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From making pairing of cyclic cohomology with K-theory for algebras with derivations to the noncommutative rotation by elliptic index theory as specialized

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Abstract

We study noncommutative geometry (NG) by Connes, as a step forward to special understanding the theory as a memory keeping exercise back to the past for a return to the future.

cohomology, cyclic cohomology, K-theory, noncommutative geometry, index theory, derivation, rotation algebra, elliptic operator

46L05, 46L06, 46L55, 46L80, 46L87, 46L85

1 Introduction

Following Connes [4] we as beginners, outsiders, or fools or not would like to study the theory of pairing of cyclic cohomology with K-theory, as of first interest, to be continued.

This is a short reviewing story like a incense burnig with shallow insight, like a pencil mightier than an apple.

This seems a path walking along the way in the middle of the bush.

This is reviewing or studying part of the theory in noncommutative geometry by Connes.

That is either notes or proofs (or sketches) of the contents notes like, together with $\star$ as an indicator, and with considerable effort to read the contents carefully and patiently for a goal ahead.

Those are like twittering and more.

The pairing of K-theory and cohomology theory is just induced by functional inserting or valuation as a duality.

The noncommutative rotation means the irrational rotation smooth or C^*-algebra as an important example in the NG theory.

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Let us begin with contents of three sections as below.

The Contents (with **References**)

2 Pairing of cyclic cohomology with K-theory for algebras

Let A be a unital algebra. Let $K_0(A)$ and $K_1(A)$ be the algebraic K-theory groups of A (cf. [10]).

By definition, $K_0(A)$ is the group by the semi-group of stable isomorphism classes of finite projective modules over A. Also, $K_1(A)$ is the quotient of the group $GL_\infty(A)$ by its commutator subgroup, where $GL_\infty(A)$ is the inductive limit of the general linear groups $GL_n(A)$ of invertible elements of the $n \times n$ matrix algebra $M_n(A)$ over A, with the embedding maps

$$GL_n(A) \ni x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A).$$

There is a pairing between the even cyclic cohomology $cH^{\mathrm{ev}}(A)$ and $K_0(A)$, and as well a pairing between the odd cyclic cohomology $cH^{\mathrm{od}}(A)$ and $K_1(A)$, where a unital algebra A is endowed with a differential structure by a derivation.

There is an equivalent approach by Karoubi [7].

The pairing satisfies

$$\langle S\varphi, e \rangle = \langle \varphi, e \rangle, \quad \varphi \in cH^*(A), e \in K_*(A).$$

Hence it is defined on $H^*(A) = cH^*(A) \otimes_{cH^*(\mathbb{C})} \mathbb{C}$.

The pairing in terms of connections and curvature is also formulated as a computational device, as done for the usual Chern character for smooth manifolds.

This implies the Morita equivalence of $cH^*(A)$ and giving an action of the ring $K_0(A)$ on $cH^*(A)$ on the case of A abelian.

$\star$ Recall some basics from Rosenberg [16].

Let A be a ring. The canonical non-unital embedding of $n \times n$ matrix algebras over A is

$$M_n(A) \to M_{n+1}(A), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

so that their inductive limit is defined as

$$M_n(A) \to M_{n+1}(A) \to \cdots \to \varinjlim M_n(A) = \cup_{n=1}^\infty M_n(A) = M_\infty(A).$$

This is a non-unital ring.

As well, the canonical unital embedding of $n \times n$ invertible matrix group over A unital is

$$GL_n(A) \to GL_{n+1}(A), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

so that their inductive limit is defined as

$$GL_n(A) \to GL_{n+1}(A) \to \cdots \to \varinjlim GL_n(A) = \cup_{n=1}^{\infty} GL_n(A) = GL_\infty(A).$$

This is a group.

Let $IP_\infty(A)$ be the set of idempotent matrices $p = p^2$ of $M_\infty(A)$.

$GL_\infty(A)$ acts on $IP_\infty(A)$ by conjugation.

Because $(apa^{-1})(apa^{-1}) = ap^2a^{-1} = apa^{-1}$.

The group $K_0(A)$ of A unital is defined to be the Grothendieck group of the semigroup $IP_\infty(A)/GL_\infty(A)$ of conjugation orbits or classes $[p]$ with diagonal sum operation $[p] \oplus [q]$

The Morita invariance holds as that $K_0(A)$ is isomorphic to $K_0(M_n(A))$ for any integer $n \geq 1$.

For A unital, let $E_n(A)$ be the subgroup of $GL_n(A)$ generated by elementary invertible matrices with 1 on the diagonal and at most one entry on the off-digaonal.

For instance,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There is the induced embedding as

$$E_n(A) \to E_{n+1}(A) \to \cdots \to \varinjlim E_n(A) = \cup_{n=1}^{\infty} E_n(A) = E_\infty(A),$$

called the group of Elementary matrices.

The group $K_1(A)$ is defined to be the quotient $GL_\infty(A)/E_\infty(A)$.

The Whitehead lemma says that the commutator subgroups of $GL_\infty(A)$ and $E_\infty(A)$ are $E_\infty(A)$, so that $E_\infty(A)$ is normal in $GL_\infty(A)$, and the quotient $GL_\infty(A)/E_\infty(A)$ is maximally abelian.

By the elementary operation, we have

$$E_\infty(A) = [E_\infty(A), E_\infty(A)] \subset [GL_\infty(A), GL_\infty(A)].$$

Moreover, for $A, B \in GL_n(A)$,

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \in GL_{2n}(A).$$

Moreover,

$$\begin{aligned} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} &= \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & A \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & -1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

with

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

It is shown by the elementary operation that any upper triangular, or lower triangular matrix with 1 on the diagonal belongs to $E_\infty(A)$.

For classes $[A], [B] \in K_1(A)$, the product $[A][B]$ is defined by $[AB]$ or $[A \oplus B]$.

Because

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$$

so that $[A \oplus B] = [AB \oplus 1] = [AB]$.

Lemma 2.1. *Assume $\varphi \in cZ^n(A)$ a cyclic cocycle $\varphi : A^{n+1} \to \mathbb{C}$ $(n+1)$-linear with $b\varphi = 0$, and let $p, q \in M_k(A)$ idempotents such that $p = uv$ and $q = vu$ for some $u, v \in M_k(A)$. Then the following cocycles*

$$\psi_1(a_0, \cdots, a_n) = (\varphi\#\mathrm{tr})(a_0, \cdots, a_n),$$
$$\psi_2(a_0, \cdots, a_n) = (\varphi\#\mathrm{tr})(va_0u, \cdots, va_nu)$$

on $\mathfrak{B} = \{x \in M_k(A) \,|\, xp = px = x\}$ differ by a coboundary, where $\varphi\#\mathrm{tr} = \varphi \otimes^\sim \mathrm{tr}$ with $M_k(A) \cong A \otimes M_k(\mathbb{C})$ and tr the canonical trace on $M_k(\mathbb{C})$.

Proof. By replacing A with $M_k(A)$, we may assume that $k = 2$, where $M_{2k}(A) \cong M_2(M_k(A))$.

We can replace p, q, u, v with

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}, \quad \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}.$$

We then have

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}.$$

We can also assume the existence of an invertible element w such that $wpw^{-1} = q$, $u = pw^{-1} = w^{-1}q$, and $v = qw = wp$.

We take as

$$w = \begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ vp & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ vuv & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ qv & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ vuv & 0 \end{pmatrix}$$

coincide, with $qv = vp = v$ on $\mathfrak{B}$. Also,

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} = \begin{pmatrix} uv & u-uq \\ 0 & 0 \end{pmatrix}$$

with $u - uq = p(u - uq) = uvu - uvuvu = uvu - uvu = 0$ on $\mathfrak{B}$. As well,

$$\begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u-pu \\ 0 & vu \end{pmatrix}$$

with $u - pu = p(1-p)u = 0$ on $\mathfrak{B}$.

And assume that

$$\begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (1-p)a + uc & (1-p)b + ud \\ va + (1-q)c & vb + (1-q)d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It then follows that

$$w^{-1} = \begin{pmatrix} 1-p & u \\ v & 1-q \end{pmatrix} = w!$$

since $(1-p)u + u(1-q) = 0$ and

$$v(1-p) + (1-q)v = v - vp + v - vuv = 0 + v - vp = 0.$$

Recall that there correspond to closed graded traces

$$(\varphi \# \mathrm{tr})^\wedge = (\varphi^\wedge \otimes \mathrm{tr}^\wedge) \circ \pi = (\varphi \otimes^\sim \mathrm{tr})^\wedge,$$

where

$$\pi : \Omega^*(A \otimes M_k(\mathbb{C})) \to \Omega^*(A) \otimes \Omega^*(M_k(\mathbb{C}))$$

is a natural homomorphism of differential graded algebras of A, $M_k(\mathbb{C})$, and $A \otimes M_k(\mathbb{C}) \cong M_k(A)$.

It is shown that $\varphi\#\mathrm{tr}$ and $(\varphi\#\mathrm{tr}) \circ \mathrm{Ad}(w)$ are in the same cohomology class.

Moreover, for $a \in \mathfrak{B}$, we have $waw^{-1} = wpapw = vau$. $\square$

Lemma 2.2. *The subalgebra $\mathfrak{B}$ of $M_k(A)$ is $pM_k(A)p$.*

Proof. For any $a \in M_k(A)$, $pap = p(pap) = (pap)p$. Thus, $pM_k(A)p \subset \mathfrak{B}$.

Conversely, if $x \in \mathfrak{B}$, then $x = px = xp$. Hence, $pxp = (px)p = xp = x$. Therefore, $\mathfrak{B} \subset pM_k(A)p$. $\square$

Proposition 2.3. *A bilinear pairing $\langle \cdot, \cdot \rangle : K_0(A) \times cH^{\mathrm{ev}}(A) \to \mathbb{C}$ between the even K-theory group $K_0(A)$ and the even cyclic cohomology $cH^{\mathrm{ev}}(A)$ for a unital differential algebra A is defined by the following equality*

$$\langle [p], [\varphi] \rangle = \frac{1}{m!} (\varphi \otimes^\sim \mathrm{tr})(p, \cdots, p), \quad p \in IP_k(A), \varphi \in cZ^{2m}(A).$$

Then we have $\langle [p], [S\varphi] \rangle = \langle [p], [\varphi] \rangle$, where $S : cH^n(A) \to cH^{n+2}(A)$ is defined as $S\varphi = \sigma \otimes^\sim \varphi = \varphi \otimes^\sim \sigma$ for $\varphi \in cZ^n(A)$ and $\sigma \in cH^2(\mathbb{C})$ the generator.

For even cyclic $2m, 2n$-cocycles φ and ψ on unital differential algebras A and B respectively, and for $p \in K_0(A)$, $q \in K_0(B)$, we have

$$\langle p \otimes q, \varphi \otimes^\sim \psi \rangle = \langle p, \varphi \rangle \langle q, \psi \rangle,$$

where $\langle \cdot, \cdot \rangle : K_0(A \otimes B) \times cH^{2(m+n)}(A \otimes B) \to \mathbb{C}$ on the left side.

Proof. If $\varphi \in cB^{2m}(A)$ with $\varphi = b\psi$ for some ψ, then $\varphi \otimes^{\sim} \mathrm{tr}$ is also a coboundary, with $\varphi \otimes^{\sim} \mathrm{tr} = b(\psi \otimes^{\sim} \mathrm{tr})$. Then

$$
\begin{aligned}
&(\varphi\#\mathrm{tr})(p, \cdots, p) = b(\psi \otimes^{\sim} \mathrm{tr})(p, \cdots, p) \quad (p^2 = p) \\
&= \sum_{j=0}^{2m} (-1)^j (\psi\#\mathrm{tr})(p, \cdots, p) \quad \text{(Odd sum!)} \\
&= (\psi\#\mathrm{tr})(p, \cdots, p) = 0
\end{aligned}
$$

since $(\psi\#\mathrm{tr})^{\lambda} = -(\psi\#\mathrm{tr})$ for any permutation λ on odds. It then follows together with the lemma above that the value $(\varphi\#\mathrm{tr})(p, \cdots, p)$ depends only on the equivalence class of p in $K_0(A)$.

Replacing p with

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

does not change the definition.

The additivity is

$$
\begin{aligned}
&\langle [p] + [q], [\varphi] \rangle = \langle [p \oplus q], [\varphi] \rangle = \frac{1}{m!} (\varphi\#\mathrm{tr})(p \oplus q, \cdots p \oplus q) \\
&= \frac{1}{m!} (\varphi\#\mathrm{tr})(p \oplus 0, \cdots p \oplus 0) + \frac{1}{m!} (\varphi\#\mathrm{tr})(0 \oplus q, \cdots 0 \oplus q) \\
&= \langle [p], [\varphi] \rangle + \langle [q], [\varphi] \rangle .
\end{aligned}
$$

As well, the equation $(\varphi_1 + \varphi_2)\#\mathrm{tr} = (\varphi_1\#\mathrm{tr}) + (\varphi_2\#\mathrm{tr})$ implies the linearity in the second variable.

Next,

$$
\begin{aligned}
&(\varphi\#\mathrm{tr})(p, \cdots, p) = (\varphi^{\wedge} \otimes \mathrm{tr}^{\wedge})(p dp \cdots dp), \\
&(S\varphi)(p, \cdots, p) = \sum_{j=1}^{2m=n} \varphi^{\wedge}(p(dp)^{j-1} p (dp)^{n-j+1}) \\
&= (m+1)\varphi(p, \cdots, p)
\end{aligned}
$$

since $p^2 = p$, so that $p(dp)p = 0$ and $p(dp)^2 = (dp)^2 p$.

$\star$ $dp = d(p^2) = (dp)p + p(dp)$. Thus,

$$p(dp)p = p(dp)p^2 + p^2(dp)p = 2(p(dp)p).$$

Hence, $p(dp)p = 0$. Also,

$$
\begin{aligned}
(dp)^2 &= (dp)p(dp) + p(dp)^2, \\
(dp)^2 &= (dp)^2 p + (dp)p(dp).
\end{aligned}
$$

Subtracting both sides yields $p(dp)^2 = (dp)^2 p$.

$$\begin{aligned}
\varphi^\wedge(pp(dp)^n) &= \varphi^\wedge(p(dp)^n) = \varphi(p,\cdots,p) \quad (j=1),\\
\varphi^\wedge(p(dp)p(dp)^{n-1}) &= \varphi^\wedge(0(dp)^{n-1}) = 0 \quad (j=2),\\
\varphi^\wedge(p(dp)^2p(dp)^{n-2}) &= \varphi^\wedge(p^2(dp)^2(dp)^{n-2}) = \varphi(p,\cdots,p) \quad (j=3),\\
\varphi^\wedge(p(dp)^{j-1}p(dp)^{n-j+1}) &= \begin{cases} \varphi(p,\cdots,p) & (j \text{ is odd}),\\ 0 & (j \text{ is even}).\end{cases}
\end{aligned}$$

Possibly, the factor by $(m+1)$ should be m? If so, the factor by $\frac{1}{m!}$ in the definition should be replaced by $\frac{1}{(m-1)!}$.

Anyway, the factor by $\frac{1}{m!}$ is also used as the symbol representing the degrees of cohomology classes. (Right?)

Moreover,

$$\begin{aligned}
&(\varphi \otimes^\sim \psi) \otimes^\sim \mathrm{tr}(p\otimes q,\cdots,p\otimes q)\\
&= (\varphi \otimes^\sim \mathrm{tr}) \otimes^\sim (\psi \otimes^\sim \mathrm{tr})(p\otimes q,\cdots,p\otimes q)\\
&= (\varphi \otimes^\sim \mathrm{tr})(p,\cdots,p)(\psi \otimes^\sim \mathrm{tr})(q,\cdots,q)
\end{aligned}$$

where a possible multiple is omitted.

If $\frac{\alpha}{(m+n)!} = \frac{1}{m!}\frac{1}{n!}$, then $\alpha = \frac{(m+n)!}{m!n!}$. □

Proposition 2.4. *A bilinear pairing $\langle\cdot,\cdot\rangle : K_1(A) \times cH^{\mathrm{od}}(A) \to \mathbb{C}$ between the odd K-theroy group $K_1(A)$ and the odd cyclic cohomology $cH^{\mathrm{od}}(A)$ for a unital differential algebra A is defined by the following equality*

$$\langle [u],[\varphi]\rangle = \frac{1}{\sqrt{2i}2^n\Gamma(\frac{n}{2}+1)}(\varphi\otimes^\sim \mathrm{tr})(u^{-1}-1,u-1,u^{-1}-1,\cdots,u-1)$$

for $u \in GL_k(A)$, $\varphi \in cZ^n(A)$.

Then we have $\langle [u],[S\varphi]\rangle = \langle [u],[\varphi]\rangle$.

For even and odd cyclic $2m$- and $2k+1$-cocycles φ, ψ on unital differential algebras A and B respectively, and for $p \in K_0(A)$ and $u \in K_1(B)$, we have

$$\langle [p\otimes u + (1-p)\otimes 1], \varphi\#\psi\rangle = \langle p,\varphi\rangle\langle u,\psi\rangle$$

where $\langle\cdot,\cdot\rangle : K_0(A)\otimes K_1(B) \times cH^{2(m+k)+1}(A\otimes B) \to \mathbb{C}$ on the left.

Proof. Let $A+\mathbb{C}1$ be the algebra obtained by adjoining a unit 1 to A unital with 1_A. Then $A+\mathbb{C}1$ is isomorphic to the (pointwise) product $A\times\mathbb{C}$ by the homomorphism $\rho : A+\mathbb{C}1 \to A\times\mathbb{C}$ defined by $\rho(a+\lambda 1) = (a+\lambda 1_A,\lambda)$ for $a+\lambda 1 = (a,\lambda) \in A+\mathbb{C}1$, and so identified or not.

