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LINEAR RELATIONS FOR BERNOULLI NUMBERS AND ITS APPLICATION TO CONGRUENCES INVOLVING HARMONIC SUMS*

Kazufumi Kimoto

Abstract

We show certain linear relations among Bernoulli numbers by using umbral calculus. As an application, we prove some congruence relations involving binomial coefficients and harmonic sums which appear in a certain supercongruence problem.

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1 Introduction

In this short note, we give a simple way to produce linear relations among Bernoulli numbers by using umbral calculus, and use it to prove some congruence relations involving binomial coefficients and harmonic sums, which appear in a certain supercongruence problem [3].

In §2, we first introduce a linear map $\psi\colon \mathbb{R}[x] \to \mathbb{R}$ which sends each monomial x^k to the Bernoulli number B_k, and describe the very basic properties of it. For any polynomial $f(x) \in \ker\psi$, the equation $\psi(f(x)) = 0$ gives a certain linear relation among Bernoulli numbers. Thus it is natural to seek a sufficient condition for a polynomial $f(x)$ to be in the kernel of this umbral map ψ. We give such a simple sufficient condition. Our calculation in §2 is essentially the same with the one given by Momiyama [2]. Actually, if we discuss over the p-adic integer ring $\mathbb{Z}_p$, then the umbral map ψ is realized as the Volkenborn integral. As we will see, however, we do not need to bring the Volkenborn integral to obtain linear relations among Bernoulli numbers in a similar manner; We only needs the standard properties on Bernoulli numbers.

In §3, by using the facts given in §2 and the von Staudt-Clausen theorem, we give several congruence relations modulo p^2, where p is an odd prime, among certain

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sums involving binomial coefficients and harmonic sums. Such congruence relations are used to reduce a certain supercongruence (i.e. a congruence relation modulo a power of p) to a lower power case.

2 Linear relations for Bernoulli numbers

We denote by B_k and $B_k(x)$ the Bernoulli numbers and Bernoulli polynomials respectively:

$$\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}, \qquad \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \frac{te^{tx}}{e^t - 1}.$$

We recall the standard facts: For any $k \in \mathbb{Z}_{\geq 0}$, we have

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j} x^j, \tag{2.1}$$

$$B_k(b) - B_k(a) = k \sum_{a \leq j < b} j^{k-1} \qquad (a, b \in \mathbb{Z},\ a < b), \tag{2.2}$$

$$(-1)^k B_k(1) = B_k. \tag{2.3}$$

2.1 A lemma for the umbral map

Define a $\mathbb{R}$-linear map $\psi \colon \mathbb{R}[x] \to \mathbb{R}$ by

$$\psi \colon \mathbb{R}[x] \ni f(x) = \sum_{k=0}^{n} a_k x^k \longmapsto \psi(f(x)) = \sum_{k=0}^{n} a_k B_k \in \mathbb{R}. \tag{2.4}$$

The following fact is an immediately consequence of the basic properties (2.1), (2.2) and (2.3).

Proposition 2.1. *For any $f(x) \in \mathbb{R}[x]$, we have*

$$\psi((x + a)^k) = B_k(a) \qquad (k \in \mathbb{Z}_{\geq 0},\ a \in \mathbb{R}), \tag{2.5}$$

$$\psi(f(x + b) - f(x + a)) = \sum_{a \leq j < b} f'(j) \qquad (a, b \in \mathbb{Z},\ a < b), \tag{2.6}$$

$$\psi(f(-x - 1)) = \psi(f(x)). \tag{2.7}$$

□

Proof. First, by the linearity of ψ and (2.1), we have

$$\psi((x + a)^k) = \sum_{j=0}^{k} \binom{k}{j} a^j \psi(x^{k-j}) = \sum_{j=0}^{k} \binom{k}{j} a^j B_{k-j} = B_k(a),$$

which is (2.5). By the linearity of ψ again, it is enough to prove (2.6) and (2.7) when $f(x) = x^k$, $k \in \mathbb{Z}_{\geq 0}$. By (2.2) and (2.3), we have

