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REMARKS AND EXAMPLES ON TWO-VARIABLE ZETA FUNCTIONS FOR GRAPHS*

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Abstract

We establish a formula for the two-variable zeta functions for graphs introduced by Lorenzini, and use it to give explicit formulas of the zeta functions for dipole graphs, doubled trees and friendship graphs. We observe that the two-variable zeta functions for these graphs have a unified formula expressed using the Tutte polynomials.

1 Introduction

The chip-firing game on graphs and its variations have been studied by many authors from various perspective. Baker and Norine [1] developed the divisor theory on graphs. They regard a chip configuration on a graph as a divisor, introduce analogs of various notions in algebraic geometry such as linear equivalence, rank, Jacobians, etc., and established a graph-theoretic version of the Riemann-Roch theorem. Motivated by this result and the study by Pellikaan [7] on the two-variable zeta function for curves over a finite field studied (which becomes the local or congruent zeta function by a suitable specialization of a variable), Lorenzini [5] introduced and studied the two-variable zeta function $Z(G, t, u)$ for a graph G (in fact, Lorenzini generalized the Riemann-Roch theory to a corank one lattice in $\mathbb{Z}^n$ equipped with a certain rank function). Like the local zeta functions, the two-variable zeta function $Z(G, t, u)$ is a rational function and has the functional equation between $Z(G, t, u)$ and $Z(G, 1/ut, u)$.

One of the purpose of the paper is to give several explicit computations for concrete families of graphs such that the genera of the members are strictly increasing. Concretely, we treat the following three families: dipole graphs, doubled trees, and friendship graphs. The two-variable zeta function $Z(G, t, u)$ of a graph G is determined by first g terms, where g is the genus of G, in the defining series of $Z(G, t, u)$, so it is easier to determine the zeta function when g is small. Actually, $Z(G, t, u)$ is immediately obtained if the genus of G is at most two (see the examples in Section 3). Thus it would be meaningful to give a concrete example of an infinite family of graphs whose genera are unbounded and their zeta functions are explicitly computed. Interestingly, the zeta functions for these have the same form of expression using their

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Tutte polynomials. To accomplish this, we give a formula for the two variable zeta functions for graphs that is convenient for our calculation, which is interesting in its own right.

The paper is organized as follows. In Section 2, we review the necessary language for the divisor theory on finite graphs and state the facts we will use in our computation of zeta functions. We also show that the contraction of a bridge does not change the Picard groups, degree and rank of divisors. This means that it is enough to deal with the graphs with no edges. In Section 3, we give the definition of Lorenzini's two-variable zeta function for a graph and state basic facts on it. In Section 4, we establish a formula for the two-variable zeta function. Our subsequent computations are based on this formula. In Section 5, we perform the calculations and give explicit formulas for the zeta functions of three families of graphs. We conclude the paper by stating several problems that arise from our observations of examples.

General conventions

For $a \in \mathbb{R}$, $\lfloor a \rfloor$ is the largest integer not exceeding a. For a statement P, define

$$\delta(P) := \begin{cases} 1 & P \text{ is true,} \\ 0 & P \text{ is false.} \end{cases}$$

We denote by $A \triangle B$ the symmetric difference of two sets A and B, that is, $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For a positive integer n, put $[n] := \{1, 2, \ldots, n\}$.

2 Divisors and linear relations on graphs

We quickly review the definition and basic properties of divisors on graphs and related notions. We refer to the textbooks [4] and [3] for detailed information. Afterward, we will briefly discuss how the contraction of a bridge affects the materials related to divisors.

2.1 Definitions and basic facts

Let $G = (V, E)$ be an undirected connected finite (multi)graph without loops. The *genus* (or the *first Betti number*) g of G is defined by $g := |E| - |V| + 1$. Since G is connected, we readily see that $g \geq 0$, and $g = 0$ if and only if G is a tree. For an edge $e \in E$, $G - e$ is the graph obtained from G by deleting e, and G/e is the graph obtained from G by contracting e. The degree of $v \in V$ is denoted by $d(v)$, and the number of edges between v and w ($v, w \in V$) is denoted by $\nu(v, w)$. An edge $e \in E$ is called a *bridge* of G if $G - e$ is not connected.

The *Tutte polynomial* $T(G, x, y)$ of a graph G is defined by

$$T(G, x, y) = T(G - e, x, y) + T(G/e, x, y)$$

if e is an edge of G which is neither a loop nor a bridge, and

$$T(G, x, y) = x^b y^l$$

if the edge set of G consists of b brideges and l loops. It is known that $T(G,1,1)$ is the number of *spanning trees* of G.

Example 2.1. Let $\mathcal{C}_n$ be the cycle graph with n vertices. Here we understand that $\mathcal{C}_1$ is a bouquet graph with one loop, and $\mathcal{C}_2$ is a dipole graph $\mathcal{D}_2$ (see §5 for dipole graphs). If $n \geq 2$, for any edge e of $\mathcal{C}_n$, we have

$$\begin{aligned}T(\mathcal{C}_n,x,y) &= T(\mathcal{C}_n - e,x,y) + T(\mathcal{C}_n/e,x,y)\\ &= T(\mathcal{P}_n,x,y) + T(\mathcal{C}_{n-1},x,y) = x^{n-1} + T(\mathcal{C}_{n-1},x,y),\end{aligned}$$

where $\mathcal{P}_n$ is a path graph with n vertices (which has only $n-1$ bridges, so $T(\mathcal{P}_n,x,y) = x^{n-1}$), and $T(\mathcal{C}_1,x,y) = y$. Therefore we have

$$T(\mathcal{C}_n,x,y) = x^{n-1} + x^{n-2} + \cdots + x + y.$$

We denote by $\mathrm{Div}(G)$ the free abelian group on V, that is, the group consisting of formal $\mathbb{Z}$-linear combination of vertices of G. An element in $\mathrm{Div}(G)$ is called a *divisor* on G. We often express the coefficient of v in $D \in \mathrm{Div}(G)$ by $D(v)$, that is,

$$D = \sum_{v\in V} D(v)v.$$

The identity element of $\mathrm{Div}(G)$ is denoted by 0. We say that $E \in \mathrm{Div}(G)$ is *effective* and write $E \geq 0$ if $E(v) \geq 0$ for all $v \in V$. We denote by $\mathrm{Div}_+(G)$ the set of all effective divisors on G. The sum of all the coefficients $D(v)$ of $D \in \mathrm{Div}(G)$ is called the *degree* of D, and is denoted by $\deg D$:

$$\deg D := \sum_{v\in V} D(v).$$

The map $\mathrm{Div}(G) \ni D \mapsto \deg D \in \mathbb{Z}$ is a homomorphism. For convenience, we put

$$\begin{aligned}\mathrm{Div}^k(G) &:= \{D \in \mathrm{Div}(G) \mid \deg D = k\},\\ \mathrm{Div}^k_+(G) &:= \mathrm{Div}^k(G) \cap \mathrm{Div}_+(G).\end{aligned}$$

$\mathrm{Div}^0(G)$ is the kernel of the degree map. Notice that

$$\mathrm{Div}^k_+(G) = \{x_1 + \cdots + x_k \mid x_1,\ldots,x_k \in V\}$$

for each $k \in \mathbb{Z}_{\geq 0}$, and $\mathrm{Div}^k_+(G) = \varnothing$ if $k < 0$.