⋆ Note that

$$\begin{aligned}
&\rho(a+\lambda 1)\rho(b+\mu 1) = (a+\lambda 1_A,\lambda)(b+\mu 1_A,\mu)\\
&= ((a+\lambda 1_A)(b+\mu 1_A),\lambda\mu) = (ab+\mu a+\lambda b+\lambda\mu 1_A,\lambda\mu)\\
&= \rho(ab+\mu a+\lambda b+\lambda\mu 1) = \rho((a+\lambda 1)(b+\mu 1)).
\end{aligned}$$

Also,

$$\rho(\mu(a+\lambda 1)) = \rho(\mu a + \mu\lambda 1) = (\mu a + \mu\lambda 1_A, \mu\lambda) = \mu\rho(a+\lambda 1).$$

As well,

$$\begin{aligned}&\rho((a+\lambda 1)+(b+\mu 1)) = \rho(a+b+(\lambda+\mu)1)\\ &= (a+b+(\lambda+\mu)1_A, \lambda+\mu) = \rho(a+\lambda 1)+\rho(b+\mu 1).\end{aligned}$$

Hence, ρ is linear as desired, and so certainly injective and surjective.

For $\varphi \in cZ^n(A)$, define $\varphi^\sim \in cZ^n(A+\mathbb{C}1)$ by the equality

$$\varphi^\sim((a_0,\lambda_0),\cdots,(a_n,\lambda_n)) = \varphi(a_0,\cdots,a_n), \quad (a_j,\lambda_j) \in A+\mathbb{C}1.$$

Indeed, $b(\varphi^\sim)=0$ is checked in the following. We have

$$\begin{aligned}&\varphi^\sim((a_0,\lambda_0),\cdots,(a_j,\lambda_j)(a_{j+1},\lambda_{j+1}),\cdots,(a_{n+1},\lambda_{n+1}))\\ &= \varphi(a_0,\cdots,a_ja_{j+1},\cdots,a_{n+1}) + \lambda_j\varphi(a_0,\cdots,a_{j-1},a_{j+1},\cdots,a_{n+1})\\ &\quad + \lambda_{j+1}\varphi(a_0,\cdots,a_j,a_{j+2},\cdots,a_{n+1}).\end{aligned}$$

Thus,

$$\begin{aligned}&b(\varphi^\sim)((a_0,\lambda_0),\cdots,(a_{n+1},\lambda_{n+1}))\\ &= \lambda_0\varphi(a_1,\cdots,a_{n+1}) + (-1)^{n-1}\lambda_0\varphi(a_{n+1},a_1,\cdots,a_n) = 0.\end{aligned}$$

$\star$ Check that

$$\begin{aligned}&b(\varphi^\sim)((a_0,\lambda_0),(a_1,\lambda_1)) = \varphi^\sim((a_0,\lambda_0)(a_1,\lambda_1)) - \varphi^\sim((a_1,\lambda_1)(a_0,\lambda_0))\\ &= \varphi(a_0a_1)+\lambda_0\varphi(a_1)+\lambda_1\varphi(a_0)-\varphi(a_1a_0)-\lambda_1\varphi(a_0)-\lambda_0\varphi(a_1)\\ &= \varphi(a_0a_1)-\varphi(a_1a_0) = 0.\end{aligned}$$

Because $cZ^0(A) = \mathrm{Tr}(A)$. Check also that

$$\begin{aligned}&b(\varphi^\sim)((a_0,\lambda_0),(a_1,\lambda_1),(a_2,\lambda_2)) = \varphi^\sim((a_0,\lambda_0)(a_1,\lambda_1),(a_2,\lambda_2))\\ &\quad - \varphi^\sim((a_0,\lambda_0),(a_1,\lambda_1)(a_2,\lambda_2)) + \varphi^\sim((a_2,\lambda_2)(a_0,\lambda_0),(a_1,\lambda_1))\\ &= \varphi(a_0a_1,a_2)+\lambda_0\varphi(a_1,a_2)+\lambda_1\varphi(a_0,a_2)\\ &\quad - \varphi(a_0,a_1a_2)-\lambda_1\varphi(a_0,a_2)-\lambda_2\varphi(a_0,a_1)\\ &\quad + \varphi(a_2a_0,a_1)+\lambda_2\varphi(a_0,a_1)+\lambda_0\varphi(a_2,a_1)\\ &= \varphi(a_0a_1,a_2)-\varphi(a_0,a_1a_2)+\varphi(a_2a_0,a_1)\\ &= (b\varphi)(a_0,a_1,a_2) = 0.\end{aligned}$$

Now for $u \in GL_1(A)$ we have

$$\begin{aligned}&\varphi(u^{-1}-1,u-1,\cdots,u^{-1}-1,u-1)\\ &= (\varphi^\sim\circ\rho^{-1})((u,1)^{-1},(u,1),\cdots(u,1)^{-1},(u,1)) = \chi(u).\end{aligned}$$

$\star$ Note that $(1,1)$ is the unit for $A \times \mathbb{C}$ with pointwise product. Thus, $(u,1)^{-1} = (u^{-1}, 1)$.

As well, $\rho^{-1}(u^{\pm 1}, 1) = \rho^{-1}(u^{\pm 1} - 1 + 1, 1) = (u^{\pm 1} - 1, 1) \in A + \mathbb{C}1$.

To show that

$$\chi(uv) = \chi(u) + \chi(v), \quad u, v \in GL_1(A)$$

we may assume that $\varphi(1, a_0, \cdots, a_{n-1}) = 0$ for $a_j \in A$ and replace χ by

$$\chi(u) = \varphi(u^{-1}, u, \cdots, u^{-1}, u).$$

$\star$ Note that

$$\begin{aligned}\chi(uv) &= \varphi((uv)^{-1} - 1, uv - 1, \cdots, (uv)^{-1} - 1, uv - 1) \\ &= \varphi((uv)^{-1}, uv - 1, \cdots, (uv)^{-1} - 1, uv - 1) \\ &\quad - \varphi(1, uv - 1, \cdots, (uv)^{-1} - 1, uv - 1).\end{aligned}$$

This process may be continued under the assumption above with cycling go.

Such a degenerating assumption is allowed or not?

With

$$U = \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} = uv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

we have

$$\begin{aligned}\chi(uv) &= (\varphi \# \mathrm{tr})(U^{-1}, U, \cdots, U^{-1}, U), \\ \chi(u) + \chi(v) &= (\varphi \# \mathrm{tr})(V^{-1}, V, \cdots, V^{-1}, V)\end{aligned}$$

with $\# = \otimes^{\sim}$ and tr the canonical trace on matrix algebras $M_2(\mathbb{C})$ or $M_k(\mathbb{C})$.

$\star$ Note that

$$\begin{aligned}\chi(uv) &= \varphi((uv)^{-1}, uv, \cdots, (uv)^{-1}, uv) \\ &= \varphi((uv)^{-1} + 1, uv + 1, \cdots, (uv)^{-1} + 1, uv + 1) \\ &= \varphi(\mathrm{tr}(U^{-1}), \mathrm{tr}(U), \cdots, \mathrm{tr}(U^{-1}), \mathrm{tr}(U)) \\ &= (\varphi \circ \mathrm{tr})(U^{-1}, U, \cdots, U^{-1}, U)\end{aligned}$$

with $\varphi \circ \mathrm{tr}$ identified with $\varphi \# \mathrm{tr}$. Indeed,

$$\begin{aligned}&(\varphi \# \mathrm{tr})(U^{-1}, U, \cdots, U^{-1}, U) \\ &= \varphi((uv)^{-1} + 1, uv + 1, \cdots, (uv)^{-1} + 1, uv + 1) \\ &= \varphi((uv)^{-1}, uv, \cdots, (uv)^{-1}, uv) = \chi(uv).\end{aligned}$$

Also,

$$\begin{aligned}&(\varphi \circ \mathrm{tr})(V^{-1}, V) = (\varphi \# \mathrm{tr})(V^{-1}, V) \\ &= \varphi(u^{-1} + v^{-1}, u + v) \\ &= \varphi(u^{-1}, u) + \varphi(u^{-1}, v) + \varphi(v^{-1}, u) + \varphi(v^{-1}, v) \\ &= \chi(u) + \chi(v).\end{aligned}$$

If so, we need to have $\varphi(u^{-1}, v) + \varphi(v^{-1}, u) = 0$, so that required as well is cycling involving inversing.

Or as a possible mixed or wrong sense in notation, tensor product multilinearity is not assumed, but used is vector space direct sum linearity?

The 2×2 matrix U is connected to V by the smooth path

$$\begin{aligned} U_t &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix} \\ &= (u \oplus 1) R_t (1 \oplus v) R_t^{-1} = (u \oplus 1) \mathrm{Ad}(R_t)(1 \oplus v) \end{aligned}$$

for $0 \leq t \leq \frac{\pi}{2}$.

$\star$ Note that

$$\begin{aligned} U_0 &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -u \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix} = \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} = U, \\ U_{\frac{\pi}{2}} &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = V. \end{aligned}$$

It is enough to check that

$$\frac{d}{dt} (\varphi \# \mathrm{tr})(U_t^{-1}, U_t, \cdots, U_t^{-1}, U_t) = 0.$$

$\star$ If so, $(\varphi \# \mathrm{tr})(U_t^{-1}, U_t, \cdots, U_t^{-1}, U_t)$ is constant with respect to t.

We have $(U_t^{-1})' = -U_t^{-1} U_t' U_t^{-1}$.

$\star$ Note that

$$\frac{d}{dt}(U_t^{-1} U_t) = (U_t^{-1})' U_t + U_t^{-1} U_t' = \frac{d}{dt} 1 = 0.$$

The desired differential zero equation follows by using the differential relation.

$\star$ Note that the chain rule implies

$$\begin{aligned} &\frac{d}{dt} (\varphi \# \mathrm{tr})(U_t^{-1}, U_t) = d(\varphi \# \mathrm{tr})(U_t^{-1}, U_t)(U_t^{-1})' + d(\varphi \# \mathrm{tr})(U_t^{-1}, U_t) U_t' \\ &= -d(\varphi \# \mathrm{tr})(U_t^{-1}, U_t) U_t^{-1} U_t' U_t^{-1} + d(\varphi \# \mathrm{tr})(U_t^{-1}, U_t) U_t'. \end{aligned}$$

But in this sense, it seems to be still incomplete to hold the equation desired.

On the other hand, we note that

$$\begin{aligned} U_t' &= (u \oplus 1) R_t' (1 \oplus v) R_t^{-1} + (u \oplus 1) R_t (1 \oplus v)(R_t^{-1})' \\ &= (u \oplus 1) \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} (1 \oplus v) R_t^{-1} + (u \oplus 1) R_t (1 \oplus v) \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \end{aligned}$$

Thus,

$$
\begin{aligned}
&U_t' U_t^{-1} = U_t' R_t (1 \oplus v^{-1}) R_t^{-1} (u^{-1} \oplus 1) \\
&= (u \oplus 1) \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix} (u^{-1} \oplus 1) \\
&+ (u \oplus 1) R_t (1 \oplus v) \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix} (1 \oplus v^{-1}) R_t^{-1} (u^{-1} \oplus 1) \\
&= (u \oplus 1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (u^{-1} \oplus 1) \\
&+ (u \oplus 1) R_t (1 \oplus v) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (1 \oplus v^{-1}) R_t^{-1} (u^{-1} \oplus 1) \\
&= \begin{pmatrix} 0 & u \\ -1 & 0 \end{pmatrix} (u^{-1} \oplus 1) + (u \oplus 1) R_t \begin{pmatrix} 0 & -1 \\ v & 0 \end{pmatrix} (1 \oplus v^{-1}) R_t^{-1} (u^{-1} \oplus 1) \\
&= \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix} + (u \oplus 1) R_t \begin{pmatrix} 0 & -v^{-1} \\ v & 0 \end{pmatrix} R_t^{-1} (u^{-1} \oplus 1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&U_t^{-1} U_t' U_t^{-1} = R_t (1 \oplus v^{-1}) R_t^{-1} (u^{-1} \oplus 1) \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix} \\
&+ R_t (1 \oplus v^{-1}) \begin{pmatrix} 0 & -v^{-1} \\ v & 0 \end{pmatrix} R_t^{-1} (u^{-1} \oplus 1) \\
&= R_t (1 \oplus v^{-1}) R_t^{-1} \begin{pmatrix} 0 & 1 \\ -u^{-1} & 0 \end{pmatrix} + R_t \begin{pmatrix} 0 & -v^{-1} \\ 1 & 0 \end{pmatrix} R_t^{-1} (u^{-1} \oplus 1).
\end{aligned}
$$

The right understanding seems to have

$$d(\varphi \# \mathrm{tr}) = (d\varphi) \otimes^{\sim} \mathrm{tr} + \varphi \otimes^{\sim} d(\mathrm{tr}) = 0.$$

It then follows that the function χ defines a homomorphism of $GL_k(A)$ to $\mathbb{C}$.

The compatibility of χ with the inclusion of $GL_k(A)$ to $GL_{k'}(A)$ with $k \leq k'$ holds, up to the class K_1.

In order to show that the value is zero for φ a coboundary, we may assume that $k = 1$, and that $\varphi = b\psi$ for $\psi \in cC^{n-1}$ and $\psi(1, a_0, \cdots, a_{n-2}) = 0$ for $a_j \in A$.

We have $b(\varphi^{\sim}) = (b\psi)^{\sim}$.

Then $b\psi(u^{-1}, u, \cdots, u^{-1}, u) = 0$ is obtained.

$\star$ Note that always

$$b\psi(u^{-1}, u) = \psi(u^{-1}u) - \psi(uu^{-1}) = \psi(1) - \psi(1) = 0.$$

Also,

$$
\begin{aligned}
&(b\psi)(u^{-1}, u, u^{-1}, u) \\
&= \psi(1, u^{-1}, u) - \psi(u^{-1}, 1, u) + \psi(u^{-1}, u, 1) - \psi(1, u, u^{-1}) \\
&= 0 - 0 + 0 - 0 = 0
\end{aligned}
$$

by just the choice of ψ. This seems to be the just reason.

We have

$$2^{n+2}\Gamma(\frac{n+2}{2}+1) = 2\cdot 2^n\Gamma(\frac{n}{2}+1+1) = (n+2)2^n\Gamma(\frac{n}{2}+1).$$

It is so required that $S\varphi = (n+2)\varphi$ on $(u^{-1}-1, u-1, \cdots, u^{-1}-1, u-1)$.

Note that $d(u^{-1}-1) = d(u^{-1})$ and $d(u-1) = du$. Also, $(u^{-1}-1)(u-1) = 2-u^{-1}-u$. As well,

$$(u^{-1}-1)d(u-1)(u^{-1}-1) = (u^{-1}-1)du(u^{-1}-1).$$

And $0 = d1 = d(u^{-1}u) = d(u^{-1})u + u^{-1}du$. Thus, $u^{-1}du = -d(u^{-1})u$. Also, $(du)u^{-1} + ud(u^{-1}) = 0$. Hence,

$$\begin{aligned}
&(u^{-1}-1)du(u^{-1}-1) = -d(u^{-1})u(u^{-1}-1) - du(u^{-1}-1)\\
&= -d(u^{-1}) + d(u^{-1})u - (du)u^{-1} + du\\
&= -d(u^{-1}) - u^{-1}du + ud(u^{-1}) + du\\
&= (u-1)d(u^{-1}) - (u^{-1}-1)du.
\end{aligned}$$

For $n = 2k+1$ odd,

$$\begin{aligned}
\Gamma(\frac{n}{2}+1) = \Gamma(k+\frac{1}{2}+1) &= (k+\frac{1}{2})\Gamma(k-1+\frac{1}{2}+1)\\
&= (k+\frac{1}{2})(k-1+\frac{1}{2})\cdots(1+\frac{1}{2})\frac{1}{2}\Gamma(\frac{1}{2})\\
&= \frac{2k+1}{2}\frac{2k-1}{2}\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi} = \frac{n!!}{2^{k+1}}\sqrt{\pi}.
\end{aligned}$$

For $2m$- and $n = 2k+1$-cocycles φ and ψ identified with their K-theory classes,

$$\begin{aligned}
&\langle p,\varphi\rangle\langle u,\psi\rangle =\\
&\frac{1}{m!}(\varphi\#\mathrm{tr})(p,\cdots,p)\frac{1}{\sqrt{2\pi i}2^k n!!}(\psi\#\mathrm{tr})(u^{-1}-1, u-1, \cdots, u^{-1}-1, u-1)\\
&= \langle p\otimes u, \varphi\#\psi\rangle.
\end{aligned}$$

On the other hand,

$$\langle (1-p)\otimes 1\rangle, \varphi\#\psi\rangle = \langle 1-p,\varphi\rangle\langle 1,\psi\rangle = 0.$$

□

Remark. The normalization of the pairing between the K_0 and even cH is uniquely specified by the conditions involving S and $\otimes$. The normalization of the pairing between the K_1 and odd cH is only specified up to an overall multiplicative constant independent of n by the condition involving S and $\otimes$.