$$\begin{aligned}
\psi(f(x+b) - f(x+a)) - \sum_{a \leq j < b} f'(j) &= \psi((x+b)^k) - \psi((x+a)^k) - \sum_{a \leq j < b} k j^{k-1} \\
&= B_k(b) - B_k(a) - k \sum_{a \leq j < b} j^{k-1} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\psi(f(-x-1)) - \psi(f(x)) &= \psi((-x-1)^k) - \psi(x^k) \\
&= (-1)^k B_k(1) - B_k = 0
\end{aligned}$$

as desired. □

If $f(x) \in \ker \psi$, then we get some linear relation $\psi(f(x)) = 0$ among Bernoulli numbers. Thus it is convenient if we have a simple sufficient condition for $f(x)$ to be killed by ψ. One such would be as follows.

Lemma 2.2. *Let L be a positive integer. Assume that $f(x) \in \mathbb{R}[x]$ satisfies the following conditions:*

(A1) $f(-x) = -f(x - L)$,

(A2) $\sum_{i=1}^{L-1} f'(-i) = 0$.

Then $\psi(f(x)) = 0$.

Proof. By (2.7) and (A1), we have

$$\psi(f(x)) = \psi(f(-x-1)) = -\psi(f(-(-x-1)-L)) = -\psi(f(x-L+1)).$$

If $L = 1$, then we have $\psi(f(x)) = 0$ at this point. When $L \geq 2$, by adding $\psi(f(x))$ to the both side and using (2.6), we get

$$2\psi(f(x)) = \psi(f(x) - f(x-L+1)) = \sum_{-L+1 \leq j < 0} f'(j) = \sum_{i=1}^{L-1} f'(-i) = 0$$

by (A2). □

We give a slightly weaker version of the lemma above. This is the main tool in our discussion below.

Lemma 2.3. *Let L be a positive integer. Assume that $F(x) \in \mathbb{R}[x]$ satisfies the following conditions:*

(B1) $F(-x) = F(x - L)$,

(B2) $\prod_{i=1}^{L-1} (x+i)^3 \mid F(x)$,

Then $\psi(F'(x)) = 0$.

Proof. It is clear that $F'(x)$ satisfies (A1) when $F(x)$ satisfies (B1). If $F(x)$ satisfies (B2), then $F''(-i) = 0$ for $i = 1, \dots, L-1$, which implies that $F'(x)$ satisfies (A2). □

2.2 Examples

We give a few examples obtained by Lemma 2.3.

Example 2.4. Let s be a non-negative integer. Put

$$F(x) = x^s (x+1)^s.$$

It is immediate to see that $F(x)$ satisfies (B1) and (B2) with $L = 1$. Hence we have $\psi(F'(x)) = 0$ by Lemma 2.3. Since

$$F'(x) = \frac{d}{dx} \sum_{k=0}^{s} \binom{s}{k} x^{s+k} = \sum_{k=0}^{s} (k+s) \binom{s}{k} x^{k+s-1},$$

we get the formula

$$\sum_{k=0}^{s} (k+s) \binom{s}{k} B_{k+s-1} = 0. \tag{2.8}$$

The formula (2.8) is due to von Ettingshausen [4]. For any $r \geq 0$, the $2r$-th derivative $F^{(2r)}(x)$ of $F(x)$ also satisfies (B1) and (B2) with $L = 1$. Since

$$\frac{F^{(2r+1)}(x)}{(2r+1)!} = \sum_{k=0}^{s} \binom{s}{k} \binom{k+s}{2r+1} x^{k+s-2r-1},$$

we also get a slightly general formula

$$\sum_{k=0}^{s} \binom{s}{k} \binom{k+s}{2r+1} B_{s-2r-1+k} = 0, \tag{2.9}$$

where we understand that $B_i = 0$ when $i < 0$. As a special case, by letting $s = 2r+1$, we have

$$\sum_{k=0}^{s} \binom{s}{k} \binom{k+s}{k} B_k = 0 \tag{2.10}$$

if s is *odd*. This equation does not hold when s is even.