Let $\mathcal{M}(G)$ be the set of all $\mathbb{Z}$-valued functions on V. Define a map $\Delta\colon \mathcal{M}(G) \to \mathrm{Div}(G)$ by

$$\Delta f := \sum_{v\in V} \Delta_v(f)v$$

with

$$\Delta_v(f) = d(v)f(v) - \sum_{w\in V} \nu(v,w)f(w).$$

We denote by $\mathrm{Prin}(G)$ the image of Δ. Notice that $\mathrm{Prin}(G)$ is a subgroup of $\mathrm{Div}^0(G)$. Two divisors $D, D' \in \mathrm{Div}(G)$ are called *linearly equivalent* if and only if $D - D' \in$

$\mathrm{Prin}(G)$. We simply write $D \sim D'$ to mean that D and D' are linearly equivalent. Define

$$L(D) := \{E \in \mathrm{Div}_+(G) \mid D \sim E\}.$$

Namely, $L(D)$ is the set of all effective divisors which are linearly equivalent to D. We define the *rank* function $r\colon \mathrm{Div}(G) \to \mathbb{Z}_{\geq -1}$ by the following conditions:

(i) If $L(D) = \varnothing$, then $r(D) := -1$.

(ii) For any $s \in \mathbb{Z}_{\geq 0}$,

$$r(D) \geq s \iff L(D-E) \neq \varnothing, \quad \forall E \in \mathrm{Div}_+^s(G).$$

By definition, we see that $r(D) = r(D')$ if $D \sim D'$. If $\deg D < 0$, then $L(D) = \varnothing$, and hence $r(D) = -1$. If $\deg D \geq 0$, then $\deg(D-E) < 0$ when $\deg E > \deg D$, which implies that $r(D) \leq \deg D$.

Let us introduce a distinguished divisor on G

$$K := \sum_{v \in V} (d(v) - 2)v,$$

which is called the *canonical divisor* on G. We see that $\deg K = 2g-2$. Remarkably, the following graph-analog of the Riemann-Roch theorem holds.

Theorem 2.2 (Baker-Norine [1]). *For any $D \in \mathrm{Div}(G)$, we have*

$$r(D) - r(K-D) = \deg D - g + 1.$$

We further define the *Picard group* and *Jacobian group* of G by

$$\mathrm{Pic}(G) := \mathrm{Div}(G)/\mathrm{Prin}(G),$$
$$\mathrm{Jac}(G) := \mathrm{Div}^0(G)/\mathrm{Prin}(G).$$

The order $|\mathrm{Jac}(G)|$ of the group $\mathrm{Jac}(G)$ is equal to the number of *spanning trees* of G by the matrix-tree theorem. In particular, $\mathrm{Jac}(G) = \{0\}$ (or $\mathrm{Div}^0(G) = \mathrm{Prin}(G)$) if and only if G is a tree. We denote by $[D]$ an element in $\mathrm{Pic}(G)$ represented by $D \in \mathrm{Div}(G)$. Notice that

$$[D] = [D'] \iff D \sim D' \implies \deg D = \deg D'.$$

For convenience, we put

$$\mathrm{Pic}^k(G) := \{[D] \in \mathrm{Pic}(G) \mid \deg D = k\}.$$

Notice that $|\mathrm{Pic}^k(G)| = |\mathrm{Jac}(G)|$ for any $k \in \mathbb{Z}$.

Remark 2.3. If the *Smith normal form* of the matrix L_1, which is obtained by deleting the first row and first column of the Laplacian matrix of G, is $\mathrm{diag}(a_1, \ldots, a_{n-1})$, then $\mathrm{Jac}(G) \cong (\mathbb{Z}/a_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/a_{n-1}\mathbb{Z})$.

For $q \in V$, a divisor $D \in \mathrm{Div}(G)$ is called *q-reduced* if

(1) $D(v) \geq 0$ for all $v \in V \setminus \{q\}$,

(2) for any $S \subset V \setminus \{q\}$, there exists a vertex $v \in V \setminus \{q\}$ such that $D(v) - \Delta(\sum_{s \in S} \delta_s)(v) < 0$.

We denote by $\mathrm{Div}(G)_q$ the set of all q-reduced divisors on G, and put

$$\mathrm{Div}^i(G)_q := \mathrm{Div}(G)_q \cap \mathrm{Div}^i(G),$$
$$\mathrm{Div}^i_+(G)_q := \mathrm{Div}^i(G)_q \cap \mathrm{Div}_+(G).$$

Remark 2.4. In the language of chip-firing game, a q-reduced divisor is a *superstable configuration with sink q*.

The following fact is crucial for our discussion (see Theorem 3.6 and Corollary 3.7 in [3]).

Theorem 2.5. *Let $q \in V$ be an arbitrary vertex. We can take the set of all the q-reduced divisors on G as a complete system of representatives of* $\mathrm{Pic}(G)$*. Namely, for any $D \in \mathrm{Div}(G)$, there exists a unique q-reduced divisor which is linearly equivalent to D. Further, if D is q-reduced, then*

$$r(D) \geq 0 \iff D(q) \geq 0.$$

Remark 2.6. Let $D \in \mathrm{Div}(G)_q$ be a q-reduced divisor. If $D(q) \geq 0$, then $D - (D(q)+1)q = \sum_{v \neq q} D(v)v - q$ is also q-reduced and its coefficient of q is negative. Thus we have $r(D) \leq D(q)$.

For any $E \in \mathrm{Div}^s_+(G)$, there uniquely exists $E' \in \mathrm{Div}^s_+(G)_q$ such that $E \sim E'$. Thus we see that

$$r(D) \geq s \iff L(D - E) \neq \varnothing, \quad \forall E \in \mathrm{Div}^s_+(G)_q.$$

2.2 Contraction of a bridge

We note here that a contraction of a bridges does not affect to the structure of the Picard groups, degrees and ranks of divisors. We begin with a simple fact.