The choice as up to the choice of the square root of $2i$ is the only choice for which the following formula holds:

$$\langle u \wedge v, \varphi \# \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle$$

where the product $\wedge : K_1(A) \times K_1(B) \to K_0(A \otimes B)$ is defined in the context of pre-C^*-algebras. To check the formula above we just needs to know that if $f \in C^\infty(\mathbb{T}^2, P_1(\mathbb{C}))$ is a degree 1 map of the 2-torus $\mathbb{T}^2$ to $P_1(\mathbb{C}) \subset M_2(\mathbb{C})$, then $\tau(f, f, f) = 2\pi i$, where τ is the cyclic 2-cocycle on $C^\infty(\mathbb{T}^2)$ given by

$$\tau(f_0, f_1, f_2) = \int_{\mathbb{T}^2} f_0 df_1 \wedge df_2, \quad f_j \in C^\infty(\mathbb{T}^2).$$

Note that the Bott generator of the K-theory of $P_1(\mathbb{C})$ corresponds to the class of $[1 - f] - 1$ (cf. [1]).

$\star$ There is an inclusion from the complex projective line $P_1(\mathbb{C})$ to the unitary group $U(2) \subset GL_2(\mathbb{C})$ defined as $f(X) = 1 - 2X$ (cf. [18]). Note that

$$P_1(\mathbb{C}) = \{X \in M_2(\mathbb{C}) \,|\, X^* = X, X^2 = X, \mathrm{tr}(X) = 1\} \cong S^2$$

where

$$S^2 \cong D = \{(x, z) \in \mathbb{R} \times \mathbb{C} \,|\, x(1 - x) = |z|^2\}$$

and

$$\begin{pmatrix} x & \overline{z} \\ z & 1 - x \end{pmatrix} = X \in P_1(\mathbb{C}).$$

$\star$ Indeed, if $X \in P_1(\mathbb{C})$, then the conditions $X = X^*$ and $\mathrm{tr}(X) = 1$ imply that

$$X = \begin{pmatrix} x & \overline{z} \\ z & 1 - x \end{pmatrix}$$

for some $x \in \mathbb{R}$ and $z \in \mathbb{C}$. Moreover, the condition

$$\begin{aligned} X^2 &= \begin{pmatrix} x & \overline{z} \\ z & 1 - x \end{pmatrix} \begin{pmatrix} x & \overline{z} \\ z & 1 - x \end{pmatrix} \\ &= \begin{pmatrix} x^2 + |z|^2 & \overline{z} \\ z & |z|^2 + (1 - x)^2 \end{pmatrix} = \begin{pmatrix} x & \overline{z} \\ z & 1 - x \end{pmatrix} = X \end{aligned}$$

implies that $|z|^2 = x - x^2 = x(1 - x)$. This equation is converted to

$$(x - \frac{1}{2})^2 + |z|^2 = \frac{1}{4}$$

so that the definition domain for X is homeomorphic to the 2-dimensional sphere S^2 by sending (x, z) to $(2x - 1, 2z) = (t, w) \in S^2$.

$\star$ For instance, let

$$f(z, w) = \begin{pmatrix} 0 & \frac{1}{zw} \\ zw & 1 \end{pmatrix} \quad (z, w) \in \mathbb{T}^2.$$

Then

$$\begin{aligned} df &= f_z dz + f_w dw \\ &= \begin{pmatrix} 0 & -\frac{1}{z^2 w} \\ w & 0 \end{pmatrix} dz + \begin{pmatrix} 0 & -\frac{1}{zw^2} \\ z & 0 \end{pmatrix} dw. \end{aligned}$$

Viewing matrices as coefficients as well as vectors if allowed, by the definition of integral of differential forms (cf. [9]) we have

$$\begin{aligned} &\int_{\mathbb{T}^2} f df \wedge df \\ &= \int_{\mathbb{T}^2} f(z,w) \begin{pmatrix} 0 & -\frac{1}{z^2 w} \\ w & 0 \end{pmatrix} dz \begin{pmatrix} 0 & -\frac{1}{zw^2} \\ z & 0 \end{pmatrix} dw \\ &\quad + \int_{\mathbb{T}^2} f(z,w) \begin{pmatrix} 0 & -\frac{1}{zw^2} \\ z & 0 \end{pmatrix} dw \begin{pmatrix} 0 & -\frac{1}{z^2 w} \\ w & 0 \end{pmatrix} dz \\ &= \int_{\mathbb{T}^2} \begin{pmatrix} 0 & \frac{1}{zw} \\ zw & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{zw} & 0 \\ 0 & -\frac{1}{zw} \end{pmatrix} dzdw \\ &\quad + \int_{\mathbb{T}^2} \begin{pmatrix} 0 & \frac{1}{zw} \\ zw & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{zw} & 0 \\ 0 & -\frac{1}{zw} \end{pmatrix} dwdz \\ &= 2 \int_{\mathbb{T}^2} \begin{pmatrix} 0 & -1 \\ -1 & -\frac{1}{zw} \end{pmatrix} dzdw \\ &= \begin{pmatrix} 0 & -2\int_{\mathbb{T}^2} 1 dzdw \\ -2\int_{\mathbb{T}^2} 1 dzdw & -2\int_{\mathbb{T}^2} \frac{1}{zw} dzdw \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -2(\int_{\mathbb{T}} \frac{1}{z} dz)^2 \end{pmatrix} = -2(2\pi i)^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

which may be identified with $8\pi^2$ (cf. [17] in part). Other normalizations as divided by such as $2\pi i$ may be taken.

Let $cH_p^*(A) = cH^*(A) \otimes_{cH^*(\mathbb{C})} \mathbb{C}$ be the periodic cyclic cohomology of A.

The only even $cH^*(\mathbb{C})$ is identified with a polynomial ring $\mathbb{C}[\sigma]$, and acts on $\mathbb{C}$ by sending $p(\sigma)$ to $p(1)$. This homomorphism of $cH^*(\mathbb{C})$ to $\mathbb{C}$ is the paring between the only even $cH^*(\mathbb{C})$ and the generator [1] of $K_0(\mathbb{C}) = \mathbb{Z}$, so that $\langle p(\sigma), [1] \rangle = p(1)$.

By construction, $cH_p^*(A)$ is the inductive limit of the cH groups $cH^n(A)$ under the S map from $cH^n(A)$ to $cH^{n+2}(A)$. Equivalently, it is the quotient of $cH^*(A)$ by the equivalence relation as $\varphi \sim S\varphi$. As well, it inherits a natural $\mathbb{Z}_2$ grading and a filtration as (modified)

$$F^q cH_p^*(A) = \mathrm{im}(F^{q+2} cH^*(A)) = S(F^{q+2} cH^*(A)).$$

$\star$ Recall that

$$F^q = \sum_{m \geq q} C^{n,m} = \sum_{m \geq q} C^{n-m} = C^{n-q} \oplus C^{n-q-1} \oplus \cdots \oplus C^0.$$

Defined as well is a canonical pairing between $cH_p^{\mathrm{ev}}(A)$ and $K_0(A)$ as well as between $cH_p^{\mathrm{od}}(A)$ and $K_1(A)$.

Corollary 2.5. *Let A be a locally convex algebra and τ a continuous even (or odd) cocycle on A. Then the paring of τ with $p \in K_0(A)$ (or with $u \in K_1(A)$) only depends upon the homotopy class of p (u respectively).*

Assume that the algebra A is endowed with a locally convex topology such that the product from $A \times A$ to A is continuous.

In other words, for any continuous semi-norm p on A, there exists a continuous semi-norm p' such that $p(ab) \leq p'(a)p'(b)$ for any $a, b \in A$.

The algebraic dual of A is replaced by the topological dual of A, both denoted as A^*.

The space of $(n+1)$-linear functionals on A is replaced by that of continuous $(n+1)$-linear functionals on A, both denoted as $C^n(A, A^*) = C^n$.

That $\varphi \in C^n$ if and only if for some continuous semi-norm p, we have $|\varphi(a_0, \cdots, a_n)| \leq p(a_0) \cdots p(a_n)$ for any $a_j \in A$.

Since the product is continuous, we have $bC^n \subset C^{n+1}$.

The formulae for the cup product of cochains only involve the product of A so that the algebraic results apply with no change to the topological ones.

A pure point to be mentioned below is the use of resolutions in the computation of the Hochschild cohomology.

We may as well assume that A is complete, still a locally convex topological algebra since C^n as well as cohomology are not affected by taking the completion.

Let B be a complete locally convex topological algebra.

A topological module over B is a locally convex vector space M which is a B-module such that the map from $B \times M$ to M by sending (b, ξ) to $b\xi$ is continuous.

We say that M is topologically projective if it is a direct summand of a topological module of the form $B \otimes E$, where E is a complete locally convex vector space and $\otimes$ means the projective tensor product (cf. [5]). In particular, M is complete as a closed subspace of $B \otimes E$.

Let M_1 and M_2 be topological B-modules which are complete as locally convex vector spaces, with $p : M_1 \to M_2$ a continuous B-linear map with a continuous $\mathbb{C}$-linear cross-section s so that $p \circ s$ is the identity map on M_2. For any continuous B-linear map $f : M \to M_2$, we can complete the diagram of continuous B- and $\mathbb{C}$-linear maps

$$\begin{tikzcd}
M_2 \arrow[r, "s", "\mathbb{C}"'] & M_1 \subset B \otimes E_1 \arrow[d, "p"', "B"] \\
M \arrow[u, "f", "B"'] \arrow[r, "f", "B"'] & M_2 \subset B \otimes E_2
\end{tikzcd}$$

to an B-linear $f^\sim : M \to M_2$ extended from $s \circ f$, as

$$f^\sim(bm) = bs(f(m)), \quad (s \circ f)(bm) = s(b(f(m))$$

in a possible sense.

Definition 2.6. Let B be a complete locally convex topological algebra and M be a topological B-module. A topological projective resolution of M is defined to be an exact sequence of projective B-modules and continuous B-linear maps as

$$M \xleftarrow{\varepsilon} M_0 \xleftarrow{b_1} M_1 \xleftarrow{b_2} M_2 \xleftarrow{b_3} \cdots$$

with $\mathbb{C}$-linear continuous homotopy $s_j : M_j \to M_{j+1}$ such that

$$\begin{tikzcd} M_j \arrow[r, "s_j"] \arrow[d, "b_j"'] & M_{j+1} \arrow[d, "b_{j+1}"] \\ M_{j-1} \arrow[r, "s_{j-1}"] & M_j \end{tikzcd}$$

with the compositions added to make $(b_{j+1} \circ s_j) + (s_{j-1} \circ b_j) = \mathrm{id}$ on M_j.

⋆ Note that $(b_{j+1} \circ s_j)(M_j)$ is contained in $\ker(b_j)$, so that $(s_{j-1} \circ b_j)(M_j)$ may not be contained in $\ker(b_j)$. If s_j is a section, then $b_{j+1} \circ s_j$ is the identity map on M_j.

The module A over $B = A \otimes A^{\mathrm{op}}$ by the opposite algebra A^{op} is given by

$$(a \otimes b')c = acb', \quad a, c \in A, b' \in A^{\mathrm{op}}.$$

⋆ Check that

$$\begin{aligned}(a_1 \otimes b_1')((a_2 \otimes b_2')c) &= a_1(a_2cb_2')b_1' = (a_1a_2)c(b_2'b_1') \\ &= (a_1a_2 \otimes b_2'b_1')c = ((a_1 \otimes b_1')(a_2 \otimes b_2'))c.\end{aligned}$$

There is the following canonical projective resolution

$$A = M \xleftarrow{\varepsilon} M_0 = B = A \otimes A^{\mathrm{op}} \xleftarrow{b_1} \cdots \xleftarrow{b_n} M_n = B \otimes (\otimes^n A)$$

where $\varepsilon(a \otimes b') = ab'$, $a \in A$, $b' \in A^{\mathrm{op}}$, and

$$\begin{aligned}&b_n(1 \otimes a_1 \otimes \cdots \otimes a_n) = (a_1 \otimes 1) \otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{j=1}^{n-1} (-1)^j (1 \otimes 1) \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n + (-1)^n (1 \otimes a_n^\circ) \otimes (a_1 \otimes \cdots \otimes a_{n-1}),\end{aligned}$$

and the usual continuous section on M_n is given by

$$s_n((a \otimes b^\circ) \otimes (a_1 \otimes \cdots \otimes a_n)) = (1 \otimes b^\circ) \otimes (a \otimes a_1 \otimes \cdots \otimes a_n).$$

⋆ Note that for $a_1 \in A$, with $a_1 = a_1^\circ \in A^{\mathrm{op}}$,

$$(\varepsilon \circ b_1)(1 \otimes a_1) = \varepsilon(a_1 \otimes 1 - 1 \otimes a_1^\circ) = a_1 - a_1 = 0.$$

Also,

$$\begin{aligned}&(b_1 \circ b_2)(1 \otimes a_1 \otimes a_2) = b_1((a_1 \otimes 1) \otimes a_2 - (1 \otimes 1) \otimes a_1a_2 + (1 \otimes a_2^\circ) \otimes a_1) \\ &= (a_1 \otimes 1)(a_2 \otimes 1 - 1 \otimes a_2^\circ) - (1 \otimes 1)(a_1a_2 \otimes 1 - 1 \otimes (a_1a_2)^\circ) + (1 \otimes a_2^\circ)(a_1 \otimes 1 - 1 \otimes a_1^\circ) \\ &= a_1a_2 \otimes 1 - a_1 \otimes a_2^\circ - a_1a_2 \otimes 1 + 1 \otimes (a_1a_2)^\circ + a_1 \otimes a_2^\circ - 1 \otimes a_2^\circ a_1^\circ = 0!\end{aligned}$$

As well,

$$\begin{aligned}
&(b_2 \circ s_1)((a \otimes b^\circ) \otimes a_1) = b_2((1 \otimes b^\circ)(a \otimes a_1)) \\
&= (1 \otimes b^\circ)((a \otimes 1) \otimes a_1 - (1 \otimes 1) \otimes aa_1 + (1 \otimes a_1^\circ) \otimes a) \\
&= (a \otimes b^\circ) \otimes a_1 - (1 \otimes b^\circ) \otimes aa_1 + (1 \otimes b^\circ a_1^\circ) \otimes a
\end{aligned}$$

to be equal to $(a \otimes b^\circ) \otimes a_1$, with allowed

$$(1 \otimes b^\circ) \otimes aa_1 = (1 \otimes b^\circ)(1 \otimes a_1^\circ) \otimes a = (1 \otimes b^\circ a_1^\circ) \otimes a.$$

Comparing that resolution with an arbitrary topological projective resolution of the module A over B we have

Lemma 2.7. *For any topological projective resolution (M_*, b_*) of the module A over $B = A \otimes A^{\mathrm{op}}$, the Hochschild cohomology $H^*(A, A^*)$ coincides with the cohomology of the Hom complex of continuous B-linear maps from M_* to A^*.*

$$\mathrm{Hom}_B(M_0, A^*) \xrightarrow{b_1^*} \mathrm{Hom}_B(M_1, A^*) \xrightarrow{b_2^*} \cdots$$

This lemma can be extended to any complete topological bimodule over A.

We may refer to [6] for more tools in the topological context refined.

$\triangledown$ Now the following notion is important in explicit computations of the pairing above and in the discussion of Morita equivalences.

This is already clear in the case where $A = C^\infty(V)$ for V a smooth manifold.

Definition 2.8. Let $\rho : A \to \Omega$ be a cycle over A, and E a finite projective module over A. A connection ∇ on E is defined to be a linear map $\nabla : E \to E \otimes_A \Omega^1$ such that

$$\nabla(\xi x) = (\nabla \xi)x + \xi \otimes d\rho(x), \quad \xi \in E, x \in A,$$

where E is a right module over A and Ω^1 is considered as a bimodule over A using the homomorphism $\rho : A \to \Omega^0$ and the ring structure of Ω^*.

$\star$ Note that

$$A \xrightarrow{\rho} \Omega^0 \xrightarrow{d} \Omega^1 \longrightarrow \cdots$$

so that $d\rho(x) = (d \circ \rho)(x)$.

$\star$ It says that ∇ is a module map like mod the second term involving cycling and differentialing, in short, cydiffing.

$\star$ It seems to look like a projective module differentiation.

Proposition 2.9. (1) *Let $p \in \mathrm{End}_A(E)$ be an idempotent and ∇ a connection on E over A. Then $\xi \mapsto (p \otimes 1)\nabla\xi$ is a connection on pE.*

(2) *Any finite projective module E admits a connection.*

(3) *The space of connections on E over A is an affine space over the vector space $V = \mathrm{Hom}_A(E, E \otimes_A \Omega^1)$.*

(4) *Any connection on E over A extends uniquely to a linear map of $E^\sim = E \otimes_A \Omega$ into itself such that*

$$\nabla(\xi \otimes w) = (\nabla\xi)w + \xi \otimes dw, \quad \xi \in E, w \in \Omega$$

in the sense that the extension sends $E \otimes_A \Omega^j$ to $E \otimes_A \Omega^{j+1} = (E \otimes_A \Omega^1) \otimes_A \Omega^j$.

Proof. (1) Multiplying the equality of the definition for ∇ by $p \otimes 1$ we have

$$(p \otimes 1)\nabla(\xi x) = (p \otimes 1)(\nabla \xi)x + p\xi \otimes d\rho(x), \quad \xi \in E, x \in A.$$

Thus, for $\eta \in pE$,

$$(p \otimes 1)\nabla(\eta x) = (p \otimes 1)(\nabla \eta)x + p\eta \otimes d\rho(x), \quad \eta \in E, x \in A$$

with $p\eta = \eta$.

(2) By (1) we can assume that $E = \mathbb{C}^k \otimes A$ for some k. With $(\xi_j)_{j=1}^k$ a canonical basis for E, define

$$\nabla(\sum \xi_j a_j) = \sum \xi_j \otimes d\rho(a_j) \in E \otimes_A \Omega^1, \quad a_j \in A.$$

$\star$ So that $\nabla(\xi_j) = 0$ to have.