Remark 2.5. By putting $s = n+1$ in (2.8), we get

$$\widetilde{B}_{2n} = -\frac{1}{n+1} \sum_{i=0}^{n-1} \binom{n+1}{i} \widetilde{B}_{n+i}, \tag{2.11}$$

where $\widetilde{B}_k = (k+1)B_k$ [1].

Example 2.6. Let m, n be non-negative integers. Put

$$F(x) = (-1)^n x^{n+1} (x+1)^{m+1} + (-1)^m x^{m+1} (x+1)^{n+1}.$$

It is immediate to see that $F(x)$ satisfies (B1) and (B2) with $L = 1$. Hence we have $\psi(F'(x)) = 0$ by Lemma 2.3. Since

$$F(x) = (-1)^n \sum_{k=0}^{m+1} \binom{m+1}{k} x^{n+k+1} - (-1)^m \sum_{k=0}^{n+1} \binom{n+1}{k} x^{m+k+1},$$

we have

$$(-1)^n \sum_{k=0}^{m+1} \binom{m+1}{k} (n+k+1) B_{n+k} + (-1)^m \sum_{k=0}^{n+1} \binom{n+1}{k} (m+k+1) B_{m+k} = 0.$$

We notice that there occurs a cancellation between the last terms in these sums: $((-1)^n + (-1)^m) B_{m+n+1} = 0$ when $m+n > 0$. Thus we get

$$(-1)^n \sum_{k=0}^{m} \binom{m+1}{k} (n+k+1) B_{n+k} + (-1)^m \sum_{k=0}^{n} \binom{n+1}{k} (m+k+1) B_{m+k} = 0.$$

This is the Momiyama's identity [2]. By the same argument as in Example 2.4, we have

$$(-1)^n \sum_{k=0}^{m+1} \binom{m+1}{k} \binom{n+k+1}{2r+1} B_{n-2r+k} \\ + (-1)^m \sum_{k=0}^{n+1} \binom{n+1}{k} \binom{m+k+1}{2r+1} B_{m-2r+k} = 0$$

for $r \geq 0$.

Remark 2.7. In general, for any positive integer L and any polynomial $p(x)$ such that $\prod_{i=1}^{L-1} (x-i)^3 \mid p(x)$,

$$F(x) = p(-x) + p(x+L)$$

satisfies (B1) and (B2). For instance, $p(x) = -x^{n+1}(1-x)^{m+1}$ and $L = 1$ give the last example.

3 Congruences involving binomial coefficients and harmonic sums

We first recall the von Staudt-Clausen theorem:

Theorem 3.1. *For any positive integer n and any odd prime p,*

$$B_{2n} + \sum_{\substack{p:\text{prime} \\ p-1 \mid 2n}} \frac{1}{p} \tag{3.1}$$

is an integer. □

As a simple consequence of the theorem, for any odd prime p and a positive integer k, we have

$$pB_k \equiv \begin{cases} -1 & p-1 \mid k \\ 0 & \text{otherwise} \end{cases} \pmod{p}. \tag{3.2}$$

This implies that if

$$f(x) = \sum_{k=0}^{N} a_k x^k \in \mathbb{Q}[x]$$

and the denominator of the coefficient a_k is not divisible by p for every k, then

$$p\psi(f(x)) = \sum_{k=0}^{N} a_k\, pB_k \equiv -\sum_{i=1}^{\left\lfloor \frac{N}{p-1} \right\rfloor} a_{(p-1)i} \pmod{p}. \tag{3.3}$$

We give a lemma for later use.

Lemma 3.2. *For any odd prime p,*

$$\frac{p^k}{k!} \equiv 0 \pmod{p^2}$$

holds for $k \geq 3$.