Lemma 2.7. *If $e = xy \in E$ is a bridge of G, then $x - y \in$* $\mathrm{Prin}(G)$*. More generally, if there is a unique path between x and y, then $x - y \in$* $\mathrm{Prin}(G)$

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the connected components of $G - e$ such that $x \in V_1$ and $y \in V_2$. If we take a function $f \in \mathcal{M}(G)$ as

$$f(v) := \begin{cases} 1 & v \in V_1, \\ 0 & v \in V_2, \end{cases}$$

then we have $\Delta(f) = x - y$. □

Corollary 2.8. *If G is not a tree, then $r(v) = 0$ for every $v \in V$.*

Proof. Suppose that there exists a vertex $v \in V$ such that $r(v) = 1$. Since $\mathrm{Div}^0_+(G) = \{0\}$, we have $v - w \sim 0$ for any $w \in V$. Hence, for any $w, w' \in V$, we have $w - w' = (v - w') - (v - w) \in \mathrm{Prin}(G)$. This implies that $\mathrm{Prin}(G) = \mathrm{Div}^0(G)$, or G is a tree. $\square$

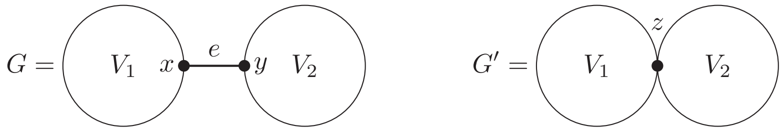
If e is a bridge of G, then the number of spanning trees (or the order of the Jacobian group) is invariant under the contraction of e, i.e. $|\mathrm{Jac}(G)| = |\mathrm{Jac}(G/e)|$. More precisely, this operation preserves the structure of the Picard group as well as degrees and ranks of divisors.

Theorem 2.9. *If e is a bridge of G, then there exists a group isomorphism $\overline{\varphi}\colon \mathrm{Pic}(G) \to \mathrm{Pic}(G/e)$ which preserves degree and rank.*

Proof. Let $G = (V, E)$ be a connected graph, $e = xy \in E$ $(x, y \in V)$ be a bridge of G, and set $G' = (V', E') = G/e$. We take $V_1, V_2 \subset V$ and $z \in V'$ so that

$$V = V_1 \sqcup \{x\} \sqcup \{y\} \sqcup V_2, \qquad V' = V_1 \sqcup \{z\} \sqcup V_2$$

are disjoint unions (see the figure below).



We denote by $d'(v)$, $\nu'(v, w)$ and $r'(D)$ the degree of the vertex $v \in V'$, the number of edges in G' joining $v, w \in V'$ and the rank of $D \in \mathrm{Div}(G')$ respectively. Define $\varphi\colon \mathrm{Div}(G) \to \mathrm{Div}(G')$ by

$$\varphi\colon \mathrm{Div}(G) \ni D = \sum_{v \in V} D(v)v \mapsto \sum_{v \in V_1 \cup V_2} D(v)v + (D(x) + D(y))z \in \mathrm{Div}(G').$$

It is immediate to see that φ preserves the degrees of the divisors. This map φ is apparently surjective, and

$$\ker \varphi = \mathbb{Z}(x - y) = \{k(x - y) \mid k \in \mathbb{Z}\} < \mathrm{Prin}(G)$$

by Lemma 2.7. Thus φ induces the isomorphism

$$\overline{\varphi}\colon \mathrm{Pic}(G) \ni [D] \mapsto [\varphi(D)] \in \mathrm{Pic}(G').$$

It is easy to check that φ gives a bijection between the set of all x-reduced divisors on G and that of all z-reduced divisors on G' (notice that $D(y) = 0$ if $D \in \mathrm{Div}(G)$ is x-reduced). Furthermore, we have

$$r(D) \geq k \iff r'(\varphi(D)) \geq k$$

for $D \in \mathrm{Div}(G)$ and $k \geq 0$. $\square$

By this theorem, we can restrict our attention to the 2-*edge-connected graphs* (i.e. graphs which have no bridges) without loosing generality when we consider the two-variable zeta functions introduced in the next section (see Lemma 3.5).

3 Two-variable zeta functions of graphs

For $D \in \mathrm{Div}(G)$, we put

$$h(D) \coloneqq r(D) + 1.$$

Notice that we have $h(D) = 0$ when $\deg D < 0$, and by Theorem 2.2, we have

$$h(D) = \deg D - g + 1$$

when $\deg D > 2g - 2$.

Lorenzini [5] introduced the two-variable zeta function of G as follows.

Definition 3.1 (two-variable zeta function of G)**.** Define

$$Z(G, t, u) \coloneqq \sum_{[D] \in \mathrm{Pic}(G)} \frac{u^{h(D)} - 1}{u - 1} t^{\deg D} = \sum_{i=0}^{\infty} b_i(G, u) t^i,$$

where we put

$$b_i(G, u) \coloneqq \sum_{[D] \in \mathrm{Pic}^i(G)} \frac{u^{h(D)} - 1}{u - 1}$$

for brevity.

Remark 3.2. In fact, in [5], Lorenzini defines a zeta function for a corank-one lattice $\Lambda \subset \mathbb{Z}^n$ equipped with a function $r \colon \Lambda \to \mathbb{Z}$ satisfying an analog of the Riemann-Roch theorem in general.

Remark 3.3. For a smooth projective curve C over the finite field $\mathbb{F}_q$ with q elements, the *local zeta function* (or *congruent zeta function*) of C is defined by

$$\zeta(C/\mathbb{F}_q, s) \coloneqq \sum_{D \geq 0} q^{-s \deg D},$$

where the sum is taken over all effective divisors D on C. It is known that

$$\zeta(C/\mathbb{F}_q, s) = \sum_{i \geq 0} \Big(\sum_{\substack{[D] \in \mathrm{Pic}(C) \\ \deg D = i}} \frac{q^{h(D)} - 1}{q - 1} \Big) T^i,$$

where $T = q^{-s}$ and $h(D) = \dim_{\mathbb{F}_q} L(D)$ is the dimension of the linear system of D.

The following is the basic facts on the zeta functions.

Theorem 3.4 (Lorenzini [5])**.** (1) *There exists a polynomial $L(G, t, u) \in \mathbb{Z}[t, u]$ such that*

$$Z(G, t, u) = \frac{L(G, t, u)}{(1 - t)(1 - ut)}. \tag{3.1}$$

(2) *$Z(G, t, u)$ satisfies the functional equation*

$$Z(G, 1/ut, u) = (ut^2)^{1-g} Z(G, t, u). \tag{3.2}$$

(3) $L(G,0,u)=1,\ L(G,1,u)=|\mathrm{Jac}(G)|$.

(4) $L(G,t,0)=t^gT(G,1,1/t)$.