If $k = 1$, then $E = \mathbb{C} \otimes A = A$, then $A \otimes_A \Omega^1 = \rho(1)\Omega^1$, and $\nabla a = \rho(1)d\rho(a)$ for $a \in A$ unital. This differs from d in general, even when $\rho(1)$ is the unit of Ω^0.

(3) It would be immediate.

$\star$ Because, for any ∇_1, ∇_2 connections on E, let $v = \nabla_2 - \nabla_1$. It then follows that v is a linear map from E to $E \otimes_A \Omega^1$ and that $v(\xi x) = (v\xi)x$ for $\xi \in E$, $x \in A$. Namely, v is an element of the vector space V.

For any $v \in V$ and ∇ a connection on E, $v + \nabla$ is also a connection on E.

If $v_{21} = \nabla_2 - \nabla_1$ and $v_{32} = \nabla_3 - \nabla_2$, then $v_{21} + v_{32} = \nabla_3 - \nabla_1 = v_{31} \in V$.

(4) By construction, $E^\sim$ is the projective module over Ω induced by the homomorphism $\rho : A \to \Omega^0$.

$\star$ Note that if A is unital, then $E^\sim = \oplus_{j \geq 0}\rho(1)\Omega^j = \rho(1)\Omega$, on which Ω can act from the right.

The uniqueness in the statement follows from that $\nabla\xi$ is defined for $\xi \in E$.

The existence does from the equality

$$\nabla(\xi a)w + \xi a \otimes dw = (\nabla \xi)aw + \xi \otimes d(aw)$$

for any $\xi \in E$, $a \in A$, and $w \in \Omega$.

$\star$ Note that

$$\nabla(\xi a)w = (\nabla \xi)aw + (\xi \otimes d\rho(a))w$$

and $d(aw) = d(\rho(a))w + a(dw)$ to have.

The left hand side in that equality is $\nabla(\xi a \otimes w)$ which is equal to $\nabla(\xi \otimes aw)$ that is equal to the right hand side, so that the equality above is well defined, to be mentioned. $\square$

$\star$ The connection ∇ seems to be a connection between E over A and $E^\sim$ over Ω.

A cycle over $\mathrm{End}_A(E)$ is now constructed as follows. Start with the graded algebra $\mathrm{End}_\Omega(E^\sim)$.

A $T \in \mathrm{End}_\Omega(E^\sim)$ is of degree k if $T(E^{\sim,j}) \subset E^{\sim,j+k}$.

For any $T \in \mathrm{End}_\Omega(E^\sim)$ of degree k, let $\delta(T) = \nabla T - (-1)^k T\nabla$.

We have

$$\nabla(\xi\omega) = (\nabla\xi)\omega + (-1)^{\deg\xi}\xi d\omega, \quad \xi \in E^{\sim}, \omega \in \Omega,$$

by definition, well defined.

It hence follows that $\delta(T) \in \mathrm{End}_\Omega(E^{\sim})$ of degree $k+1$.

By construction, δ is a graded derivation of $\mathrm{End}_\Omega(E^{\sim})$.

$\star$ For T of degree k and S of degree l, TS of degree $k+l$,

$$\begin{aligned}
&\delta(T)S + (-1)^k T\delta(S) = (\nabla T - (-1)^k T\nabla)S + (-1)^k T(\nabla S - (-1)^l S\nabla) \\
&= \nabla TS - (-1)^{k+l} TS\nabla = \delta(TS).
\end{aligned}$$

This is the reason for the definition of δ.

Since $E^{\sim} = \rho(1)\Omega$ is a finite projective module over Ω, the (closed) graded trace $\int : \Omega^n \to \mathbb{C}$ defines a trace on the graded algebra $\mathrm{End}_\Omega(E^{\sim})$, denoted by the same as $\int$.

Lemma 2.10. *We have $\int \delta(T) = 0$ for any $T \in \mathrm{End}_\Omega(E^{\sim})$ of degree $n-1$.*

Proof. We first replace the connection ∇ by $\nabla' = \nabla + \Gamma$ for $\Gamma \in \mathrm{Hom}_A(E, E \otimes_A \Omega^1)$. Then the corresponding extension to $E^{\sim}$ is given by $\nabla' = \nabla + \Gamma^{\sim}$, where $\Gamma^{\sim} \in \mathrm{End}_\Omega(E^{\sim})$ of degree 1. Thus, it is enough to prove the lemma for some connection on E.

We can assume that $E = pA^k$ for some projection p of $M_k(A)$ and that ∇ is given by $(p \otimes 1)\nabla_0$ on $pE = pA^k = E$ for a connection ∇_0 on A^k.

Then for $T \in \mathrm{End}(E^{\sim}) \subset \mathrm{End}(E_0^{\sim})$ with $E_0 = A^k$,

$$\begin{aligned}
\delta(T) &= \nabla T - (-1)^{\deg T} T\nabla = \nabla T - (-1)^{\deg T}(T\nabla)(p \otimes 1) \\
&= (p \otimes 1)\nabla_0 T - (-1)^{\deg T} T(p \otimes 1)\nabla_0(p \otimes 1) \\
&= (p \otimes 1)\nabla_0 T(p \otimes 1) - (-1)^{\deg T}(p \otimes 1)T\nabla_0(p \otimes 1) \\
&= (p \otimes 1)(\nabla_0 T - (-1)^{\deg T} T\nabla_0)(p \otimes 1) = (p \otimes 1)\delta_0(T)(p \otimes 1).
\end{aligned}$$

As well,

$$\begin{aligned}
\delta_0(T) &= \nabla_0 T - (-1)^{\deg T} T\nabla_0 \\
&= \nabla_0(p \otimes 1)T(p \otimes 1) - (-1)^{\deg T}(p \otimes 1)T(p \otimes 1)\nabla_0 \\
&= \delta_0((p \otimes 1)T(p \otimes 1)).
\end{aligned}$$

Also, the derivation rule implies

$$\begin{aligned}
&\delta_0((p \otimes 1)T(p \otimes 1)) = \delta_0(p \otimes 1)T(p \otimes 1) + (p \otimes 1)\delta_0(T(p \otimes 1)) \\
&= \delta_0(p \otimes 1)T + (p \otimes 1)\delta_0(T)(p \otimes 1) + (-1)^{\deg T}(p \otimes 1)T\delta_0(p \otimes 1) \\
&= \delta_0(p \otimes 1)T + \delta(T) + (-1)^{\deg T} T\delta_0(p \otimes 1).
\end{aligned}$$

Thus, reduced is to the case where $E = A^k = \mathbb{C}^k \otimes A$, with ∇ given by $\nabla(\sum_j \xi_j a_j) = \sum_j \xi_j \otimes d\rho(a_j) \in E \otimes_A \Omega^1$, for (ξ_j) the canonical basis of E.

Let $k = 1$ so $A \otimes_A \Omega^1 = \rho(1)\Omega^1$ and let $p = \rho(1)$. Then we have $E^\sim = p\Omega$, $\mathrm{End}_\Omega(E^\sim) = p\Omega p$, $\nabla(a) = pd\rho(a)$, and $\delta(a) = p(da)p = p(d\rho(a))p$. Thus,

$$\delta(a) = d(pap) - (dp)a - (-1)^{\deg a} adp.$$

With it, by closedness and graded traceness for $\int$,

$$\int \delta(a) = \int \{d(pap) - (dp)a - (-1)^{\deg a} adp\} = 0.$$

Note that $\int \circ d = 0$, $\deg a = 0$ and $\int (dp)a = (-1)^{\deg dp} \int adp = -\int adp$. $\square$

We may have a sort of cycle over $\mathrm{End}_A(E)$ by taking the obvious homomorphism of $\mathrm{End}_A(E)$ in $\mathrm{End}_\Omega(E^\sim)$ with δ like the differential and $\int$ the integral. In fact, the crucial property of δ as $\delta^2 = 0$ is not satisfied.

Proposition 2.11. (1) *The map $\theta = \nabla^2 = \nabla \circ \nabla$ of $E^\sim$ to $E^\sim$ becomes an endomorphism so that $\theta \in \mathrm{End}_\Omega(E^\sim)$ and*

$$\delta^2(T) = \theta T - T\theta, \quad T \in \mathrm{End}_\Omega(E^\sim) = \Omega'.$$

(2) *We have*

$$\langle [E], [\tau] \rangle = \frac{1}{m!} \int \theta^m, \quad n = 2m,$$

where $[E] \in K_0(A)$, and τ is the character of Ω.

Proof. (1) Use the rules

$$\nabla(\eta\omega) = (\nabla\eta)\omega + (-1)^{\deg \eta} \eta d\omega, \quad d^2 = 0$$

to check that $\nabla^2(\eta\omega) = \nabla^2(\eta)\omega$.

$\star$ Note that

$$\begin{aligned}
&\nabla^2(\eta\omega) = \nabla(\nabla(\eta\omega)) \\
&= \nabla((\nabla\eta)\omega) + (-1)^{\deg \eta} \nabla(\eta d\omega) \\
&= \nabla^2(\eta)\omega + (-1)^{\deg \nabla\eta} (\nabla\eta) d\omega + (-1)^{\deg \eta} (\nabla\eta) d\omega + (-1)^{2\deg \eta} \eta d^2\omega \\
&= \nabla^2(\eta)\omega,
\end{aligned}$$

where $\deg \nabla\eta = \deg \eta + 1$.

$\star$ As well, for $T \in \mathrm{End}_\Omega(E^\sim)$ of degree k,

$$\begin{aligned}
&\delta^2(T) = \delta(\delta(T)) \\
&= \delta(\nabla T) - (-1)^k \delta(T\nabla) \\
&= \nabla^2 T - (-1)^{k+1} (\nabla T)\nabla - (-1)^k \nabla(T\nabla) - (-1)^k (-(-1)^{k+1})(T\nabla)\nabla \\
&= \nabla^2 T - T\nabla^2 = \theta T - T\theta,
\end{aligned}$$

where $(\nabla T)\nabla = \nabla(T\nabla)$.

(2) It is shown that

$$\int \theta^m = \int \nabla^{2m} = \int \nabla^n$$

is independent of the connection ∇.

The result is checked by taking the connection on $E = A^k$ as well as $E = pA^k$ as $\nabla(\sum \xi_j a_j) = \sum \xi_j \otimes d\rho(a_j) \in E \otimes_A \Omega^1$.

For $E = pA^k$ with $p \in M_k(A)$ a projection,

$$\langle [E], [\tau] \rangle = \langle [p], [\tau] \rangle = \frac{1}{m!}(\tau \# \mathrm{tr})(p, \cdots, p)$$

for $\tau \in cZ^{2m}(A)$ with $2m = n$.

$\star$ It seems by definition that

$$(\tau \# \mathrm{tr})(p, \cdots, p) = \int \nabla^n$$

or just $\langle [E], [\tau] \rangle = \int \nabla^n$, which does involve τ or $\tau \#\mathrm{tr}$, by independence for ∇.

Let $\nabla' = \nabla + \Gamma$, where Γ is an endomorphism of degree 1 of $E^{\sim}$.

Check that the derivative of $\int \theta_t^m$ is 0, where θ_t corresponds to $\nabla_t = \nabla + t\Gamma$.

$\star$ We have

$$\begin{aligned}\theta_t &= (\nabla_t)^2 = (\nabla + t\Gamma)^2 \\ &= \nabla^2 + t(\nabla\Gamma + \Gamma\nabla) + t^2\Gamma^2.\end{aligned}$$

Thus, $\frac{d}{dt}\theta_t = \nabla\Gamma + \Gamma\nabla + 2t\Gamma^2$. Since Γ has degree 1, then

$$\delta(\Gamma) = \nabla\Gamma - (-1)\Gamma\nabla = \left(\frac{d}{dt}\theta_t\right)_{t=0}.$$

On the other hand, the derivative rule for product implies

$$\frac{d}{dt}\int \theta_t^m = \sum_{j=0}^{m-1} \int \theta_t^j \left(\frac{d}{dt}\theta_t\right) \theta_t^{m-1-j}.$$

In particular, at $t = 0$,

$$\begin{aligned}\left(\frac{d}{dt}\int \theta_t^m\right)_{t=0} &= \sum_{j=0}^{m-1} \int \theta_0^j \left(\frac{d}{dt}\theta_t\right)_{t=0} \theta_0^{m-1-j} \\ &= \sum_{j=0}^{m-1} \int \nabla^{2j} \delta(\Gamma) \nabla^{2(m-1-j)} \\ &= \sum_{j=0}^{m-1} \int \nabla^{2j} (\nabla\Gamma + \Gamma\nabla) \nabla^{2(m-1-j)}.\end{aligned}$$

$\star$ We also have, for $\Gamma\theta^{m-1}$ of degree $2m-1 = n-1$,

$$0 = \int \delta(\Gamma\theta^{m-1}) = \int \delta(\Gamma\nabla^{2(m-1)})$$
$$= \int (\nabla\Gamma\nabla^{2(m-1)} + \Gamma\nabla^{2(m-1)}\nabla),$$

which corresponds to the 0-th term of the sum above.

Similarly, we have, for $\theta^j\Gamma\theta^{m-1-j}$ of degree $2m-1 = n-1$,

$$0 = \int \delta(\theta^j\Gamma\theta^{m-1-j}) = \int \delta(\nabla^{2j}\Gamma\nabla^{2(m-1-j)})$$
$$= \int (\nabla\nabla^{2j}\Gamma\nabla^{2(m-1^j)} + \nabla^{2j}\Gamma\nabla^{2(m-1-j)}\nabla),$$

which corresponds to the j-th term of the sum above.

It then follows that $\int \theta_t^m$ is independent of t.

Thus, connections can be replaced with ∇'.

It then says that it's an independent day for ∇. $\square$

$\star$ Note that $\delta^2(\nabla) = \theta\nabla - \nabla\theta = \nabla^3 - \nabla^3 = 0$. Also, $\delta^2(\mathrm{id}) = 0$. But if T is a certain projection of degree 0 with range degree $\leq k$, which is certainly linear, then θT has range degree $\leq k+2$, while $T\theta$ has range degree $\leq k$. It then follows that δ^2 is not zero.

Lemma 2.12. *Let $(\Omega' = \mathrm{End}_\Omega(E^\sim), \delta, \theta, \int)$ be a quadruple such that Ω' is a graded algebra, δ a graded derivation of degree 1 of Ω', and $\theta \in (\Omega')^2$, satisfying $\delta(\theta) = 0$ and $\delta^2(\omega) = \theta\omega - \omega\theta$ for $\omega \in \Omega'$. Then constructed canonically is a cycle by adjoining to Ω' an element X of degree 1 with $dX = 0$, such that $X^2 = \theta$, $\omega_1 X \omega_2 = 0$ for any $\omega_1, \omega_2 \in \Omega'$.*

$\star$ We might take as $\nabla = X$. For instance, $\nabla a = \rho(1)d\rho(a)$.

Proof. Let Ω'' be the graded algebra obtained by adjoining X to Ω'. Any element ω of Ω'' has the form

$$\omega = \omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X, \quad \omega_{ij} \in \Omega'$$
$$= \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \in M_2(\Omega').$$

Thus, Ω'' is identified with $M_2(\Omega')$ as a vector space.

The product for Ω'' is given by

$$\omega\omega' = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} \omega'_{11} & \omega'_{12} \\ \omega'_{21} & \omega'_{22} \end{pmatrix}$$
$$= \begin{pmatrix} \omega_{11} & \omega_{12}\theta \\ \omega_{21} & \omega_{22}\theta \end{pmatrix} \begin{pmatrix} \omega'_{11} & \omega'_{12} \\ \omega'_{21} & \omega'_{22} \end{pmatrix} = \begin{pmatrix} \omega_{11}\omega'_{11} + \omega_{12}\theta\omega'_{21} & \omega_{11}\omega'_{12} + \omega_{12}\theta\omega'_{22} \\ \omega_{21}\omega'_{11} + \omega_{22}\theta\omega'_{21} & \omega_{21}\omega'_{12} + \omega_{22}\theta\omega'_{22} \end{pmatrix}.$$

⋆ Indeed, we have

$$\begin{aligned}
&\omega\omega' = (\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X)(\omega'_{11} + \omega'_{12}X + X\omega'_{21} + X\omega'_{22}X) \\
&= \omega_{11}\omega'_{11} + \omega_{11}\omega'_{12}X + \omega_{11}X\omega'_{21} + \omega_{11}X\omega'_{22}X \\
&+ \omega_{12}X\omega'_{11} + \omega_{12}X\omega'_{12}X + \omega_{12}XX\omega'_{21} + \omega_{12}XX\omega'_{22}X \\
&+ X\omega_{21}\omega'_{11} + X\omega_{21}\omega'_{12}X + X\omega_{21}X\omega'_{21} + X\omega_{21}X\omega'_{22}X \\
&+ X\omega_{22}X\omega'_{11} + X\omega_{22}X\omega'_{12}X + X\omega_{22}XX\omega'_{21} + X\omega_{22}XX\omega'_{22}X \\
&= \omega_{11}\omega'_{11} + \omega_{11}\omega'_{12}X + 0 + 0 \\
&+ 0 + 0 + \omega_{12}\theta\omega'_{21} + \omega_{12}\theta\omega'_{22}X \\
&+ X\omega_{21}\omega'_{11} + X\omega_{21}\omega'_{12}X + 0 + 0 \\
&+ 0 + 0 + X\omega_{22}\theta\omega'_{21} + X\omega_{22}\theta\omega'_{22}X \\
&= \begin{pmatrix} \omega_{11}\omega'_{11} + \omega_{12}\theta\omega'_{21} & \omega_{11}\omega'_{12} + \omega_{12}\theta\omega'_{22} \\ \omega_{21}\omega'_{11} + \omega_{22}\theta\omega'_{21} & \omega_{21}\omega'_{12} + \omega_{22}\theta\omega'_{22} \end{pmatrix}
\end{aligned}$$

done to the matrix equality.