Proof. Let us denote by $\nu_p(x)$ the p-adic valuation of $x \in \mathbb{Q} \setminus \{0\}$, that is,

$$x = p^{\nu_p(x)} \frac{a}{b}, \qquad a, b \in \mathbb{Z},\ p \nmid a,\ p \nmid b.$$

It is well known that

$$\nu_p(k!) = \sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor.$$

Hence we have

$$\nu_p\Big(\frac{p^k}{k!}\Big) = k - \sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor \geq k - \sum_{i \geq 1} \frac{k}{p^i} \geq 3\Big(1 - \frac{p}{p-1}\Big) \geq \frac{3}{2} > 1$$

as desired. □

3.1 Results

In what follows, we fix an odd prime p, and put $m = \frac{p-1}{2}$ for short. We denote by H_n the harmonic sum, i.e. $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Theorem 3.3. *If $p \geq 5$, then*

$$\sum_{k=0}^{m} \binom{m+k}{k}^4 (H_{m+k} - H_k) \equiv 0 \pmod{p^2}, \tag{3.4}$$

$$\sum_{k=0}^{m} \binom{m+k}{k}^6 (H_{m+k} - H_k) \equiv 0 \pmod{p^2}. \tag{3.5}$$

Proof. For a positive integer s, define

$$F_s(x) := \binom{x+m}{m}^s = \sum_{i=0}^{sm} e_i^{(s)} x^i.$$

Notice that the denominator of the coefficient $e_i^{(s)} \in \mathbb{Q}$ is not divisible by p for every i. Since

$$\frac{F_s'(x)}{F_s(x)} = s \sum_{i=1}^{m} \frac{1}{x+i},$$

we have

$$\sum_{k=0}^{m} \binom{m+k}{k}^s (H_{m+k} - H_k) = \frac{1}{s} \sum_{k=0}^{m} F_s'(k).$$

Thus it is enough to prove

$$\frac{1}{s} \sum_{k=0}^{m} F_s'(k) \equiv 0 \pmod{p^2}$$

for $s = 4, 6$. Notice that

$$\frac{1}{s} F_s'(x) = \binom{x+m}{m}^{s-1} \sum_{i=1}^{m} \frac{1}{m!} \prod_{\substack{1 \le j \le m \\ j \ne i}} (x+j),$$

so that the denominator of every coefficient $\frac{ie_i^{(m)}}{s}$ of $\frac{1}{s}F_s'(x) \in \mathbb{Q}[x]$ is not divisible by p regardless of whether s is divisible by p or not.

For a while, we only suppose that s is even, $s \ge 4$ and $p \nmid s$ (notice that $s = 4, 6$ satisfy this condition). Since

$$\frac{1}{s} F_s'(k) = \binom{k+m}{m}^{s-1} \sum_{i=1}^{m} \frac{1}{m!} \prod_{\substack{1 \le j \le m \\ j \ne i}} (k+j) \equiv 0 \pmod{p^{s-1}}$$

if $m+1 \le k \le p-1$, we have

$$\frac{1}{s} \sum_{k=0}^{m} F_s'(k) \equiv \frac{1}{s} \sum_{k=0}^{p-1} F_s'(k) \pmod{p^2}.$$

By (2.6) and Lemma 3.2, we have

$$\begin{aligned} \sum_{k=0}^{p-1} F_s'(k) &= \psi(F_s(x+p) - F_s(x)) \\ &= \sum_{k=1}^{sm} \frac{p^k}{k!} \psi(F_s^{(k)}(x)) \\ &\equiv p\psi(F_s'(x)) + \frac{p^2}{2} \psi(F_s''(x)) \pmod{p^2}. \end{aligned}$$

We see that $F_s(x)$ satisfies (B1) and (B2) with $L = m+1$. Indeed,

$$F_s(x) = \prod_{i=1}^{m} \frac{(x+i)^s}{i^s}$$

is divisible by $\prod_{i=1}^{m}(x+i)^3$ since we assume $s \geq 4$, and

$$F_s(-x) = \binom{-x+m}{m}^s = \binom{(x-m-1)+m}{m}^s = F_s(x-m-1)$$

by the relation $\binom{-a}{m} = (-1)^m \binom{a-m+1}{m}$ and the assumption that s is even. Hence we have

$$\psi(F_s'(x)) = 0$$

by Lemma 2.3. By using (3.3), we get

$$\begin{aligned} p\psi(F_s''(x)) &= p\psi\Big(\sum_{i=0}^{sm-2} (i+2)(i+1)e_{i+2}^{(s)} x^i\Big) \\ &\equiv -\sum_{i=1}^{\lfloor \frac{sm-2}{2m} \rfloor} (2im+2)(2im+1)e_{2im+2}^{(s)} \pmod p \\ &\equiv -\sum_{i=1}^{\frac{s}{2}-1} (i-1)(i-2)e_{2im+2}^{(s)} \pmod p. \end{aligned}$$