By Theorem 2.9, we obtain the

Lemma 3.5. *If e is a bridge of G, then $Z(G,t,u)=Z(G/e,t,u)$.*

Put $N=|\mathrm{Jac}(G)|$. Notice that

$$b_i(G,u)=N\frac{u^{i-g+1}-1}{u-1}$$

when $i>2g-2$. Hence

$$\begin{aligned}Z(G,t,u)&=\sum_{i=0}^{2g-2}b_i(G,u)t^i+\frac{Nt^{2g-1}}{u-1}\Big(\frac{u^g}{1-ut}-\frac{1}{1-t}\Big)\\&=\sum_{i=0}^{2g-2}b_i(G,u)t^i-N\sum_{i=1}^{g-1}\frac{u^i-1}{u-1}t^{i+g-1}+\frac{Nt^g}{(1-t)(1-ut)}.\end{aligned}$$

By Theorem 2.2, we see that

$$b_{2g-2-i}(G,u)=u^{g-1-i}b_i(G,u)+N\frac{u^{g-1-i}-1}{u-1}$$

for $0\le i\le g-1$, it follows that

$$Z(G,t,u)=\sum_{i=0}^{g-1}b_i(G,u)t^i+\sum_{i=1}^{g-1}u^ib_{g-1-i}(G,u)t^{g-1+i}+\frac{Nt^g}{(1-t)(1-ut)}. \tag{3.3}$$

Thus, the zeta function $Z(G,t,u)$ is determined by the polynomials $b_i(G,u)$ ($i=0,1,\dots,g-1$). This implies that the larger the genus g of G, the harder it may become to compute the zeta function $Z(G,t,u)$.

We see that $b_0(G,u)=1$ in general. If G is 2-edge-connected, then we have $b_1(G,u)=|V|$. Thus the two-variable zeta function of a 2-edge connected graph whose genus is at most 2 is completely determined by the numbers of its vertices and spanning trees as we show in the examples below.

Example 3.6 ($g=0$)**.** If G is a tree, then we readily have

$$Z(G,t,u)=\frac{1}{(1-t)(1-ut)}$$

by (3.3). This is an analog of the local zeta function for the projective line $\mathbb{P}^1$.

Example 3.7 ($g=1$)**.** If $G=\mathcal{C}_n$ is a cycle graph with n vertices, then we have

$$Z(G,t,u)=1+\frac{nt}{(1-t)(1-ut)}=\frac{1+(n-u-1)t+ut^2}{(1-t)(1-ut)}$$

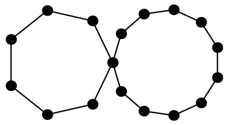
by (3.3). This is an analog of the local zeta function for an elliptic curve.

Example 3.8 ($g = 2$)**.** If $g = 2$ and G is 2-edge connected, then we have

$$\begin{aligned} Z(G,t,u) &= 1 + nt + ut^2 + \frac{Nt^2}{(1-t)(1-ut)} \\ &= \frac{1 + (n-u-1)t + (N-n+2u-nu)t^2 + u(n-u-1)t^3 + u^2t^4}{(1-t)(1-ut)} \end{aligned}$$

by (3.3), where $n = |V|$ and $N = |\mathrm{Jac}(G)|$.

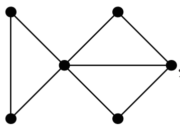
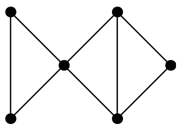
Remark 3.9. Let $G = \mathcal{C}_m + \mathcal{C}_n$ be a *vertex sum*[1] of two cycle graphs $\mathcal{C}_m$ and $\mathcal{C}_n$, that is, G is a graph obtained by gluing $\mathcal{C}_m$ and $\mathcal{C}_n$ at one vertex (see the figure below for the case $m = 7$, $n = 11$).



It is easy to see that $\mathrm{Jac}(G) \cong \mathrm{Jac}(\mathcal{C}_m) \times \mathrm{Jac}(\mathcal{C}_n)$ and $T(G,x,y) = T(\mathcal{C}_m,x,y)T(\mathcal{C}_n,x,y)$. On the other hand, an extra term appears in the relation between (the numerators of) the zeta functions:

$$L(G,t,u) = L(\mathcal{C}_m,t,u)L(\mathcal{C}_n,t,u) + ut(1-t)(1-ut).$$

In general, if $G = G_1 + G_2$ is a vertex sum of two graphs G_1 and G_2, then we have $\mathrm{Jac}(G) \cong \mathrm{Jac}(G_1) \times \mathrm{Jac}(G_2)$ and $T(G,x,y) = T(G_1,x,y)T(G_2,x,y)$, but the relation between the zeta functions is rather complicated. In general, $Z(G,t,u)$ is not determined by $Z(G_1,t,u)$ and $Z(G_2,t,u)$ alone, but depends on which vertex G_1 and G_2 are glued at. For instance (see Example 3.1 in [2]), the following two graphs

$G_1 =$ , $G_2 =$ 

have the same Tutte polynomials and isomorphic Jacobian groups, but

$$Z(G_2,t,u) = Z(G_1,t,u) + ut^2.$$

However, we can show that

$$L(G,t,u) \equiv L(G_1,t,u)L(G_2,t,u) \pmod{ut(1-t)(1-ut)} \tag{3.4}$$

holds in $\mathbb{Z}[t,u]$. See §4.3.

[1] To be precise, the vertices to be glued to define the vertex sum should be indicated, but the choice of such vertices is irrelevant in the present context, so we omit it.

4 A formula for two-variable zeta functions

4.1 A formula for two-variable zeta functions

Let us prepare a formula for $Z(G,t,u)$ which is useful in our calculation. By rewriting the summand in $b_i(G,u)$ as

$$\frac{u^{h(D)}-1}{u-1} = \sum_{k=0}^{r(D)} u^k = \sum_{k=0}^{\infty} \delta(r(D) \geq k) u^k,$$

we have

$$\begin{aligned} Z(G,t,u) &= \sum_{i=0}^{\infty} \sum_{D \in \mathrm{Div}^i(G)_q} \sum_{k=0}^{\infty} \delta(r(D) \geq k) u^k t^i \\ &= \sum_{i=0}^{\infty} \sum_{D \in \mathrm{Div}^0(G)_q} \sum_{k=0}^{\infty} \delta(r(D+iq) \geq k) u^k t^i. \end{aligned}$$

Here we use the fact that $\mathrm{Div}^i(G)_q = \mathrm{Div}^0(G)_q + iq$ in the second equality. For a divisor $D \in \mathrm{Div}(G)$, we denote by $\mathrm{Red}_q(D)$ the unique q-reduced divisor which is linearly equivalent to D. We see that

$$\begin{aligned} & r(D+iq) \geq k \\ \Longleftrightarrow\ & r((D+iq) - (D'+kq)) \geq 0,\ \forall D' \in \mathrm{Div}^0(G)_q, D'(q)+k \geq 0 \\ \Longleftrightarrow\ & \mathrm{Red}_q(D-D')(q) + i - k \geq 0,\ \forall D' \in \mathrm{Div}^0(G)_q, D'(q)+k \geq 0. \end{aligned}$$