The grading is obtained by considering X as an element of degree 1. Then the matrix (ω_{ij}) is of degree k when ω_{11} is of degree k, ω_{12}, ω_{21} of degree $k-1$, and ω_{22} of degree $k-2$.

⋆ It is so that all the four terms of ω as such an original from

$$\sum_{i,j=1}^{2} X^{i-1}\omega_{ij}X^{j-1}$$

have degree k.

Checked is that Ω'' is a graded algebra containing Ω'.

⋆ Indeed,

$$\Omega' = \begin{pmatrix} \Omega' & 0 \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} \Omega' & \Omega' \\ \Omega' & \Omega' \end{pmatrix} = \Omega''.$$

Namely, ordering is given as

$$(\Omega'')^k = \begin{pmatrix} (\Omega')^k & (\Omega')^{k-1} \\ (\Omega')^{k-1} & (\Omega')^{k-2} \end{pmatrix}.$$

The differential d is given by the conditions

$$d\omega = \delta(\omega) + X\omega + (-1)^{\deg\omega}\omega X = \begin{pmatrix} \delta(\omega) & (-1)^{\deg\omega}\omega \\ \omega & 0 \end{pmatrix}$$

for $\omega \in \Omega' \subset \Omega''$, and $dX = 0$. Moreover, obtained is that

$$\begin{aligned}
&d\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} \delta(\omega_{11}) & \delta(\omega_{12}) \\ -\delta(\omega_{21}) & -\delta(\omega_{22}) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & -\theta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} + (-1)^{\deg\omega}\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\theta & 0 \end{pmatrix} \\
&= \begin{pmatrix} \delta(\omega_{11}) & \delta(\omega_{12}) \\ -\delta(\omega_{21}) & -\delta(\omega_{22}) \end{pmatrix} + \begin{pmatrix} -\theta\omega_{21} + (-1)^{\deg\omega+1}\omega_{12}\theta & -\theta\omega_{22} + (-1)^{\deg\omega}\omega_{11} \\ \omega_{11} + (-1)^{\deg\omega+1}\omega_{22}\theta & \omega_{12} + (-1)^{\deg\omega}\omega_{21} \end{pmatrix}.
\end{aligned}$$

Checked is that the last second term as two terms of three on the right define graded derivations of Ω'' and that $d^2 = 0$.

$\star$ That left hand side in the equation above with ω homogeneous of degree k is computed as

$$
\begin{aligned}
& d\omega = d \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} = d\omega_{11} + d(\omega_{12}X) + d(X\omega_{21}) + d(X\omega_{22}X) \\
& = \delta(\omega_{11}) + X\omega_{11} + (-1)^k \omega_{11} X \\
& + \delta(\omega_{12}X) + X\omega_{12}X + (-1)^k \omega_{12} X^2 \\
& + \delta(X\omega_{21}) + X^2\omega_{21} + (-1)^k X\omega_{21} X \\
& + \delta(X\omega_{22}X) + X^2\omega_{22}X + (-1)^k X\omega_{22}X^2 \\
& = \delta(\omega_{11}) + X\omega_{11} + (-1)^k \omega_{11} X \\
& + \delta(\omega_{12})X + (-1)^{k-1}\omega_{12}\delta(X) + X\omega_{12}X + (-1)^k \omega_{12}\theta \\
& + \delta(X)\omega_{21} + (-1)X\delta(\omega_{21}) + \theta\omega_{21} + (-1)^k X\omega_{21}X \\
& + \delta(X)\omega_{22}X + (-1)X\delta(\omega_{22}X) + \theta\omega_{22}X + (-1)^k X\omega_{22}\theta
\end{aligned}
$$

with $\delta(\omega_{22}X) = \delta(\omega_{22})X + (-1)^{k-2}\omega_{22}\delta(X)$ and with

$$0 = dX = \delta(X) + X^2 - X^2 = \delta(X)$$

(but $\delta(X) = X^2 - (-1)X^2 = 2\theta$, not assumed so that $X \neq \nabla$ but $X^2 = \theta$, is this correct?), so that the last side of 4 lines is transformed to

$$\begin{pmatrix} \delta(\omega_{11}) & \delta(\omega_{12}) \\ -\delta(\omega_{21}) & -\delta(\omega_{22}) \end{pmatrix} + \begin{pmatrix} (-1)^k\omega_{12}\theta + \theta\omega_{21} & (-1)^k\omega_{11} + \theta\omega_{22} \\ \omega_{11} + (-1)^k\omega_{22}\theta & \omega_{12} + (-1)^k\omega_{21} \end{pmatrix}$$

and the second term is transformed to

$$\begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} + (-1)^k \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}$$

as a possible computation. Therefore, our checking computation shows that the minus sign of θ in the original formula should be removed, where $\delta(X) = 0$ is assumed (as a possible assumption?).

Let $d^{\sim}\omega$ be the last sum term. We may indeed check that

$$
\begin{aligned}
& (d^{\sim})^2\omega = d^{\sim}\left(\begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} + (-1)^k \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} \right) \\
& = \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} + (-1)^{k+1} \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} \\
& + (-1)^k \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} + (-1)^{2k+1} \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}^2 \\
& = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \omega + (-1)\omega \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} = 0!
\end{aligned}
$$

Also, for ω, ω' of degree k, l,

$$d^{\sim}(\omega\omega') = \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \omega\omega' + (-1)^{k+l}\omega\omega' \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}$$

and

$$d^{\sim}(\omega)\omega' + (-1)^k \omega d^{\sim}(\omega') = \left(\begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \omega + (-1)^k \omega \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} \right) \omega'$$
$$+ (-1)^k \omega \left(\begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \omega' + (-1)^l \omega' \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} \right)$$
$$= \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \omega\omega' + (-1)^k \omega \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} \cdot \omega' + (-1)^k \omega \cdot \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \omega' + (-1)^{k+l} \omega\omega' \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}$$

with

$$\begin{aligned} &\omega \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} \cdot \omega' + (-1)^k \omega \cdot \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \omega' \\ &= \omega \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \omega' + (-1)^k \omega \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \omega' \\ &= \omega \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} \omega' + (-1)^k \omega \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} \omega' \end{aligned}$$

which is zero only for k odd. There so might involve degree sign in the product rule from the first?

As well,

$$d \begin{pmatrix} \delta(\omega_{11}) & \delta(\omega_{12}) \\ -\delta(\omega_{21}) & -\delta(\omega_{22}) \end{pmatrix} = \begin{pmatrix} \delta^2(\omega_{11}) & \delta^2(\omega_{12}) \\ (-1)^2\delta^2(\omega_{21}) & (-1)^2\delta^2(\omega_{22}) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta(\omega_{11}) & \delta(\omega_{12}) \\ -\delta(\omega_{21}) & -\delta(\omega_{22}) \end{pmatrix} + (-1)^{k+1} \begin{pmatrix} \delta(\omega_{11}) & \delta(\omega_{12}) \\ -\delta(\omega_{21}) & -\delta(\omega_{22}) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\theta\delta(\omega_{21}) + (-1)^{k+1}\delta(\omega_{12})\theta & -\theta\delta(\omega_{22}) + (-1)^{k+1}\delta(\omega_{11}) \\ \delta(\omega_{11}) + (-1)^k\delta(\omega_{22})\theta & \delta(\omega_{12}) + (-1)^k\delta(\omega_{21}) \end{pmatrix}$$

.

Let d_δ be the first term part of d involving δ. Then

$$\begin{aligned} &d_\delta \left(\begin{pmatrix} (-1)^k\omega_{12}\theta + \theta\omega_{21} & (-1)^k\omega_{11} + \theta\omega_{22} \\ \omega_{11} + (-1)^k\omega_{22}\theta & \omega_{12} + (-1)^k\omega_{21} \end{pmatrix} \right) \\ &= \begin{pmatrix} (-1)^k\delta(\omega_{12})\theta + (-1)^2\theta\delta(\omega_{21}) & (-1)^k\delta(\omega_{11}) + (-1)^2\theta\delta(\omega_{22}) \\ -\delta(\omega_{11}) + (-1)^{k+1}\delta(\omega_{22})\theta & -\delta(\omega_{12}) + (-1)^{k+1}\delta(\omega_{21}) \end{pmatrix}. \end{aligned}$$

Summing up, it follows by cancellation that $d^2 = 0$.
It is checked finally that the following equality

$$\int (\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X) = \int \omega_{11} - (-1)^{\deg\omega} \int \omega_{22}\theta$$

defines a closed graded trace.

⋆ Closedness follows by

$$\begin{aligned}\int d\omega &= \int \delta(\omega) + \int (X\omega + (-1)^{\deg\omega}\omega X)\\ &= 0 + (-1)^{\deg\omega}\int \omega X + (-1)^{\deg\omega}\int \omega X = 0.\end{aligned}$$

If so, multiplication by -1 is necessary at $(-1)^{\deg\omega}$ in the definition for $d\omega$.

Note that

$$\int X\omega_{22}X = (-1)^{\deg\omega - 2 + 1}\int \omega X^2 = -(-1)^{\deg\omega}\int \omega_{22}\theta.$$

Also, $\int \omega_{12}X = -\int X\omega_{12}$. Thus, for the equality of the trace we may assume from the first that ω is symmetric in the sense that $\omega_{12} = \omega_{21}$? □

Corollary 2.13. *Let $\rho : A \to \Omega$ be a cycle over A, E a finite projective module over A, and $A' = \mathrm{End}_A(E)$, $E^\sim$ over Ω. To each connection ∇ on E corresponds canonically a cycle $\rho' : A' \to \Omega'' = M_2(\Omega') = M_2(\mathrm{End}_\Omega(E^\sim))$ (or $\mathrm{Sym}_2(\Omega')$) over A'.*

It can be shown that the character $\tau' \in cZ^n(A')$ of that new cycle has a class $[\tau'] \in cH^n(A')$ independent of the choice of the connection ∇, which coincides with the class given by the first lemma. It is checked that a reciprocity formula takes care of Morita equivalence.

Corollary 2.14. *Let A and B be unital algebras and E an (A, B)-bimodule, finite projective on both sides, wiht $A = \mathrm{End}_B(E)$ and $B = \mathrm{End}_A(E)$. Then $cH^*(A)$ is canonically isomorphic to $cH^*(B)$.*

Proof. The diagram involving a cycle Ω and a homomorphism ρ

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes_A \Omega^1 \\ \uparrow & & \\ A & \xrightarrow{\rho} & \Omega \end{array}$$

corresponds to

$$B = \mathrm{End}_A(E) \xrightarrow{\rho'} \Omega'' = M_2(\Omega') = M_2(\mathrm{End}_\Omega(E^\sim)).$$

As well, A and B can be exchanged in the correspondence. Moreover, that is extended to the context in cyclic cohomology. □

Let A be an abelian algebra and E a finite projective module over A. Then there is a homomorphism from A to $A' = \mathrm{End}_A(E)$.

⋆ if $E = \mathbb{C}^k \otimes A$, then we may define $\varphi(a)(\xi \otimes b) = \xi \otimes ba \in E$ for $a \in A$.

Corollary 2.15. *Let A be an abelian algebra. Then $cH^*(A)$ is a module over the ring $K_0(A)$ in a natural manner.*

Proof. We have $cH^*(A') = cH^*(\mathrm{End}_A(E))$ mapped to $cH^*(A)$ by restriction to A.

$$A \xrightarrow{\varphi} A' = \mathrm{End}_A(E) \xrightarrow{\rho'} \Omega'$$

A projection p in A' represents a class of $K_0(A)$, and E can be replaced with pE in the diagram. □

Example 2.16. Let M be a compact oriented smooth manifold. Let $A = C^\infty(M)$ and Ω be the cycle over A given by the ordinary de Rham complex and integration of forms of degree n. Let $V \to M$ be a complex vector bundle over M and $E = C^\infty(M, V)$ the corresponding finite projective module over A. Then the notion of connection coincides with the usual one.

$$\begin{array}{ccc} E = C^\infty(M,V) & \xrightarrow{\nabla} & C^\infty(M,V) \otimes_{C^\infty(M)} \Omega^1 \\ \uparrow & & \\ A = C^\infty(M) & \longrightarrow & \Omega. \end{array}$$

Then there is a cocycle $\tau \in cZ^n(A)$ associated to ∇ canonically.

Proposition 2.17. *Let ω_k be the differential form of degree $2k$ on M which gives the component of degree $2k$ of the Chern character of the bundle $V \to M$ with connection ∇. Namely, $\omega_k = \frac{1}{k!}\mathrm{tr}(\theta^k)$, where θ is the curvature form. Then we have the equality $\tau = \sum S^k \omega_k^\sim$, where $\omega_k^\sim \in cZ^{n-2k}(A)$ is given by*

$$\omega_k^\sim(f_0, \cdots, f_{n-2k}) = \int f_0 df_1 \wedge \cdots \wedge df_{n-2k} \wedge \omega_k,$$

$f_j \in A = C^\infty(M)$, and where τ is the restriction to A of the character of the cycle associated to the bundle V, the connection ∇, and the de Rham cycle of A

⋆ Note that $S^k\omega_k^\sim$ is in $cH^{(n-2k)+2k}(A) = cH^n(A)$ mapped by the S map k-times in cH level. Note as well that $H^*(C^\infty(M))$ is isomorphic to $H_*(M, \mathbb{C})$ with filtration by dimension.

In that example of $A = C^\infty(M)$, the pairing between cyclic cohomology and $K_0(A)$ is given back by the ordinary Chern character of vector bundles.

Example 2.18. Let $\mathfrak{A}_\theta$ the irrational rotation C^*-algebra for an irrational θ angle on the circle as a $\mathbb{Z}$-action. The smooth norm-dense subalgebra A_θ of $\mathfrak{A}_\theta$ is stable under holomorphic functional calculus. By a result of Pimsner and Voiculescu, we have $K_0(A_\theta) = \mathbb{Z}^2$ ([13]). An explicit description of finite projective modules over A_θ is given by Connes [3]. It is obtained by Rieffel [15] that all the finite projective modules over A_θ are classified up to isomorphism.

Example 2.19. Let $(p, q) \in \mathbb{Z}^2$ be a pair of relatively prime integers with $q > 0$ and $p = 0$ allowed. Constructed is a finite projective module $E = E_{p,q}$ over A_θ as follows. Let $Sw(\mathbb{R})$ be the usual Schwartz space of (rapidly decreasing or compactly supported, smooth) complex-valued functions on the real line. Let V_1 and V_2 be operators on $Sw(\mathbb{R})$ defined by

$$(V_1\xi)(s) = \xi(s - \varepsilon), \quad \varepsilon = \frac{p}{q} - \theta, \quad s \in \mathbb{R}, \quad \xi \in Sw(\mathbb{R}),$$
$$(V_2\xi)(s) = e^{2\pi i s}\xi(s) = e(s)\xi(s).$$

It then follows that

$$(V_2V_1\xi)(s) = e(s)(V_1\xi)(s) = e(s)\xi(s - \varepsilon)$$
$$(V_1V_2\xi)(s) = (V_2\xi)(s - \varepsilon) = e(s - \varepsilon)\xi(s - \varepsilon)$$

so that $V_2V_1 = e(\varepsilon)V_1V_2$ holds.

Let K be a finite-dimensional Hilbert space with unitary operators w_1 and w_2 on K such that

$$w_2w_1 = \overline{e(\frac{p}{q})}w_1w_2, \quad w_1^q = w_2^q = 1.$$

Namely, K is viewed as a finite-dimensional representation of the Heisenberg commutation relations for the finite cyclic group $\mathbb{Z}/q\mathbb{Z}$.. It is decomposed into a direct sum of d equivalent irreducible representations.

Let $E = Sw(\mathbb{R}) \otimes K$. We have as operators on E,

$$(V_2 \otimes w_2)(V_1 \otimes w_1) = (V_2V_1) \otimes w_2w_1$$
$$= e(\varepsilon)V_1V_2 \otimes e(-\frac{p}{q})w_1w_2 = e(-\theta)V_1V_2 \otimes w_1w_2 = \overline{\lambda}(V_1 \otimes w_1)(V_2 \otimes w_2)$$

with $\lambda = e(\theta)$.

E becomes a right A_θ-module as follows. Define $\xi U_1 = (V_1 \otimes w_1)\xi$ and $\xi U_2 = (V_2 \otimes w_2)\xi$ for any $\xi \in E = Sw(\mathbb{R}) \otimes K$, where U_1, U_2 are unitary generators of A_θ with the relation $U_2U_1 = e(\theta)U_1U_2$.

$\star$ Note that

$$(\xi U_1)U_2 = (V_2 \otimes w_2)(\xi U_1) = (V_2 \otimes w_2)(V_1 \otimes w_1)\xi$$
$$= \overline{\lambda}(V_1V_2 \otimes w_1w_2)\xi$$

which should be $\xi(U_1U_2)$ by definition. Also,

$$(\xi U_2)U_1 = (V_1 \otimes w_1)(\xi U_2) = (V_1 \otimes w_1)(V_2 \otimes w_2)\xi$$
$$= \lambda(V_2V_1 \otimes w_2w_1)\xi$$

which should be $\xi(U_2U_1)$ by definition. If so, certainly $\xi(U_2U_1)$ coincides with $\xi(\lambda U_1U_2) = \lambda\xi(U_1U_2) = (V_1V_2 \otimes w_1w_2)\xi$.

