\sinh\frac{t}{2}) \\ &= \cosh^2\frac{t}{2}\sigma_\lambda(u) - \sinh^2\frac{t}{2}F\sigma_\lambda(u)F \\ &\quad + \cosh\frac{t}{2}\sinh\frac{t}{2}(\sigma_\lambda(u)F - F\sigma_\lambda(u)) \end{aligned}$$

with

$$\begin{aligned} &F\sigma_\lambda(u)F = FuF + \lambda F(Fu - uF)F = FuF + \lambda(uF - Fu). \\ &\sigma_\lambda(u)F - F\sigma_\lambda(u) = uF - Fu + \lambda((Fu - uF)F - F(Fu - uF)) \\ &= uF - Fu + \lambda(2FuF - 2u). \end{aligned}$$

Thus, the inner conjugated is converted to

$$\begin{aligned} &(\cosh^2\frac{t}{2})u - (\sinh^2\frac{t}{2})FuF + \lambda\{\cosh^2\frac{t}{2}[F, u] + \sinh^2\frac{t}{2}[F, u]\} \\ &\quad + \cosh\frac{t}{2}\sinh\frac{t}{2}\{uF - Fu + \lambda(2FuF - 2u)\}. \end{aligned}$$

Note that

$$e^{-s} = e^{\frac{1}{2}\log(1-4\lambda^2)} = \sqrt{1 - 4\lambda^2}.$$

As well,

$$
\begin{aligned}
&(\cosh\frac{t}{2} - F\sinh\frac{t}{2})\sigma_\lambda(F)(\cosh\frac{t}{2} + F\sinh\frac{t}{2}) \\
&= (\cosh^2\frac{t}{2})F - \sinh^2\frac{t}{2}F + \cosh\frac{t}{2}\sinh\frac{t}{2}(1-1) = F!
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&(\cosh\frac{t}{2} - F\sinh\frac{t}{2})\sigma_\lambda(\gamma)(\cosh\frac{t}{2} + F\sinh\frac{t}{2}) \\
&= (\cosh^2\frac{t}{2})\sigma_\lambda(\gamma) - \sinh^2\frac{t}{2}F\sigma_\lambda(\gamma)F + \cosh\frac{t}{2}\sinh\frac{t}{2}(\sigma_\lambda(\gamma)F - F\sigma_\lambda(\gamma))
\end{aligned}
$$

with

$$
\begin{aligned}
&F\gamma(1-2\lambda F)F = -\gamma + 2\lambda\gamma F = -\gamma(1-2\lambda F). \\
&\gamma(1-2\lambda F)F - F\gamma(1-2\lambda F) = 2\gamma F - 4\lambda\gamma.
\end{aligned}
$$

Thus, the inner conjugated is converted to

$$
\begin{aligned}
&(\cosh^2\frac{t}{2} + \sinh^2\frac{t}{2})\frac{1}{\sqrt{1-4\lambda^2}}\gamma(1-2\lambda F) \\
&+ \cosh\frac{t}{2}\sinh\frac{t}{2}\frac{1}{\sqrt{1-4\lambda^2}}(2\gamma F - 4\lambda\gamma).
\end{aligned}
$$

Our mission impossible of checking the contents suitably or patiently this time as a sort of continuation of [14] as well as [15] is yet incomplete towards the end in a few pages left, involving the last reformuation lemma, proposition, and theorem, which may not be continued to be done.

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