Define

$$\mu_k(D) := \max\bigl\{-\mathrm{Red}_q(D-D')(q) \bigm| D' \in \mathrm{Div}^0(G)_q, D'(q)+k \geq 0\bigr\}$$

for $k \geq 0$ and $D \in \mathrm{Div}^0(G)$. By definition, we have

$$\mu_0(D) \leq \mu_1(D) \leq \mu_2(D) \leq \dots,$$

and

$$\mu_0(D) = \max\bigl\{-\mathrm{Red}_q(D-D')(q) \bigm| D' \in \mathrm{Div}^0(G)_q, D'(q) \geq 0\bigr\} = -D(q) \geq 0 \quad (4.1)$$

since $\mathrm{Div}^0(G)_q = \{0\}$. Notice that

$$r(D+iq) \geq k \iff \mu_k(D) \leq i-k.$$

Therefore we have

$$\begin{aligned} Z(G,t,u) &= \sum_{i=0}^{\infty} \sum_{D \in \mathrm{Div}^0(G)_q} \sum_{k=0}^{\infty} \delta(\mu_k(D) \leq i-k) u^k t^i \\ &= \sum_{D \in \mathrm{Div}^0(G)_q} \sum_{k=0}^{\infty} u^k \sum_{i=0}^{\infty} \delta(\mu_k(D) \leq i-k) t^i. \end{aligned}$$

Thus we obtain the following formula for $Z(G,t,u)$.

Theorem 4.1. *We have*

$$Z(G,t,u) = \frac{1}{1-t} \sum_{D\in \mathrm{Div}^0(G)_q} \sum_{k=0}^{\infty} u^k t^{k+\mu_k(D)}. \tag{4.2}$$

By this formula, the computation of $Z(G,t,u)$ is reduced to that of $\mu_k(D)$ for each q-reduced divisor $D \in \mathrm{Div}^0(G)_q$ of degree 0.

4.2 Remarks on Theorem 4.1

We denote by $\mathrm{Crit}(G)_q$ the set of all *critical configurations* on G with sink q having $2|E| - |V|$ chips. Let $D \in \mathrm{Div}^0(G)_q$ be a q-reduced divisor of degree 0. If we define θ by $\theta(v) := d(v) - 1 - D(v)$ for $v \in V$, then $\theta \in \mathrm{Crit}(G)_q$. This gives a bijection between $\mathrm{Div}^0(G)_q$ and $\mathrm{Crit}(G)_q$.

It is known by Merino [6] that if we put

$$\mathrm{level}(\theta) := \sum_{v\neq q} \theta(v) - |E| + d(q)$$

for $\theta \in \mathrm{Crit}(G)_q$, then we have

$$0 \leq \mathrm{level}(\theta) \leq g$$

and

$$\sum_{\theta\in \mathrm{Crit}(G)_q} y^{\mathrm{level}(\theta)} = T(G,1,y). \tag{4.3}$$

From this, it follows that $0 \leq -D(q) \leq g$ since $\mathrm{level}(\theta) = D(q) + g$ when θ and D are related as above. Hence we have

$$0 \leq \mu_k(D) \leq g \tag{4.4}$$

for $D \in \mathrm{Div}^0(G)_q$. Furthermore, if $k \geq g$, then $D'(q) + k \geq D'(q) + g \geq 0$ for any $D' \in \mathrm{Div}^0(G)_q$. This implies that when $k \geq g$, we have

$$\begin{aligned} \mu_k(D) &= \max\bigl\{-\mathrm{Red}_q(D - D')(q) \bigm| D' \in \mathrm{Div}^0(G)_q\bigr\} \\ &= \max\bigl\{-D'(q) \bigm| D' \in \mathrm{Div}^0(G)_q\bigr\} = g. \end{aligned}$$

Let

$$L_k(G,t) := \sum_{D\in \mathrm{Div}^0(G)_q} t^{\mu_k(D)} \tag{4.5}$$

be the generating function of $\mu_k(D)$ for $D \in \mathrm{Div}^0(G)_q$. We notice that $L_k(G,t) = |\mathrm{Jac}(G)|t^g$ if $k \geq g$. It follows that

$$Z(G,t,u) = \frac{1}{1-t} \sum_{k=0}^{\infty} (ut)^k L_k(G,t), \tag{4.6}$$

so that the polynomial $L(G,t,u)$ in Theorem 3.4 is given by

$$L(G,t,u) = (1-t)(1-ut)Z(G,t,u)$$

$$= \sum_{k=0}^{\infty} (ut)^k L_k(G,t) - \sum_{k=0}^{\infty} (ut)^{k+1} L_k(G,t)$$
$$= L_0(G,t) + \sum_{k=1}^{\infty} (ut)^k \big(L_k(G,t) - L_{k-1}(G,t)\big)$$
$$= t^g T(G,1,1/t) + \sum_{k=1}^{g} (ut)^k \big(L_k(G,t) - L_{k-1}(G,t)\big). \tag{4.7}$$

Here $L_0(G,t) = t^g T(G,1,1/t)$ is immediate from (4.3) and (4.1). This recovers (4) of Theorem 3.4 by letting $u = 0$. We also note that

$$L(G,t,1/t) = L_0(G,t) + \sum_{k=1}^{g} \big(L_k(G,t) - L_{k-1}(G,t)\big) = L_g(G,t) = |\mathrm{Jac}(G)|t^g. \tag{4.8}$$

4.3 Remarks on vertex sums of two graphs

Let G be a vertex sum of two graphs G_1 and G_2, and g, g_1, g_2 be the genera of G, G_1, G_2 respectively. Notice that $g = g_1 + g_2$. Since

$$t^g T(G,1,1/t) = t^{g_1} T(G_1,1,1/t) t^{g_2} T(G_2,1,1/t),$$

we see that

$$L(G,t,u) \equiv L(G_1,t,u) L(G_2,t,u) \pmod{ut}$$

by (4.7), and

$$L(G,1,u) - L(G_1,1,u)L(G_2,1,u) = |\mathrm{Jac}(G)| - |\mathrm{Jac}(G_1)||\mathrm{Jac}(G_2)| = 0,$$
$$L(G,t,1/t) - L(G_1,t,1/t)L(G_2,t,1/t) = |\mathrm{Jac}(G)|t^g - |\mathrm{Jac}(G_1)|t^{g_1}|\mathrm{Jac}(G_2)|t^{g_2} = 0$$

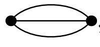
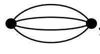
by (3) of Theorem 3.4 and (4.8), which imply that $L(G,t,u) - L(G_1,t,u)L(G_2,t,u)$ is divisible by $1-t$ and $1-ut$. Since ut, $1-t$ and $1-ut$ are relatively prime in $\mathbb{Z}[t,u]$, we obtain the congruence (3.4).

5 Examples of two-variable zeta functions

In this section we consider three examples of infinite families of graphs: dipole graphs, doubled trees, friendship graphs. For each family, we give an explicit formula of the two-variable zeta functions. In view of the question proposed by Lorenzini [5] as to whether it is possible for two connected graphs having the same Tutte polynomials to have the different zeta functions or non-isomorphic Jacobians, and the negative answer to this by Clancy-Leake-Payne [2], we include the Tutte polynomials and Jacobians in a remark for each example.