This shows that the definition equality compatible with the representation of A_θ.

It is checked that the elements ξa for any $a \in A_\theta$ belong to E using the general properties of nuclear spaces.

$\star$ For $\xi = f \otimes \eta$ with $f \in Sw(\mathbb{R})$ and $\eta \in K$, certainly enough we have $V_j f \otimes w_j \eta$ belonging to E, without the general property.

It is obtained by Connes [3] that the right module $E = Sw(\mathbb{R}) \otimes K$ over A_θ is finite and projective. Moreover, it is shown by Rieffel [15] that

Theorem 2.20. *Let E be a finite projective module over A_θ. Then either E is free so that $E = A_\theta^p$ for some positive integer p, or E is isomorphic to $Sw(\mathbb{R})\otimes K$ with module structure as above.*

We may now recall that $H^{\mathrm{ev}}(A_\theta) = H^2(A_\theta)$ is a vector space of dimension 2 with basis of $S\tau$ and φ, where τ is the canonical trace of A_θ and φ is the cyclic 2-cocycle given by

$$\varphi(x_0, x_1, x_2) = \frac{1}{2\pi i}\tau(x_0(\delta_1(x_1)\delta_2(x_2) - \delta_2(x_1)\delta_1(x_2)))$$

for $x_j \in A_\theta$, where δ_1, δ_2 are the natural commuting derivations of A_θ.

$\star$ Note that by definition,

$$\delta_1(U_1^{n_1}U_2^{n_2}) = 2\pi i n_1 U_1^{n_1}U_2^{n_2}, \quad \delta_2(U_1^{n_1}U_2^{n_2}) = 2\pi i n_2 U_1^{n_1}U_2^{n_2}.$$

Since we have

$$(U_1^{n_1}U_2^{n_2})(U_1^{m_1}U_2^{m_2}) = U_1^{n_1}e(\theta)^{n_2 m_1}U_1^{m_1}U_2^{n_2}U_2^{m_2} = e(\theta)^{n_2 m_1}U_1^{n_1+m_1}U_2^{n_2+m_2},$$

then

$$\begin{aligned}
&\delta_1((U_1^{n_1}U_2^{n_2})(U_1^{m_1}U_2^{m_2})) = e(\theta)^{n_2 m_1}2\pi i(n_1+m_1)U_1^{n_1+m_1}U_2^{n_2+m_2}, \\
&\delta_1(U_1^{n_1}U_2^{n_2})U_1^{m_1}U_2^{m_2} + U_1^{n_1}U_2^{n_2}\delta_1(U_1^{m_1}U_2^{m_2}) \\
&= 2\pi i n_1(U_1^{n_1}U_2^{n_2})(U_1^{m_1}U_2^{m_2}) + (U_1^{n_1}U_2^{n_2})2\pi i m_1(U_1^{m_1}U_2^{m_2})
\end{aligned}$$

and both of which certainly coincides. Hence, δ_1 is a derivation, and similarly so is δ_2. Also,

$$\delta_1(\delta_2(U_1^{n_1}U_2^{n_2})) = \delta_1(2\pi i n_2 U_1^{n_1}U_2^{n_2}) = (2\pi i)^2 n_1 n_2 U_1^{n_1}U_2^{n_2},$$

which certainly coincides with $\delta_2(\delta_1(U_1^{n_1}U_2^{n_2}))$ showing that δ_j are commuting.

The pairing of K_0 with $S\tau$ is the same as that with the trace τ, and is given by the Murray and Neumann dimension as (exchanged)

$$\langle E_{p,q}, \tau\rangle = p - \theta q.$$

To computing the pairing of K_0 with φ we use the following 2-dimensional cycle with character φ, as follows.

As a graded algebra, Ω^* as a cycle is the tensor product of A_θ by the exterior algebra $\wedge^*\mathbb{C}^2$ of the 2-dimensional vector space $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$.

The differential d is uniquely specified by the equality

$$d(a \otimes \alpha) = \delta_1(a) \otimes (e_1 \wedge \alpha) + \delta_2(a) \otimes (e_2 \wedge \alpha)$$

for $a \in A_\theta$, $\alpha \in \wedge^*\mathbb{C}^2$.

The graded trace from Ω^2 to $\mathbb{C}$ is given as well

$$a \otimes (e_1 \wedge e_2) \mapsto \frac{1}{2\pi i}\tau(a) \in \mathbb{C}.$$

A connection (or covariant differentiation) ∇ on a finite projective module E over A_θ (tensored with $\wedge^*\mathbb{C}^2$) is given by a pair of covariant differentials ∇_j as well as $\nabla_j\xi$, satisfying as definition

$$\nabla_j(\xi a) = (\nabla_j\xi)a + \xi\delta_j(a), \quad \xi \in E, a \in A_\theta, j = 1, 2$$

(cf. [8]).

$\star$ Defined is just as $\nabla = (\nabla_1 \otimes e_1) + (\nabla_2 \otimes e_2)$.

The curvature $\Theta = \nabla^2$ is identified as

$$(\nabla_1\nabla_2 - \nabla_2\nabla_1) \otimes (e_1 \wedge e_2).$$

$\star$ In fact, as desired,

$$\begin{aligned}\nabla^2 &= \nabla_1^2 \otimes (e_1 \wedge e_1) + \nabla_1\nabla_2 \otimes (e_1 \wedge e_2) + \nabla_2\nabla_1 \otimes (e_2 \wedge e_1) + \nabla_2^2 \otimes (e_2 \wedge e_2)\\ &= (\nabla_1\nabla_2 - \nabla_2\nabla_1) \otimes (e_1 \wedge e_2)\end{aligned}$$

with $e_j \wedge e_j = 0 = (e_1 \wedge e_2) + (e_2 \wedge e_1)$.

$\star$ Note that

$$\begin{aligned}\nabla_k(\nabla_j(\xi a)) &= \nabla_k((\nabla_j\xi)a) + \nabla_k(\xi\delta_j(a))\\ &= \nabla_k((\nabla_j\xi))a + (\nabla_j\xi)\delta_k(a) + \nabla_k(\xi)\delta_j(a) + \xi\delta_k(\delta_j(a)).\end{aligned}$$

Proposition 2.21. ([3]). *Let $E = E_{p,q} = Sw(\mathbb{R}) \otimes K = Sw(\mathbb{R}, K)$ viewed as the algebra of K-valued Schwartz functions on $\mathbb{R}$ be the finite projective module over A_θ. A connection ∇ on E is defined as a formula by*

$$(\nabla_1\xi)(s) = \frac{2\pi i}{\varepsilon}s\xi(s), \quad (\nabla_2\xi)(s) = \frac{d\xi}{ds}(s), \quad \xi \in E$$

where $\varepsilon = \frac{p}{q} - \theta$, $(p, q) \in \mathbb{Z}^2$ relatively prime integers, $q > 0$.

The curvature of this connection is constant, equal to $-\frac{2\pi i}{\varepsilon} \otimes (e_1 \wedge e_2)$.

$\star$ Note that

$$\begin{aligned}\nabla_1(\xi U_1)(s) &= \frac{2\pi i}{\varepsilon}s(\xi U_1)(s) = \frac{2\pi i}{\varepsilon}s(V_1 \otimes w_1)\xi(s)\\ &= \frac{2\pi i}{\varepsilon}sw_1(\xi(s-\varepsilon)),\\ ((\nabla_1\xi)U_1)(s) &= w_1(\nabla_1\xi)(s-\varepsilon) = \frac{2\pi i}{\varepsilon}(s-\varepsilon)w_1(\xi(s-\varepsilon)).\end{aligned}$$

Subtracting both sides implies

$$\nabla_1(\xi U_1)(s) - ((\nabla_1\xi)U_1)(s) = 2\pi i(\xi U_1)(s)$$

with $2\pi i U_1 = (2\pi i)1U_1^1 = \delta_1(U_1)$. Also note that

$$\begin{aligned}\nabla_1(\xi U_2)(s) &= \frac{2\pi i}{\varepsilon}s(\xi U_2)(s) = \frac{2\pi i}{\varepsilon}s(V_2 \otimes w_2)\xi(s)\\ &= \frac{2\pi i}{\varepsilon}se(s)w_2(\xi(s)),\\ ((\nabla_1\xi)U_2)(s) &= e(s)w_2(\nabla_1\xi)(s) = \frac{2\pi i}{\varepsilon}e(s)sw_2(\xi(s)).\end{aligned}$$

Subtracting both sides implies

$$\nabla_1(\xi U_2)(s) - ((\nabla_1\xi)U_2)(s) = 0$$

with $\delta_1(U_2) = \delta_1(U_1^0 U_2) = 2\pi i 0 U_1^0 U_2 = 0$. It then follows that ∇_1 is certainly a connection.

$\star$ Similarly, note that

$$\begin{aligned}\nabla_2(\xi U_1)(s) &= \frac{d}{ds}(\xi U_1)(s) = \frac{d}{ds}w_1(\xi(s-\varepsilon)).\\ ((\nabla_2\xi)U_1)(s) &= w_1(\nabla_2\xi)(s-\varepsilon) = w_1\frac{d}{ds}\xi(s-\varepsilon).\end{aligned}$$

Subtracting both sides implies

$$\nabla_2(\xi U_1)(s) - ((\nabla_2\xi)U_1)(s) = \frac{d}{ds}(\xi U_1)(s) - (\frac{d\xi}{ds}U_1)(s) = 0$$

with $\frac{d}{ds}w_1 = w_1\frac{d}{ds}$ since w_1 is a unitary on K but a constant matrix and with $\delta_2(U_1) = \delta_2(U_1U_2^0) = 2\pi i 0 U_1 U_2^0 = 0$. Moreover,

$$\begin{aligned}\nabla_2(\xi U_2)(s) &= \frac{d}{ds}(\xi U_2)(s) = \frac{d}{ds}(e(s)w_2(\xi(s))).\\ ((\nabla_2\xi)U_2)(s) &= e(s)w_2(\nabla_2\xi)(s) = e(s)w_2\frac{d}{ds}\xi(s).\end{aligned}$$

Subtracting both sides implies

$$\nabla_2(\xi U_2)(s) - ((\nabla_2\xi)U_2)(s) = 2\pi i e(s)w_2(\xi(s)) = 2\pi i(\xi U_2)(s)$$

with $\delta_2(U_2) = 2\pi i U_2$. It then follows that ∇_2 is certainly a connection!

$\star$ The connection ∇ but on $E \otimes (\wedge^*\mathbb{C}^2)$ may be defined as $\nabla_1 \otimes e_1 + \nabla_2 \otimes e_2$.

$\star$ The curvature $\Theta = \nabla^2 = (\nabla_1\nabla_2 - \nabla_2\nabla_1) \otimes (e_1 \wedge e_2)$ is computed as in the following. In particular,

$$\begin{aligned}(\nabla_1(\nabla_2\xi))(s) &= \frac{2\pi i}{\varepsilon}s(\nabla_2\xi)(s) = \frac{2\pi i}{\varepsilon}s\frac{d\xi}{ds}(s),\\ (\nabla_2(\nabla_1\xi))(s) &= \frac{d}{ds}(\nabla_1\xi)(s) = \frac{d}{ds}\left(\frac{2\pi i}{\varepsilon}s\xi(s)\right)\\ &= \frac{2\pi i}{\varepsilon}\xi(s) + \frac{2\pi i}{\varepsilon}s\frac{d\xi}{ds}(s).\end{aligned}$$

Subtracting both sides implies

$$(\nabla_1\nabla_2 - \nabla_2\nabla_1)(\xi)(s) = -\frac{2\pi i}{\varepsilon}\xi(s).$$

That's it.

Lemma 2.22. *The value of the pairing of $E = E_{p,q}$ and φ the character of Ω as the cyclic 2-cocycle by the graded trace $\int$ on $E \otimes (\wedge^*\mathbb{C}^2)$ or Ω^2 is given by*

$$\begin{aligned}\langle E_{p,q}, \varphi\rangle &= \frac{1}{1!}\int \theta = \int (-\frac{2\pi i}{\varepsilon})1 \otimes (e_1 \wedge e_2) \\ &= \frac{1}{2\pi i}(-\frac{2\pi i}{\varepsilon})\tau(1) = -\frac{1}{\varepsilon}(p - \theta q) \\ &= -\frac{1}{\frac{p}{q} - \theta}(p - \theta q) = -q \in \mathbb{Z}.\end{aligned}$$

Corollary 2.23. *Let $\varphi \in cH^2(A_\theta)$ be the cyclic 2-cocycle as a class. Then*

$$\langle K_0(A_\theta), \varphi\rangle \subset \mathbb{Z}.$$

Proof. Note that $K_0(A_\theta) = \mathbb{Z}^2$ as of $E_{p,q}$ for $(p,q) \in \mathbb{Z} \times \mathbb{Z}_+$. $\square$

$\star$ For $E = A_\theta^p$, $\langle E, \varphi\rangle$ is computed as $\frac{1}{1!}\varphi(1,1,1)\mathrm{tr}(1_p) = \frac{1}{2\pi i}\tau(0)p = 0$.

Remark. The filtration of even $H^{\mathrm{ev}}(A_\theta)$ for $\theta \notin \mathbb{Q}$ by dimensions is not compatible with the lattice dual to $K_0(A_\theta)$, since the zero dimensional class of τ does not pair integrally with K-theory K_0.

$\star$ Namely, $\langle E_{p,q}, \tau\rangle = p - \theta q \notin \mathbb{Z}$.

3 The noncommutative rotation by elliptic index theory

The rotation C^*-algebra $\mathfrak{A}_\theta$ is the universal C^*-algebra generated by two unitaries U_1 and U_2 such that $U_2U_1 = \lambda U_1U_2$ with $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$.

The subalgebra A_θ as the smooth structure of $\mathfrak{A}_\theta$ is defined to be the set of all $\sum_{n,m} a_{nm}U_1^nU_2^m$ with $a = (a_{nm}) \in Sw(\mathbb{Z}^2)$, where $Sw(\mathbb{Z}^2)$ is the linear space of sequences of rapid decay on $\mathbb{Z}^2$ such that the sequence $((|n|^k + |m|^k)|a_{nm}|)$ is bounded for any positive $k > 0$.

Example 3.1. Let $(a_{nm}) = (e^{-|n|-|m|})$. Then, certainly,

$$\begin{aligned}(|n|^k + |m|^k)e^{-|n|-|m|} &= \frac{|n|^k}{e^{|n|+|m|}} + \frac{|m|^k}{e^{|n|+|m|}} \\ &\le \frac{|n|^k}{e^{|n|}} + \frac{|m|^k}{e^{|m|}} \to 0 \quad (|n|, |m|) \to (\infty, \infty).\end{aligned}$$

The smooth subalgebra A_θ has stability under holomorphic functional calculus in $\mathfrak{A}_\theta$.

The Schwartz space $E = Sw(\mathbb{R})$ as a space of sections for smooth vector bundles over A_θ, as on which elliptic operators act, consists of smooth functions ξ on $\mathbb{R}$ with derivatives of rapid decay.

The right action of A_θ on E is given by $(\xi U_1)(s) = \xi(s+\theta)$ (or $\xi(s-\theta)$) and $(\xi U_2)(s) = e^{2\pi i s}\xi(s)$ for $\xi \in E$, $s \in \mathbb{R}$.

The vector fields on A_θ are given by the derivations δ_j, $j = 1, 2$ defined by $\delta_j(U_k) = 0$ for $j \neq k$ and $\delta_j(U_j) = 2\pi i U_j$.

The covariant derivatives (or connections) as lifting to the smooth sections $\xi \in E$ are given by $(\nabla_1 \xi)(s) = -\frac{2\pi i}{\varepsilon} s\xi(s)$ and $(\nabla_2\xi)(s) = \frac{d\xi}{ds}$.

Namely,

$$\nabla_j(\xi x) = (\nabla_j \xi)x + \xi\delta_j(x)$$

is satisfied.

Remark. In Fourier analysis, adding (or subtracting) a constant in a variable as an operator is transformed to multiplying the corresponding rotation with another variable times the constant as an operator, under the Fourier transform. As well, differentiating a function as an operator is transformed to multiplying another variable times i as an operator.

We consider the operators D on $E = Sw(\mathbb{R})$ of the form as a finite sum

$$D = \sum C_{\alpha,\beta} \nabla_1^\alpha \nabla_2^\beta$$

where the coefficients $C_{\alpha,\beta}$ are operators of order 0 on E, which are elements of $\mathrm{End}_{A_\theta}(E)$.

In the commutative case of $\theta = 1$, the operators D above are all the ordinary differential operators on the sections of the bundle as a nontrivial line bundle.

The operators in the general case can be treated in the same way as in the commutative case as the elliptic theory available.

Note that such an operator D is viewed as an arbitrary element of the algebra of operators on $Sw(\mathbb{R}) \subset L^2(\mathbb{R})$ the Hilbert space, generated by the multiplication operator T by s a variable defined as $(T\xi)(s) = s\xi(s)$, $s \in \mathbb{R}$, but $T = -\frac{\theta}{2\pi i}\nabla_1$, the differential operator $\frac{d}{ds}$, where $\frac{d}{ds} = \nabla_2$, the multiplication operator $R_{\frac{1}{\theta}}$ by a rotation by $\frac{1}{\theta}$ defines as $(R_{\frac{1}{\theta}}\xi)(s) = e^{\frac{2\pi i s}{\theta}}\xi(s)$, and the finite difference operator Δ as $(\Delta\xi)(s) = \xi(s+1) - \xi(s)$.