This is congruent to 0 modulo p if $s \leq 6$. Thus we have

$$\sum_{k=0}^{p-1} F_s'(k) \equiv p\psi(F_s'(x)) + \frac{p}{2} \cdot p\psi(F_s''(x)) \equiv 0 \pmod{p^2}$$

for $s = 4, 6$ as desired. □

Remark 3.4. In general, it is *not* true that

$$\sum_{k=0}^{m} \binom{m+k}{k}^s (H_{m+k} - H_k) \equiv 0 \pmod{p^2}$$

when s is even and $s \neq 4, 6$. When $s = 2$, we have

$$\sum_{k=0}^{m} \binom{m+k}{k}^2 (H_{m+k} - H_k) \equiv \frac{p}{2}\psi(F_2'(x)) \pmod{p^2}.$$

We see that $p\psi(F_2'(x)) \equiv 0 \pmod p$, but $p\psi(F_2'(x)) \not\equiv 0 \pmod{p^2}$ in general. When $s > 6$, we have

$$\sum_{k=0}^{m} \binom{m+k}{k}^s (H_{m+k} - H_k) \equiv -\frac{1}{s} \sum_{i=3}^{\frac{s}{2}-1} \binom{i-1}{2} e_{2im+2}^{(s)} \pmod{p^2},$$

which is not congruent to 0 modulo p^2 in general.

Corollary 3.5. *For any positive even integer s, we have*

$$\sum_{k=0}^{m} \binom{m+k}{k}^s (H_{m+k} - H_k) \equiv 0 \pmod{p}.$$

Proof. By the same discussion as in the proof above, we have

$$\sum_{k=0}^{m} \binom{m+k}{k}^s (H_{m+k} - H_k) = \frac{1}{s}\sum_{k=0}^{m} F_s'(k) \equiv \frac{1}{s}\sum_{k=0}^{p-1} F_s'(k) \equiv p\psi(F_s'(x)/s) \pmod{p}.$$

When $s \geq 4$, we have $\psi(F_s'(x)/s) = 0$. When $s = 2$, we directly have

$$p\psi(F_2'(x)) = \sum_{k=1}^{2m} k e_k^{(2)} p B_{k-1} \equiv 0 \pmod{p}.$$

□

3.2 An application

Lemma 3.6.

$$\binom{m+k}{k} \equiv (-1)^k \binom{m}{k} \Big(1 + p(H_{m+k} - H_m)\Big) \pmod{p^2}.$$

Proof. We have

$$\begin{aligned}
\binom{m+k}{k} &= \binom{m}{k} \prod_{j=0}^{k-1} \frac{m+j+1}{m-j} \\
&= (-1)^k \binom{m}{k} \prod_{j=0}^{k-1} \frac{1 + \frac{p}{2}(j+\frac{1}{2})^{-1}}{1 - \frac{p}{2}(j+\frac{1}{2})^{-1}} \\
&\equiv (-1)^k \binom{m}{k} \Big(1 + p\sum_{j=0}^{k-1} \frac{1}{j+\frac{1}{2}}\Big) \pmod{p^2}.
\end{aligned}$$

Since

$$H_{m+k} - H_m = \sum_{j=0}^{k-1} \frac{1}{m+j+1} \equiv \sum_{j=0}^{k-1} \frac{1}{j+\frac{1}{2}} \pmod{p},$$

we have the conclusion. □

By the lemma, for any $s \geq 1$, we have

$$\binom{m+k}{k}^{2s} \equiv \binom{m}{k}^{2s} \Big(1 + 2sp(H_{m+k} - H_m)\Big) \pmod{p^2}.$$