5.1 Dipole graphs

For each positive integer m, let $\mathcal{D}_m$ be the *dipole graph* of size m, that is, a graph with two vertices which are connected by m edges:

$\mathcal{D}_1 =$, $\mathcal{D}_2 =$ , $\mathcal{D}_3 =$ , $\mathcal{D}_4 =$ , $\ldots$

The graph $\mathcal{D}_m$ is the simplest graph whose genus is $m-1$, as well as the simplest graph which is *m-edge-connected*.

Remark 5.1. The Tutte polynomial and the Jacobian group of $\mathcal{D}_m$ are given by

$$T(\mathcal{D}_m, x, y) = x + y + y^2 + \cdots + y^{m-1}, \quad \mathrm{Jac}(\mathcal{D}_m) \cong \mathbb{Z}/m\mathbb{Z}.$$

Example 5.2.

$$\begin{aligned}
Z(\mathcal{D}_1, t, u) &= \frac{1}{(1-t)(1-ut)}, \\
Z(\mathcal{D}_2, t, u) &= 1 + \frac{2t}{(1-t)(1-ut)}, \\
Z(\mathcal{D}_3, t, u) &= 1 + 2t + ut^2 + \frac{3t^2}{(1-t)(1-ut)}, \\
Z(\mathcal{D}_4, t, u) &= 1 + 2t + (3+u)t^2 + 2ut^3 + u^2t^4 + \frac{4t^3}{(1-t)(1-ut)}.
\end{aligned}$$

Let $V = V(\mathcal{D}_m) = \{v, q\}$. We see that $mv - mq$ generates $\mathrm{Prin}(\mathcal{D}_m)$. We define

$$D(a) := a(v-q)$$

for $a \in \mathbb{Z}$. It is readily seen that

$$\mathrm{Div}^0(\mathcal{D}_m)_q = \{D(a) \mid 0 \le a < m\}.$$

Lemma 5.3. *For $k \ge 0$, we have*

$$\mu_k(D(a)) = \max\{(a-b) \bmod m \mid 0 \le b \le \min\{m-1, k\}\} = \begin{cases} a & k \le a, \\ m-1 & k > a. \end{cases} \tag{5.1}$$

Proof. We see that

$$D(a) - D(b) = (a-b)(v-q) \sim D((a-b) \bmod m)$$

for any $0 \le b < m$. Hence

$$\begin{aligned}
\mu_k(D(a)) &= \max\{-\mathrm{Red}_q(D(a) - D(b))(q) \mid 0 \le b < m, -b + k \ge 0\} \\
&= \max\{(a-b) \bmod m \mid 0 \le b \le \min\{m-1, k\}\}
\end{aligned}$$

as desired. □

Theorem 5.4. *The zeta function of $\mathcal{D}_m$ is given by*

$$Z(\mathcal{D}_m, t, u) = \frac{1}{(1-t)^2(1-ut^2)} - \frac{(1-u)t^m}{(1-t)^2(1-ut)^2} - \frac{u(ut^2)^m}{(1-ut)^2(1-ut^2)}. \tag{5.2}$$

Proof. We compute the zeta function via the expression (4.2):

$$Z(\mathcal{D}_m, t, u) = \frac{1}{1-t} \sum_{0 \le a < m} \sum_{k=0}^{\infty} u^k t^{k+\mu_k(D(a))}.$$

For a fixed a,

$$\sum_{k=0}^{\infty} u^k t^{k+\mu_k(D(a))} = \sum_{k=0}^{a} u^k t^{k+a} + \sum_{k=a+1}^{\infty} u^k t^{k+m-1}$$
$$= \frac{t^a - ut(ut^2)^a + ut^m(ut)^a}{1-ut}.$$

Hence we get

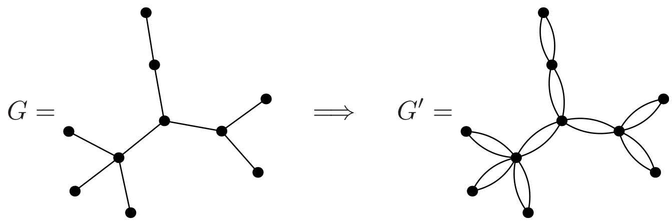
$$\begin{aligned}
&Z(\mathcal{D}_m, t, u) \\
&= \frac{1}{(1-t)(1-ut)} \sum_{0\le a<m} \Big(t^a - ut(ut^2)^a + ut^m(ut)^a\Big) \\
&= \frac{1}{(1-t)(1-ut)} \left(\frac{1-t^m}{1-t} - ut\frac{1-(ut^2)^m}{1-ut^2} + ut^m\frac{1-(ut)^m}{1-ut}\right) \\
&= \frac{1}{(1-t)(1-ut)} \left(\frac{1-ut}{(1-t)(1-ut^2)} - \frac{(1-u)t^m}{(1-t)(1-ut)} - \frac{u(1-t)(ut^2)^m}{(1-ut)(1-ut^2)}\right) \\
&= \frac{1-ut}{(1-t)^2(1-ut^2)} - \frac{(1-u)t^m}{(1-t)^2(1-ut)^2} - \frac{u(ut^2)^m}{(1-ut)^2(1-ut^2)}.
\end{aligned}$$

□

5.2 Doubled trees

Fix a positive integer m, and let G be a tree with m edges and $m+1$ vertices. Let G' be a graph which is obtained from G by replacing all its edges to double edges. Then the genus of G' is m. Notice that $2v \sim 2w$ in $\mathrm{Div}(G')$ for any $v, w \in V(G') = V(G)$.

Example 5.5. For instance,



Remark 5.6. The Tutte polynomial and the Jacobian group of G' are given by

$$T(G'; x, y) = (x+y)^m, \quad \mathrm{Jac}(G') \cong (\mathbb{Z}/2\mathbb{Z})^m.$$

Fix a vertex q of G'. For each $S \subset V \setminus \{q\}$, define

$$D(S) := \sum_{v\in S} (v-q).$$

Then we have

$$\mathrm{Div}^0(G')_q = \{D(S) \mid S \subset V \setminus \{q\}\}.$$

Lemma 5.7. *For $k \geq 0$,*

$$\mu_k(D(S)) = \min\{|S| + k, m\}. \tag{5.3}$$

Proof. Since

$$\begin{aligned} D(S) - D(T) &= \sum_{v \in S}(v - q) - \sum_{v \in T}(v - q) \\ &= \sum_{v \in S \triangle T}(v - q) - 2\sum_{v \in T \setminus S}(v - q) \sim \sum_{v \in S \triangle T}(v - q), \end{aligned}$$

we see that

$$\mu_k(D(S)) = \max\{|S \triangle T| \mid T \subset V \setminus \{q\},\ |T| \leq k\}.$$