In fact, the algebra $\mathrm{End}_{A_\theta}(E)$ of endomorphisms of E is isomorphic to $A_{\frac{1}{\theta}}$ generated by (certain) unitaries V_1 an V_2 such that $V_2V_1 = e^{\frac{2\pi i}{\theta}}V_1V_2$.

Indeed, the formula is given as represented by

$$(V_1\xi)(s) = \xi(s+1), \quad (V_2\xi)(s) = e^{-\frac{2\pi i s}{\theta}}\xi(s).$$

Thus, in particular, we have $R_{\frac{1}{\theta}} = V_2^{-1}$.

$\star$ Note that

$$(V_1(V_2\xi))(s) = (V_2\xi)(s+1) = e^{-\frac{2\pi i(s+1)}{\theta}}\xi(s+1)$$
$$(V_2(V_1\xi))(s) = e^{-\frac{2\pi i s}{\theta}}(V_1\xi)(s) = e^{-\frac{2\pi i s}{\theta}}\xi(s+1).$$

Thus, $e^{\frac{2\pi i}{\theta}} V_1 V_2 = V_2 V_1$.

It follows that the operator Δ is recovered from V_1 and the identity operator 1.

Moreover, the operator $\frac{d}{ds}$ is recovered from Δ and V_1 with taking limit. Indeed, note that

$$\begin{aligned}(\Delta^2 \xi)(s) &= \Delta(\xi(s+1) - \xi(s)) \\ &= (\xi(s+2) - \xi(s+1)) - (\xi(s+1) - \xi(s)) \\ &= \xi(s+2) - \xi(s) - 2(V_1^2 \xi)(s).\end{aligned}$$

We also have $(V_1^2 - 1)(\xi)(s) = \xi(s+2) - \xi(s)$. By induction, $(V_1^k - 1)(\xi)(s) = \xi(s+k) - \xi(s)$ for $k \in \mathbb{Z}$. Functional Calculus would implies that this is valid for k real.

Furthermore, $\frac{d}{ds}$ may be identified with is as the multiplication operator as under the Fourier transform.

Allowed is infinite sums like $\sum a_{n,m} V_1^n V_2^m$ for $a = (a_{n,m}) \in Sw(\mathbb{Z}^2)$.

In particular, included are arbitrary smooth functions with period θ acting by multiplication on $Sw(\mathbb{R})$.

As analogy as in the commutative case,

Definition 3.2. An operator D on $E = Sw(\mathbb{R})$ has order $\leq n$ for $n \in \mathbb{N}$ if

$$D = \sum_{0 \leq \alpha+\beta \leq n} C_{\alpha,\beta} \nabla_1^\alpha \nabla_2^\beta, \quad C_{\alpha,\beta} \in \mathrm{End}_{A_\theta}(E) = A_{\frac{1}{\theta}}.$$

The symbol $\sigma_n(D)$ of order n for D with order $\leq n$ is defined as the map σ from the circle S^1 to $A_{\frac{1}{\theta}}$ defined by

$$\sigma(\eta_1, \eta_2) = \sum_{\alpha+\beta=n} \eta_1^\alpha \eta_2^\beta C_{\alpha,\beta}, \quad \eta_1, \eta_2 \in \mathbb{R}, \quad \eta_1^2 + \eta_2^2 = 1.$$

We say that D is elliptic if $\sigma(\eta_1, \eta_2) \in A_{\frac{1}{\theta}}$ is invertible for any $(\eta_1, \eta_2) \in S^1 \subset \mathbb{R}^2$.

Remark. Since $A_{\frac{1}{\theta}}$ is stable under holomorphic functional calculus in the C^*-algebra $\mathfrak{A}_{\frac{1}{\theta}}$ also defined as the norm closure in the C^*-algebra $\mathbb{B}(L^2(\mathbb{R}))$ of all bounded operators on $L^2(\mathbb{R})$ the Hilbert space, the statement as the definition for being elliptic above can be equivalently replaced by saying that: $\sigma(\eta_1, \eta_2)$ is an invertible operator on $L^2(\mathbb{R})$ for any $(\eta_1, \eta_2) \in S^1$.

As analogue as one of the main results in the classical theory of elliptic differential operators, it is obtained by [3] that

Theorem 3.3. *Let D be an elliptic operator on E as well as $L^2(\mathbb{R})$ in the sense above. Then the space* $\ker(D) = \{\xi \in L^2(\mathbb{R}) \mid D\xi = 0\}$ *is finite dimensional. As well,* $\ker(D)$ *in this sense is contained in* $Sw(\mathbb{R})$.

Proof. $\star$ Note that

$$(\nabla_1^2\xi)(s) = (-\frac{2\pi i}{\varepsilon}s)^2\xi(s) = -\frac{4\pi^2}{\varepsilon^2}s^2\xi(s),$$
$$(\nabla_2^2\xi)(s) = \frac{d^2}{ds^2}\xi(s).$$

It follows that $\nabla_1^2 + \nabla_2^2$ is not the identity operator.

But if $\frac{d}{ds}$ may be identified with is as under the Fourier transform, then $\frac{d^2}{ds^2}$ may be done with $-s^2$.

However, it or that are unnecessary by the reason given later soon.

$\star$ We let $\mathfrak{D}$ be the algebra generated by $A_{\frac{1}{\theta}}$ and ∇_1, ∇_2 and the C^*-algebra $\mathbb{K}$ of all compact operators on $L^2(\mathbb{R})$. Since $A_{\frac{1}{\theta}}$ is unital, then $\mathfrak{D}$ is unital.

Note that

$$\begin{aligned}(\nabla_1(\nabla_2\xi))(s) &= -\frac{2\pi i}{\varepsilon}s\frac{d\xi}{ds},\\ (\nabla_2(\nabla_1\xi))(s) &= \frac{d}{ds}(-\frac{2\pi i}{\varepsilon}s\xi(s))\\ &\quad - \frac{2\pi i}{\varepsilon}\left(\xi(s) + s\frac{d\xi}{ds}(s)\right).\end{aligned}$$

Therefore, $\nabla_2\nabla_1 = -\frac{2\pi i}{\varepsilon}1 + \nabla_1\nabla_2$ on E as well as $L^2(\mathbb{R})$.

$\star$ There is the following short exact sequence of C^*-algebras:

$$0 \to \mathbb{K} \to \mathbb{B} = \mathbb{B}(L^2(\mathbb{R})) \to \mathbb{B}/\mathbb{K} \to 0.$$

By definition as well as the Fredholm operator theory, a bounded operator T of $\mathbb{B} = \mathbb{B}(H)$ on a Hilbert space H is said to be a Fredholm operator if its image in the quotient Calkin algebra $\mathbb{B}/\mathbb{K}$ is invertible. Equivalently, both $\ker(T)$ and $\ker(T^*)$ for T^* the adjoint operator of T with respect to the inner product on H^2 are finite dimensional and $T(H)$ is closed (cf. [12]).

$\star$ The symbol map denoted as σ sending each D with order $\leq n$ to $\sigma_n(D)$ induces a (involutive) $*$-homomorphism Σ from $\mathfrak{D}$ to $C(S^1, A_{\frac{1}{\theta}})$ which is viewed in $\mathbb{B}/\mathbb{K}$, by substituting unbounded ∇_1, ∇_2 (in this case) with commuting variables such as η_1, η_2 with domain (any) compact, for the quotient to be unital, as a reason of taking S^1 as a domain. Note that ∇_1, ∇_2 are not compact, and $A_{\frac{1}{\theta}}$ has zero intersection with $\mathbb{K}$. In fact, the differential operator ∇_2 is an unbounded operator, and the multiplication operator ∇_1 is also unbounded, just as on variable domain unbounded such as $\mathbb{R}$, but is bounded on variable domain bounded.

$\star$ It seems to certainly require that D is bounded to be Fredholm. It is possible to have such a case as if all the coefficients of D are non-trivial, so bounded.

It then in particular follows that D elliptic has kernel finite dimensional.

Since $E = Sw(\mathbb{R})$ is not closed in $L^2(\mathbb{R})$, the second statement follows from finite dimensionality of the kernel of D. $\square$

It then follows that the Fredholm index of D is defined by

$$\operatorname{index}(D) = \dim\ker(D) - \dim\ker(D^*) \in \mathbb{Z}.$$

Equivalently, $\dim\ker(D^*)$ can be replaced with the dimension $\dim\operatorname{coker}(D)$ of the cokernel $\operatorname{coker}(D)$ as the codimension of the image $T(D)$ of D on H (or $L^2(\mathbb{R})$) as the dimension of the quotient of H by $T(D)$.

An index formula for $\operatorname{index}(D)$ analytic defined, but just as in Linear Algebra or Functional Analysis, with Calculus by numerical minus operation, is given as another explicit formula in other (geometric or noncommutative geometric) words, in the following.

The index formula involves a specific cyclic cocycle on the algebra $C^\infty(S^1, A_{\frac{1}{\theta}})$ of symbols.

Recall the cyclic cocycles τ_0 and τ_2 corresponding to the generators of $cH^{\mathrm{ev}} = cH^0 \oplus cH^2$ for $A_{\frac{1}{\theta}}$.

The canonical trace τ_0 on $A_{\frac{1}{\theta}}$ is defined by

$$\tau_0(\sum a_{n,m}V_1^n V_2^m) = a_{0,0}, \quad a = (a_{n,m}) \in Sw(\mathbb{Z}^2).$$

The trace τ_0 is the Murray and Neumann trace on the hyperfiniete type II_1 factor generated by V_1 and V_2 in $L^2(\mathbb{R})$.

The τ_2 is defined as the character of the following cycle on $A_{\frac{1}{\theta}}$. Let $\Omega^* = A_{\frac{1}{\theta}} \otimes (\wedge^*\mathbb{C}^2)$ the tensor product of $A_{\frac{1}{\theta}}$ by the graded exterior algebra $\wedge^*\mathbb{C}^2$ of $\mathbb{C}^2$, as a cycle. The differential d on Ω^* is given by

$$d(a \otimes \alpha) = \delta_1(a)e_1 \wedge \alpha + \delta_2(a)e_2 \wedge \alpha, \quad a \in A_{\frac{1}{\theta}}, \quad \alpha \in \wedge^*\mathbb{C}^2$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$, and δ_j $(j = 1, 2)$ are derivations of $A_{1\theta}$ defined as

$$\delta_j(V_k) = 0 \quad (j \neq k), \quad \delta_j(V_j) = 2\pi i V_j \quad (j = 1, 2).$$

The closed graded trace $\int : \Omega^2 \to \mathbb{C}$ is then given by $\int a \otimes (e_1 \wedge e_2) = \tau_0(a)$ for $a \in A_{\frac{1}{\theta}}$.

In fact, the formula for τ_2 is

$$\tau_2(a_0, a_1, a_2) = \tau_0(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))).$$

We have

$$\frac{1}{2\pi i}\tau_2(E, E, E) = q, \quad E \in K_0(A_{\frac{1}{\theta}})$$

where $q \in \mathbb{Z}$ is uniquely determined by

$$\tau_0(E) = p + q\left(\frac{1}{\theta} - \left[\frac{1}{\theta}\right]\right), \quad p, q \in \mathbb{Z}.$$

Let ρ be the fundamental class $[S^1]$ of the circle S^1. Namely, this is the cyclic 1-cocycle on $C^\infty(S^1)$ given by

$$\rho(f_0, f_1) = \int_{S^1} f_0 df_1, \quad f_0, f_1 \in C^\infty(S^1).$$

Equivalently, it is named as the de Rham differential algebra and integral.

$\star$ Note that $d(f_0f_1) = d(f_0)f_1 + f_0d(f_1) = f_1df_0 + f_0df_1$. Since we have $\int_{S^1} d(f_0f_1) = 0$ as closedness, then $\rho(f_0, f_1) = -\rho(f_1, f_0)$ as cycling is obtained.

Then natural cyclic cocycles τ_1 and τ_3 of cH^1 and cH^3 on the algebra $B = C^\infty(S^1, A_{\frac{1}{\theta}}) = C^\infty(S^1) \otimes A_{\frac{1}{\theta}}$ as an algebraic tensor product of symbols $\sum C_{\alpha,\beta}\xi_1^\alpha\xi_2^\beta$ for $C_{\alpha,\beta} \in A_{\frac{1}{\theta}}$, $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1^2 + \xi_2^2 = 1$ are obtained by letting $\tau_1 = \rho\#\tau_0$ and $\tau_3 = \rho\#\tau_2$ as like tensors.

It is shown by a straightforward calculation that

$$\tau_1(f_0, f_1) = \int_{S^1} \tau_0(f_0(t)\frac{df_1}{dt}(t))dt,$$
$$\tau_3(f_0, f_1, f_2, f_3) = \int_{S^1} \tau_0(f_0df_1 \wedge df_2 \wedge df_3)dt,$$

where

$$(df_1 \wedge df_2 \wedge df_3)(t) \in C^\infty(S^1, A_{\frac{1}{\theta}}), \quad t \in S^1$$
$$= \sum_{\sigma\in\mathfrak{S}_3} \varepsilon(\sigma)\partial_{\sigma(1)}f_1(t)\partial_{\sigma(2)}f_2(t)\partial_{\sigma(3)}f_3(t),$$

where $\varepsilon(\sigma)$ is the signature of the permutation $\sigma \in \mathfrak{S}_3$ of $\{1, 2, 3\}$, and the derivations ∂_j, $j = 1, 2, 3$ are defined by

$$\partial_1 f = \delta_1 f, \quad \partial_2 f = \delta_2 f, \quad \partial_3 f = \frac{df}{dt}, \quad f \in C^\infty(S^1, A_{\frac{1}{\theta}}).$$

$\star$ It τ_1 looks like $\tau_0 \circ \rho_1^\sim$ as a composition, where $\rho_1^\sim$ is the same as ρ but complex-valued smooth functions in domain for ρ are replaced by $A_{\frac{1}{\theta}}$-valued, operator functions for $\rho_1^\sim$. As well, τ_3 may be such a composition of $\rho_3^\sim$ extended from $\rho_1^\sim$ by τ_0.

Any element a of $A_{\frac{1}{\theta}}$ can be written as a Laurent series in V_1 as $a = \sum_{n\in\mathbb{Z}} f_nV_1^n$, where each $f_n = \sum a_{nm}V_2^m$ is viewed as a periodic function of period θ in $C^\infty(\mathbb{R}/\theta\mathbb{Z})$.

The algebraic rule for those elements are given by that of crossed product (like). Namely, (correct?)

$$ab = \sum_{n,m\in\mathbb{Z}} f_n\alpha^n(g_m)V_1^m$$

where $b = \sum_{n\in\mathbb{Z}} g_mV_1^m$, $\alpha(g)(s) = g(s + 1)$ for $s \in \mathbb{R}$, $g \in C^\infty(\mathbb{R}/\theta\mathbb{Z})$.

$\star$ We have $V_1 = \alpha$, but not $\mathrm{Ad}(V_1^n) = \alpha^n$?

The normalized trace τ_0 is given by the integral of f_0 over period θ interval. Namely,

$$\tau_0(\sum_{n\in\mathbb{Z}} f_nV_1^n) = \frac{1}{\theta}\int_0^\theta f_0(s)ds.$$

Similarly, an arbitrary element $f \in C^\infty(S^1, A_{\frac{1}{\theta}})$ can be written as a Laurent series $f = \sum_{n\in\mathbb{Z}} f_n V_1^n$, where each f_n is a doubly periodic function $f_n(t,s)$, $t \in S^1$, $s \in \mathbb{R}/\theta\mathbb{Z}$.

Then the derivatives of f by ∂_1, ∂_2, ∂_3 are computed as

$$\partial_1 f = \sum_{n\in\mathbb{Z}} 2\pi i n f_n V_1^n,$$
$$\partial_2 f = \sum_{n\in\mathbb{Z}} (-\theta)\frac{\partial f_n}{\partial s} V_1^n,$$
$$\partial_3 f = \sum_{n\in\mathbb{Z}} \frac{\partial f_n}{\partial t} V_1^n.$$

$\star$ Here, V_2 may be identified with $e^{\frac{-2\pi i s}{\theta}}$ as a multiple function, so that

$$\partial_2 V_2 = 2\pi i V_2 = (-\theta)\frac{\partial}{\partial s} e^{\frac{-2\pi i s}{\theta}}.$$

Therefore, ∂_2 on V_2 may be identified with $-\theta\frac{\partial}{\partial s}$. As well,

$$\partial_2 V_2^m = 2\pi i m V_2^m = (-\theta)\frac{\partial}{\partial s} e^{\frac{-2\pi i m s}{\theta}}!$$

for any $m \in \mathbb{Z}$. That's it.

It then follows that the following formulas hold:

$$\tau_1(f_0, f_1) = \frac{1}{\theta}\int_0^\theta ds \int_{S^1} (f_0 \partial_3 f_1)_0(t,s) dt,$$
$$\tau_3(f_0, f_1, f_2, f_3) = \frac{1}{\theta}\int_0^\theta ds \int_{S^1} (f_0 df_1 \wedge df_2 \wedge df_3)_0(t,s) dt.$$

$\star$ Note that $\tau_0(f) = \frac{1}{\theta}\int_0^\theta f_0(s) ds$ for $f = \sum_{n\in\mathbb{Z}} f_n V_1^n$, and that $\tau_1 = \tau \circ \rho_1^{\sim}$ and $\tau_3 = \tau \circ \rho_3^{\sim}$.