Hence we have

$$\sum_{k=0}^{m}\binom{m+k}{k}^{2s}(H_{m+k}-H_k)$$
$$\equiv \sum_{k=0}^{m}\binom{m}{k}^{2s}(H_{m+k}-H_k)+2sp\sum_{k=0}^{m}\binom{m}{k}^{2s}H_{m+k}(H_{m+k}-H_k)$$
$$-2spH_m\sum_{k=0}^{m}\binom{m}{k}^{2s}(H_{m+k}-H_k) \pmod{p^2}.$$

Using Corollary 3.5, this implies that

$$\sum_{k=0}^{m}\binom{m}{k}^{2s}(H_{m+k}-H_k)\equiv 0 \pmod{p}, \tag{3.6}$$

and hence

$$\sum_{k=0}^{m}\binom{m+k}{k}^{2s}(H_{m+k}-H_k)$$
$$\equiv \sum_{k=0}^{m}\binom{m}{k}^{2s}(H_{m+k}-H_k)+2sp\sum_{k=0}^{m}\binom{m}{k}^{2s}H_{m+k}(H_{m+k}-H_k) \pmod{p^2}.$$

Especially, when $s=2,3$, Theorem 3.3 allows us to obtain the following expressions:

Proposition 3.7. *We have*

$$\sum_{k=0}^{m}\binom{m}{k}^{4}(H_{m+k}-H_k)\equiv -4p\sum_{k=0}^{m}\binom{m}{k}^{4}H_{m+k}(H_{m+k}-H_k) \pmod{p^2}, \tag{3.7}$$
$$\sum_{k=0}^{m}\binom{m}{k}^{6}(H_{m+k}-H_k)\equiv -6p\sum_{k=0}^{m}\binom{m}{k}^{6}H_{m+k}(H_{m+k}-H_k) \pmod{p^2}. \tag{3.8}$$

These exhibit the p-divisibility of the sums in an explicit manner. These formulas could be used to reduce the analysis of the mod p^2 behavior of the sums in the left-hand sides to that of the mod p behavior of the corresponding sums in the right-hand sides.

3.3 Related conjectural congruences

In the final position, we give several conjectures on congruences involving *odd* powers of binomial coefficients and harmonic sums which we found by numerical experiments.

Conjecture 3.8. *If* $p\equiv 1 \pmod{4}$ *and* $p>5$, *then*

$$\sum_{k=0}^{m}\binom{m+k}{k}^{3}(H_{m+k}-H_k)\equiv 0 \pmod{p^2}, \tag{3.9}$$
$$\sum_{k=0}^{m}\binom{m+k}{k}^{5}(H_{m+k}-H_k)\equiv 0 \pmod{p^2}. \tag{3.10}$$

Conjecture 3.9. *If* $p \equiv 3 \pmod 4$, *then*

$$\sum_{k=0}^{m} \binom{m+k}{k}^5 (H_{m+k}^{(2)} - H_k^{(2)} - 5(H_{m+k} - H_k)^2) \equiv 0 \pmod{p^2}, \tag{3.11}$$

$$\sum_{k=0}^{m} \binom{m+k}{k}^7 (H_{m+k}^{(2)} - H_k^{(2)} - 7(H_{m+k} - H_k)^2) \equiv 0 \pmod{p^2}, \tag{3.12}$$

where $H_n^{(2)} = \sum_{i=1}^{n} \frac{1}{i^2}$.

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References

[1] M. Kaneko, A recurrence formula for the Bernoulli numbers, Proc. Japan Acad. Ser. A Math. Sci. **71** (1995), 192–193.

[2] H. Momiyama, A new recurrence formula for Bernoulli numbers. Fibonacci Quart. **39** (2001), 285–288.

[3] R. Osburn, private communication, 2022.

[4] A. von Ettingshausen, *Vorlesungen über die höhere Mathematik*, vol. 1, Wien, 1827.

Department of Mathematical Sciences
Faculty of Science
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
JAPAN