If $k \leq m - |S|$, then we can take $T \subset V \setminus \{q\}$ such that $S \cap T = \varnothing$ and $|T| = k$, which attains the maximum $|S| + k$ of $|S \triangle T|$. Otherwise, we see that $\mu(S, k) = m$. Thus we get

$$\mu_k(D(S)) = \min\{|S| + k, m\}$$

as desired. □

Theorem 5.8. *The zeta function of G' is given by*

$$Z(G', t, u) = \frac{1}{1 - ut^2}\left(\frac{ut^{m+1}(1 + ut)^m}{1 - ut} + \frac{(1 + t)^m}{1 - t}\right). \tag{5.4}$$

Proof. We compute the zeta function via the expression (4.2):

$$Z(G', t, u) = \frac{1}{1 - t}\sum_{S \subset V \setminus \{q\}}\sum_{k=0}^{\infty} u^k t^{k + \mu_k(D(S))}.$$

By Lemma 5.7, we have

$$\begin{aligned} \sum_{k=0}^{\infty} u^k t^{k + \mu_k(D(S))} &= \sum_{k=0}^{m-s+1} u^k t^{2k+s} + \sum_{k=m-s}^{\infty} u^k t^{k+m} \\ &= t^s\frac{1 - (ut^2)^{m-s}}{1 - ut^2} + \frac{t^m(ut^2)^{m-s}}{1 - ut} \end{aligned}$$

for each $S \subset V \setminus \{q\}$ with $s = |S|$. Since there are $\binom{m}{s}$ subsets of $V \setminus \{q\}$ whose cardinalities are s, we get

$$\begin{aligned} Z(G', t, u) &= \frac{1}{1 - t}\sum_{s=0}^{m}\binom{m}{s}\left(t^s\frac{1 - (ut^2)^{m-s}}{1 - ut^2} + \frac{t^m(ut^2)^{m-s}}{1 - ut}\right) \\ &= \frac{1}{1 - t}\left(\frac{(1 + t)^m - (t + ut^2)^m}{1 - ut^2} + \frac{t^m(1 + ut^2)^m}{1 - ut}\right) \\ &= \frac{1}{1 - ut^2}\left(\frac{ut^{m+1}(1 + ut)^m}{1 - ut} + \frac{(1 + t)^m}{1 - t}\right). \end{aligned}$$ □

Remark 5.9. Let G_1, G_2 be trees with m_1 and m_2 edges respectively, G'_1, G'_2 be the corresponding doubled trees, and $G' = G'_1 + G'_2$ be a vertex sum of G'_1 and G'_2. Notice that G' is a doubled tree obtained from the corresponding vertex sum of G_1 and G_2, that is, $G' = (G_1 + G_2)'$. We have

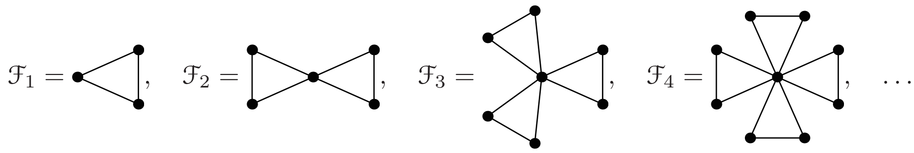
$$L(G', t, u) = L(G'_1, t, u)L(G'_2, t, u) + ut(1-t)(1-ut)R_{m_1}(t, u)R_{m_2}(t, u)$$

with

$$R_m(t, u) = \frac{t^m(1+ut)^m - (1+t)^m}{1 - ut^2}.$$

5.3 Friendship graphs

The *friendship graph* $\mathcal{F}_m$ is a *simple* graph constructed by gluing m triangles $\mathcal{C}_3$ at a common one vertex:



$\mathcal{F}_m$ has $2m+1$ vertices, $3m$ edges and 3^m spanning trees, and the genus of $\mathcal{F}_m$ is m.

Remark 5.10. The Tutte polynomial and the Jacobian group of $\mathcal{F}_m$ are given by

$$T(\mathcal{F}_m, x, y) = (x^2 + x + y)^m, \quad \operatorname{Jac}(\mathcal{F}_m) \cong (\mathbb{Z}/3\mathbb{Z})^m.$$

Let us set $V = \{q, v_1, w_1, \ldots, v_m, w_m\}$, where each $\{q, v_j, w_j\}$ forms a sub-triangle. Notice that

$$2v_j \sim w_j + q, \qquad 2w_j \sim v_j + q.$$

Example 5.11.

$$\begin{aligned}
Z(\mathcal{F}_1, t, u) &= 1 + \frac{3t}{(1-t)(1-ut)}, \\
Z(\mathcal{F}_2, t, u) &= 1 + 5t + ut^2 + \frac{3^2t^2}{(1-t)(1-ut)}, \\
Z(\mathcal{F}_3, t, u) &= 1 + 7t + (19+u)t^2 + 7ut^3 + u^2t^4 + \frac{3^3t^3}{(1-t)(1-ut)}, \\
Z(\mathcal{F}_4, t, u) &= 1 + 9t + (33+u)t^2 + (65+9u)t^3 + u(33+u)t^4 \\
&\qquad + 9u^2t^5 + u^3t^6 + \frac{3^4t^4}{(1-t)(1-ut)}.
\end{aligned}$$

The q-reduced divisors of degree 0 on $\mathcal{F}_m$ are

$$\operatorname{Div}^0(\mathcal{F}_m)_q = \{D(A, B) \mid A, B \subset [m],\ A \cap B = \varnothing\},$$

where

$$D(A, B) := \sum_{a \in A}(v_a - q) + \sum_{b \in B}(w_b - q), \qquad A, B \subset [m].$$

Notice that $-D(A, B)(q) = |A| + |B|$.

Lemma 5.12. *For $k \geq 0$,*

$$\mu_k(D(A,B)) = \min\{|A| + |B| + k, m\}. \tag{5.5}$$

Proof. We have

$$D(A,B) - D(A',B') \sim D(A'',B'')$$

with

$$\begin{aligned} A'' &= (A \setminus (A' \cup B')) \sqcup (B' \setminus (A \cup B)) \sqcup (A' \cap B), \\ B'' &= (B \setminus (A' \cup B')) \sqcup (A' \setminus (A \cup B)) \sqcup (A \cap B'). \end{aligned}$$

Notice that $A'' \cap B'' = \varnothing$ and $A'' \cup B'' = (A \cup B \cup A' \cup B') \setminus ((A \cap A') \cup (B \cap B'))$. Then

$$-D(A'',B'')(q) = |A \cup B \cup A' \cup B'| - |A \cap A'| - |B \cap B'|.$$

This attains the maximum if we take A' and B' such that $A' \cup B'$ is a maximal subset of $[m] \setminus (A \cup B)$, and then $-D(A'',B'')(q) = \min\{|A| + |B| + k, m\}$. ☐

Theorem 5.13. *The zeta function of $\mathcal{F}_m$ $(m \geq 1)$ is given by*

$$Z(\mathcal{F}_m, t, u) = \frac{(1+2t)^m}{(1-t)(1-ut^2)} + \frac{ut^{m+1}(2+ut)^m}{(1-ut)(1-ut^2)}. \tag{5.6}$$