An analogue of the Atiyah-Singer index theorem for the operator D elliptic is obtained by Connes [3] that

Theorem 3.4. *Let $D = \sum_{\alpha+\beta\leq n} C_{\alpha,\beta} \nabla_1^\alpha \nabla_2^\beta$ be an elliptic operator of order $\leq n$, and let its principal symbol of order n be also written as*

$$\sigma_n(D)(t) = \sigma(t) = \sum_{\alpha+\beta=n} C_{\alpha,\beta} \cos^\alpha t \sin^\beta t, \quad t \in [0, 2\pi].$$

Then

$$\mathrm{index}(D) = \frac{1}{6\theta(2\pi i)^2}\tau_3\left(\frac{1}{\sigma}, \sigma, \frac{1}{\sigma}, \sigma\right) - \frac{1}{2\pi i}\tau_1\left(\frac{1}{\sigma}, \sigma\right).$$

Remark. This is an index theorem for operators highly non-local such as D on $Sw(\mathbb{R})$.

$\star$ The invertible symbol σ makes a class of K-theory K_1. Also, K_1-group has pairing with odd cyclic cohomology by inserting. Moreover, suitable normalization can make a complex number into an integer.

As in the classical rock like case of differential operators on manifolds, there are a few corollaries by this index formula as follows.

Corollary 3.5. (1) *If the index* $\mathrm{index}(D)$ *is positive, then there exist non-trivial solutions of the equation* $Df = 0$, $f \in Sw(\mathbb{R})$ $(f \neq 0)$.

(2) *For any invertible* $f \in C^\infty(S^1, A_{\frac{1}{\theta}})$, *the following quantity:*

$$\frac{-1}{24\theta\pi^2}\tau_3\left(\frac{1}{\sigma}, \sigma, \frac{1}{\sigma}, \sigma\right) - \frac{1}{2\pi i}\tau_1\left(\frac{1}{\sigma}, \sigma\right) \in \mathbb{C}$$

must be an integer.

The integrality of $\frac{1}{2\pi i}\tau_2$ means as an explanation that $\langle K_0(A_{\frac{1}{\theta}}), \tau_2\rangle$ is contained in $2\pi i\mathbb{Z}$.

Example 3.6. There are non-trivial examples of elliptic operators D of the form above, only requiring information about periodic functions $g_j \in C^\infty(\mathbb{R}/\theta\mathbb{Z})$ with $\mathbb{R}/\theta\mathbb{Z} = S^1$ compact involved in the formula below. That is defined as

$$(Df)(s) = sf(s) - \sum_{k=-1}^{1} g_k(s)\frac{df}{ds}(s+k), \quad f \in Sw(\mathbb{R})$$

where there is a condition on g_k such that for some $f_k \in C(\mathbb{R}/\theta\mathbb{Z})$, we have $\sum_{k=-1}^{1} |f_k(s) - g_k(s)| < 1$ for any $s \in \mathbb{R}/\theta\mathbb{Z}$.

In fact, we choose a closed interval $I = [a, b]$ in the circle $\mathbb{T} = \mathbb{R}/\theta\mathbb{Z}$, such that $I \cap (I+1) = \emptyset$. Let f be a continuous map from I to $[-1, 1]$ such that $f(a) = -1$ and $f(b) = 1$. Define $f_0 \in C(\mathbb{T})$ by $f_0 = f$ on I, $f_0 = 1$ on $[b, a+1]$, $f_0(s) = -f(s-1)$ on $[a+1, b+1]$ so that $f_0(a+1) = -f(a) = 1$ and $f_0(b+1) = -f(b) = -1$, and $f_0 = -1$ on $[b+1, a]$. Let $f_1 \in C(\mathbb{T})$ have support in I and satisfy $f_0^2 + f_1^2 = 1$ on I. Then $f_{-1} \in C(\mathbb{T})$ is defined by $f_{-1}(s) = f_1(s-1)$ for $s \in \mathbb{T}$.

That is Powers-Rieffel idempotent(?).

$\star$ In particular, we have $f_{-1}^2 + f_0^2 + f_1^2 = 1$ on the circle. That makes a functional 2-sphere.

$\star$ Do we have p as the PR projection such that

$$(f_{-1} + f_0 + f_1)^2 = f_{-1} + f_0 + f_1 = p?$$

Since p^2 on I is

$$(f_0 + f_1)^2 = f_0^2 + f_1^2 + 2f_0f_1 = 1 + 2f_0f_1 = 1 + 2f_0\sqrt{1 - f_0^2},$$

then if so, that should be equal to $f_0 + f_1 = f_0 + \sqrt{1 - f_0^2}$ on I. Therefore $1 + \sqrt{1 - f_0^2} = f_0$. But this does not hold.

$\star$ It seems necessary to involve V_1 and V_2 by Functional Calculus.

Corollary 3.7. *Let $g_j \in C^\infty(\mathbb{R}/\theta\mathbb{Z})$ be continuous functions on the circle S^1 satisfying the conditions above. Then the operator D defined in the example above is elliptic in the sense of Definition above. The index of D is given by*

$$\mathrm{index}(D) = 1 + 2\left[\frac{1}{\theta}\right] \in \mathbb{Z}.$$

The kernel equation $Df = 0$ admits at least $1 + 2[\frac{1}{\theta}]$ linearly independent solutions f of $Sw(\mathbb{R})$.

An existence theorem as well as a regularity result are obtained.

$\star$ Note that

$$\dim\ker(D) = \mathrm{index}(D) + \dim\ker(D^*) \geq 1 + 2[\frac{1}{\theta}] \geq 1.$$

in that case. This implies the existence of at least 1 solution. Finiteness of index implies regularity as finite dimensionality of the kernel of D.

$\star$ The operator D can be written as

$$D = -\frac{\theta}{2\pi i}\nabla_1 - g_{-1}V_1^{-1}\nabla_2 - g_0\nabla_2 - g_1V_1\nabla_2,$$

so that D has order 1 in the sense of Definition above. It looks bounded since the domain is compact and g_0 is not equal to 1 since f_0 is not. Thus,

$$\sigma_1(D) = -\frac{\theta}{2\pi i}\eta_1 - g_{-1}V_1^{-1}\eta_2 - g_0\eta_2 - g_1V_1\eta_2.$$

Is this invertible? The corollary above says that that is true.

Note that the adjoint of D is given by

$$\sigma_1(D)^* = \frac{\theta}{2\pi i}\eta_1 - g_{-1}V_1\eta_2 - g_0\eta_2 - g_1V_1^{-1}\eta_2$$

if g_j are real valued functions. We then compute that

$$\begin{aligned}
\sigma_1(D)\sigma_1(D)^* &= \frac{\theta^2}{4\pi^2}\eta_1^2 + g_{-1}^2\eta_2^2 + g_0^2\eta_2^2 + g_1^2\eta_2^2 \\
&\quad + \frac{\theta}{2\pi i}(g_{-1}V_1 + g_0 + g_1V_1^{-1})\eta_1\eta_2 \\
&\quad - \frac{\theta}{2\pi i}g_{-1}V_1^{-1}\eta_1\eta_2 + (g_{-1}V_1^{-1}g_0 + g_{-1}g_1(V_1^{-1})^2)\eta_2^2 \\
&\quad - \frac{\theta}{2\pi i}g_0\eta_1\eta_2 + (g_0g_{-1}V_1 + g_0g_1V_1^{-1})\eta_2^2 \\
&\quad - \frac{\theta}{2\pi i}g_1V_1\eta_1\eta_2 + (g_1g_{-1}V_1^2 + g_1V_1g_0)\eta_2^2 \\
&= \frac{\theta^2}{4\pi^2}\eta_1^2 + (g_{-1}^2 + g_0^2 + g_1^2)\eta_2^2 \\
&\quad + \frac{\theta}{2\pi i}((g_{-1} - g_1)(V_1 - V_1^{-1}))\eta_1\eta_2 \\
&\quad + \{g_0(g_{-1} + g_1)(V_1 + V_1^{-1}) + g_{-1}g_1(V_1^2 + (V_1^{-1})^2)\}\eta_2^2.
\end{aligned}$$

Since $f_1f_{-1} = 0$, then $g_1g_{-1} = 0$ is assumed. Since $f_{-1}^2 + f_0^2 + f_1^2 = 1$, we may assume that $g_{-1}^2 + g_0^2 + g_1^2 = 1 \pm \varepsilon 1 > 0$ for some $\varepsilon > 0$. As a possible choice, if we can assume that both the norms of the second and third terms with respect to $\eta_1\eta_2$ and η_2^2 respectively by choosing g_{-1} and g_1 suitably by shifting used are sufficiently small enough, then it follows that $\sigma_1(D)\sigma_1(D)^*$ is invertible. Then $\sigma_1(D)$ is invertible with inverse $\sigma_1(D)^*[\sigma_1(D)\sigma_1(D)^*]^{-1}$.

By the way, for $z = \cos t + i\sin t \in \mathbb{T}$, we have

$$z - \frac{1}{z} = z - \overline{z} = 2i\mathrm{Im}(z) = 2i\sin t.$$

Also $z + \overline{z} = 2\mathrm{Re}(z) = 2\cos t$.

Anyhow, making both the norms small seems to be a bit difficult.

$\star$ There is also a short exact sequence of (topological) algebras as in the following:

$$0 \longrightarrow \ker(\Sigma) = \mathfrak{I} \longrightarrow \mathfrak{D} \xrightarrow[\sigma]{\Sigma} \Sigma(\mathfrak{D}) \longrightarrow 0$$

There should be the six-term exact sequence of K-theory groups for the algebras as follows:

$$\begin{tikzcd}
K_0(\mathfrak{I}) \arrow[r] & K_0(\mathfrak{D}) \arrow[r, "\Sigma_*", "\sigma_*"'] & K_0(\Sigma(\mathfrak{D})) \arrow[d] \\
K_1(\Sigma(\mathfrak{D})) \arrow[u, "\partial"] & K_1(\mathfrak{D}) \arrow[l, "\Sigma_*"', "\sigma_*"] & K_1(\mathfrak{I}). \arrow[l]
\end{tikzcd}$$

In particular, the K-theory class $[\sigma(D)]$ is obtained in $K_1(\Sigma(\mathfrak{D}))$ by ellipticity. Then the K-theory index map ∂ maps it to $\partial[\sigma(D)]$ in $K_0(\mathfrak{I})$ as a generalized index, but a vector in general. This K-theory class in this case may be identified with index(D) an integer, as another language.

The function $[\frac{1}{\theta}]$ in the corollary above with θ as a variable is discontinuous when (or at) $\theta \in \frac{1}{\mathbb{N}} \subset (0,1]$.

$\star$ If $\theta = \frac{1}{k}$, then $\frac{1}{\theta} = k \in \mathbb{N}$. Thus, $[\frac{1}{\theta}] = k$.

Also, if $k \leq \frac{1}{\theta} < k+1$, then $[\frac{1}{\theta}] = k$ and $\frac{1}{k+1} < \theta \leq \frac{1}{k}$.

The reason why this does not entail a contradiction is that when $\frac{1}{\theta}$ gets close to such an integer k, it becomes more difficult to find an interval I in $\mathbb{R}/\theta\mathbb{Z}$ such that I and $I+1$ are disjoint.

It is impossible for $\frac{1}{\theta} = k \in \mathbb{N}$.

$\star$ When θ is rational, $A_{\frac{1}{\theta}}$ becomes homogeneous, not-simple, and has all irreducible representations finite dimensional. That seems to be the reason.

The proof of the corollary above is a straightforward application of the theorem above. Equivalently, it follows from the computation of τ_2 on the Powers-Rieffel idemponent of $A_{\frac{1}{\theta}}$ (cf. [14] and [3]).

Example 3.8. There are many examples of elliptic operators with nontrivial indices constructed as follows. Let

$$(Df)(s) = sf(s) - (T\frac{df}{ds})(s),$$

where $T \in A_{\frac{1}{\theta}}$ as an operator is self-adjoint and invertible. It implies that

$$D = -\frac{\theta}{2\pi i}\nabla_1 - T\nabla_2$$

is elliptic.

$\star$ We have

$$\sigma_1(D) = -\frac{\theta}{2\pi i}\eta_1 - T\eta_2 = -\frac{\theta}{2\pi i}\cos t - (\sin t)T$$

for $(\eta_1, \eta_2) = (\cos t, \sin t) \in S^1$. Since T is self-adjoint and invertible, then the spectrum $\mathrm{sp}(T) \subset \mathbb{C}$ of T is contained in $\mathbb{R}$ and does not contain zero. It then follows that the spectrum of $\sigma_1(D)$ does not contain zero by spectral theory, so that $\sigma_1(D)$ is invertible, equivalently that D is elliptic.

$\star$ We have

$$-\frac{2\pi i}{\theta}\sigma_1(D) = \cos t + i(\frac{2\pi}{\theta})T\sin t.$$

We may replace $\frac{2\pi}{\theta}T$ with T to obtain $\sigma(t) = \cos t + iT\sin t$.

The index formula of the theorem above is reduced in that case with $\sigma(t)$ as principal symbol to

$$\mathrm{index}(D) = \frac{1}{2\pi i}\frac{2}{\theta}\tau_2(E,E,E) - \tau_0(2E-1),$$

where E is the spectral projection of T belonging to the interval $[0,\infty)$, namely, $E = \chi_{[0,\infty)}(T)$.

$\star$ Why reduced?

$\star$ Since we have the characteristic function $\chi_{[0,\infty)} = \chi_{[0,\infty)}^2 = \chi_{[0,\infty)}^*$, then $E = E^2 = E^*$ is satisfied by Functional calculus.

We have $E \in A_{\frac{1}{\theta}}$ because the algebra is stable under holomorphic functional calculus (HFC).

$\star$ The reason for that is that T is self-adjoint and invertible, and bounded, so that the spectrum is closed, contained in $\mathbb{R}$, and does not contain zero. The Spectral theory implies that $\chi_{[0,\infty)}(T)$ with $\chi_{[0,\infty)}$ restricted to the spectrum of T is defined and in $A_{\frac{1}{\theta}}$ or $\mathfrak{A}_{\frac{1}{\theta}}$.

For $q = \frac{1}{2\pi i}\tau_2(E, E, E) > 0$, the above formula can be rewritten as

$$\mathrm{index}(D) = 1 + 2\left[\frac{q}{\theta}\right].$$

$\star$ Note that with $\tau_0(E) = q(\frac{1}{\theta} - \left[\frac{1}{\theta}\right])$, in this case we have

$$\begin{aligned}\mathrm{index}(D) &= 2\frac{q}{\theta} - 2\tau_0(E) + 1 \\ &= 1 + 2\frac{q}{\theta} - 2q(\frac{1}{\theta} - \left[\frac{1}{\theta}\right]) = 1 + 2q\left[\frac{1}{\theta}\right].\end{aligned}$$

The Chern character τ_2 is used successfully to label the gaps of self-adjoint operators by J. Bellissard.

Example 3.9. The Peierls operator as a non-trivial example of a self-adjoint invertible $T \in A_{\frac{1}{\theta}}$ is given by

$$(Tf)(s) = f(s+1) + f(s-1) + 2\cos(\frac{2\pi s}{\theta})f(s) + \lambda f(s),$$

which is invertible for any λ outside a nowhere dense Cantor set K when θ is a Liouville number ([2]).

The index changes discontinuously on K.

Refer to [2] and [11] for more information on the operator T and gap labeling.

$\star$ Note that

$$T = V_1 + V_1^* + 2\cos(\frac{2\pi s}{\theta}) + \lambda 1 \in A_{\frac{1}{\theta}}$$

with $s \in \mathbb{R} \bmod \theta$ and

$$2\cos(\frac{2\pi s}{\theta}) = e^{\frac{2\pi i s}{\theta}} + e^{-\frac{2\pi i s}{\theta}},$$

which may be viewed as an element of the smooth or C^*-algebra generated by V_1 or V_2.

$\star$ Note also that since we have the norm estimate:

$$\|V_1 + V_1^*\| \leq \|V_1\| + \|V_1^*\| = 2$$

the spectrum of $V_1 + V_1^*$ is contained in the closed interval $[-2, 2] \subset \mathbb{R}$. Also, the spectrum of the function $2\cos(\frac{2\pi s}{\theta})$ as range is equal to $[-2, 2]$.

As a corollary of the gap labeling of the Peierls operator and of the theorem above, the following is obtained:

Corollary 3.10. *Let θ be a Liouville number and N an integer. Then there exists $\lambda \in (-2, 2)$ such that the following difference differential equation on $\mathbb{R}$:*

$$sf(s) = f'(s+1) + f'(s-1) + (2\cos\frac{2\pi s}{\theta} + \lambda)f'(s)$$

admits at least N linearly independent solutions f in $Sw(\mathbb{R})$.

$\star$ Note that the functional differential equation above can be written as the operator equation

$$\begin{aligned}
-\frac{\theta}{2\pi i}\nabla_1 f &= V_1\nabla_2 f + V_1^*\nabla_2 f + (2\cos\frac{2\pi s}{\theta} + \lambda)\nabla_2 f \\
&= [V_1\nabla_2 + V_1^*\nabla_2 + (2\cos\frac{2\pi s}{\theta} + \lambda)\nabla_2]f \\
&= T\nabla_2 f
\end{aligned}$$

with $T \in A_{\frac{1}{\theta}}$ self-adjoint and invertible. Namely, f belongs to the kernel of the operator $\frac{\theta}{2\pi i}\nabla_1 + T\nabla_2 = D$. The symbol $\sigma_1(D)$ is given by

$$\frac{\theta}{2\pi i}\cos t + T\sin t.$$

It then follows that D is elliptic by the same reason as before. Hence D is Fredholm so that its kernel is finite dimensional.

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