Proof. We compute the zeta function via the expression (4.2):

$$Z(\mathcal{F}_m, t, u) = \frac{1}{1-t} \sum_{\substack{A,B \subset [m] \\ A \cap B = \varnothing}} \sum_{k=0}^{\infty} u^k t^{k+\mu_k(D(A,B))}.$$

By Lemma 5.12, we have

$$\begin{aligned} \sum_{k=0}^{\infty} u^k t^{k+\mu_k(D(A,B))} &= \sum_{k=0}^{\infty} u^k t^{k+\min\{s+k,m\}} \\ &= \sum_{k=0}^{m-s-1} u^k t^{2k+s} + \sum_{k=m-s}^{\infty} u^k t^{k+m} \\ &= t^s \frac{1-(ut^2)^{m-s}}{1-ut^2} + \frac{t^m (ut)^{m-s}}{1-ut} \end{aligned}$$

for each pair (A,B) of disjoint subsets of $[m]$ with $s = |A| + |B|$. Since there are $2^s\binom{m}{s}$ such pairs, we get

$$\begin{aligned} Z(\mathcal{F}_m, t, u) &= \frac{1}{1-t} \sum_{s=0}^{m} 2^s \binom{m}{s} \left(t^s \frac{1-(ut^2)^{m-s}}{1-ut^2} + \frac{t^m (ut)^{m-s}}{1-ut} \right) \\ &= \frac{1}{(1-t)(1-ut^2)} \sum_{s=0}^{m} \binom{m}{s} (2t)^s \end{aligned}$$

$$- \frac{ut^{m+1}}{(1-ut)(1-ut^2)} \sum_{s=0}^{m} \binom{m}{s} 2^s (ut)^{m-s}$$

$$= \frac{(1+2t)^m}{(1-t)(1-ut^2)} - \frac{ut^{m+1}(2+ut)^m}{(1-ut)(1-ut^2)}. \qquad \square$$

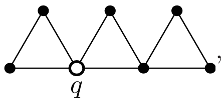
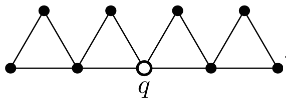
Remark 5.14. We can regard $\mathcal{F}_{m_1+m_2} = \mathcal{F}_{m_1} + \mathcal{F}_{m_2}$ by gluing $\mathcal{F}_{m_1}$ and $\mathcal{F}_{m_2}$ at q. We have

$$L(\mathcal{F}_{m_1+m_2}, t, u) = L(\mathcal{F}_{m_1}, t, u)L(\mathcal{F}_{m_2}, t, u) + ut(1-t)(1-ut)R_{m_1}(t, u)R_{m_2}(t, u)$$

with

$$R_m(t, u) = \frac{t^m(2+ut)^m - (1+2t)^m}{1-ut^2}.$$

Remark 5.15. Let us consider the graphs

$\mathcal{F}'_3 =$ , $\quad \mathcal{F}'_4 =$ .

We can regard $\mathcal{F}'_3 = \mathcal{F}_1 + \mathcal{F}_2$ and $\mathcal{F}'_4 = \mathcal{F}_2 + \mathcal{F}_2$ (by gluing at q in the figure above). We have

$$\begin{aligned} Z(\mathcal{F}'_3, t, u) &= 1 + 7t + 19t^2 + 7ut^3 + u^2t^4 + \frac{3^3t^3}{(1-t)(1-ut)} \\ &= Z(\mathcal{F}_3, t, u) - ut^2, \\ Z(\mathcal{F}'_4, t, u) &= 1 + 9t + 33t^2 + (65 + 2u)t^3 + 33ut^4 \\ &\qquad + 9u^2t^5 + u^3t^6 + \frac{3^4t^4}{(1-t)(1-ut)} \\ &= Z(\mathcal{F}_4, t, u) - ut^2 - 7ut^3 - u^2t^4. \end{aligned}$$

5.4 Observations and problems

We can restate the calculation results obtained above in a slightly different manner as follows:

$$Z(\mathcal{D}_m, t, u) = \frac{1-t^m}{(1-t)^2(1-ut^2)} + \frac{ut^m(1-(ut)^m)}{(1-ut)^2(1-ut^2)}, \tag{5.2$'$}$$

$$Z(G', t, u) = \frac{(1+t)^m}{(1-t)(1-ut^2)} + \frac{ut^{m+1}(1+ut)^m}{(1-ut)(1-ut^2)}, \tag{5.4$'$}$$

$$Z(\mathcal{F}_m, t, u) = \frac{(1+2t)^m}{(1-t)(1-ut^2)} + \frac{ut^{m+1}(2+ut)^m}{(1-ut)(1-ut^2)}. \tag{5.6$'$}$$

We notice that these are written in a unified way, that is, if G is one of the graph above, then

$$Z(G, t, u) = \frac{L_0(G, t)}{(1-t)(1-ut^2)} + \frac{ut^{g+1}T(G, 1, ut)}{(1-ut)(1-ut^2)}, \tag{5.7}$$

where g is the genus of G. It is remarkable that $Z(G,t,u)$ is written in terms of the Tutte polynomial in these cases. However, this equality (5.7) does not hold for general graphs. For instance, we have

$$\begin{aligned} Z(\mathcal{K}_4,t,u) &= 1+4t+10t^2+4ut^3+u^2t^4+\frac{16t^3}{(1-t)(1-ut)} \\ &= \frac{L_0(\mathcal{K}_4,t)}{(1-t)(1-ut^2)}+\frac{ut^{3+1}T(\mathcal{K}_4,1,ut)}{(1-ut)(1-ut^2)}-ut^2, \end{aligned}$$

where $\mathcal{K}_4$ is the complete graph with four vertices. In fact, it would be possible that G and G' have the same Tutte polynomials but $Z(G,t,u) \neq Z(G',t,u)$ (see [2]), so one cannot expect the zeta function $Z(G,t,u)$ to be expressed in terms of (several specializations of) the Tutte polynomials $T(G,x,y)$ of G alone in general. We therefore propose the following problems concerning (5.7).

Problem 5.16. Characterize the graphs whose two-variable zeta function satisfies the equation (5.7). Notice that we may restrict our consideration to 2-edge connected graphs since both $Z(G,t,u)$ and the right hand side of (5.7) are invariant under the contraction of bridges.

Problem 5.17. Find a formula for $Z(G,t,u)$ like (5.7) that holds for a broader class of graphs.

It would also be an interesting problem to find other infinite families of 2-edge connected graphs whose genera are unbounded and the two-variable zeta function of each member is determined in a closed form. The candidates that come to mind easily include complete graphs and complete bipartite graphs, wheel graphs, multiplied trees, vertex sums of dipole graphs (special cases of multiplied trees), etc.

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