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VULNERABILITY OF DIENE-THABET-YUSUF'S CUBIC MULTIVARIATE SIGNATURE SCHEME *

Yasufumi Hashimoto

Abstract

In 2020, Diene, Thabet and Yusuf proposed a new multivariate signature scheme whose public key is a set of multivariate "cubic" polynomials over a finite field. In the present paper, we show how to recover its equivalent secret key.

Keywords. multivariate public-key cryptosystems, cubic polynomials

1 Introduction

A multivariate public key cryptosystem is a cryptosystem whose public key is a set of multivariate non-linear polynomials over a finite field, and has been considered to be a candidate of post-quantum cryptography. In fact, in NIST's standardization project of post-quantum cryptography, Rainbow [3] and GeMSS [2] were selected as a finalist and an alternative candidate respectively in the final (third) round [12].

Most multivariate public key cryptosystems, including these two signature schemes, are constructed by quadratic polynomials. One of the reasons why there have been few schemes with (over) cubic polynomials is that the number of coefficients in cubic polynomials is much more than that in quadratic polynomials and then the key size is much larger. While there might be a cubic type scheme which is secure enough to compensate for the disadvantage in efficiency, we do not have such schemes at the present time (see e.g. [9, 4, 10, 1, 11, 6, 7]). Recently, Diene-Thabet-Yusuf [5] proposed a multivariate signature scheme using cubic polynomials, whose signature generations are fast enough. However, such a structure for speeding up the signature generation has yielded a vulnerability. In the present paper, we show that how to recover its equivalent secret key of this signature scheme efficiently.

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2 Diene-Thabet-Yusuf's signature scheme

We first describe the construction of Diene-Thabet-Yusuf's signature scheme [5].

Let q be a power of prime, $\mathbf{F}_q$ a finite field of order q and $r, m, n \geq 1$ integers with $m := r^2$, $n := 2r^2 = 2m$. Denote by $k_1(\mathbf{x}), \ldots, k_n(\mathbf{x})$ linear polynomials of $\mathbf{x} = {}^t(x_1, \ldots, x_n)$ and put

$$P = P(\mathbf{x}) := \begin{pmatrix} k_1(\mathbf{x}) \cdot k_{m+1}(\mathbf{x}) & k_{r+1}(\mathbf{x}) \cdot k_{m+r+1}(\mathbf{x}) & \cdots & k_{m-r+1}(\mathbf{x}) \cdot k_{n-r+1}(\mathbf{x}) \\ k_2(\mathbf{x}) \cdot k_{m+2}(\mathbf{x}) & k_{r+2}(\mathbf{x}) \cdot k_{m+r+2}(\mathbf{x}) & \cdots & k_{m-r+2}(\mathbf{x}) \cdot k_{n-r+2}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ k_r(\mathbf{x}) \cdot k_{m+r}(\mathbf{x}) & k_{2r}(\mathbf{x}) \cdot k_{m+2r}(\mathbf{x}) & \cdots & k_m(\mathbf{x}) \cdot k_n(\mathbf{x}) \end{pmatrix}.$$

Generate an $r \times r$ matrix $M = M(\mathbf{x})$ whose entries are (constants or) linear polynomials of $\mathbf{x}$ such that the entries of M^{-1} are also (constants or) linear polynomials of $\mathbf{x}$. Define the cubic polynomial map $G : \mathbf{F}_q^n \to \mathbf{F}_q^m$, $G(\mathbf{x}) = {}^t(g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$ by

$$\begin{pmatrix} g_1(\mathbf{x}) & \cdots & g_{m-r+1}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ g_r(\mathbf{x}) & \cdots & g_m(\mathbf{x}) \end{pmatrix} = M(\mathbf{x}) \cdot P(\mathbf{x}).$$

Diene-Thabet-Yusuf's signature scheme is as follows [5].

Secret key: Two invertible affine maps $S : \mathbf{F}_q^n \to \mathbf{F}_q^n$, $T : \mathbf{F}_q^m \to \mathbf{F}_q^m$ and polynomial matrices P, M.

Public key: The cubic polynomial map

$$F := T \circ G \circ S : \mathbf{F}_q^n \to \mathbf{F}_q^m.$$

Signature generation: For a message $\mathbf{m} \in \mathbf{F}_q^m$, compute $\mathbf{y} = (y_1, \ldots, y_m) := T^{-1}(\mathbf{m})$. Next choose $u_1, \ldots, u_m \in \mathbf{F}_q$ randomly and find $\mathbf{x} \in \mathbf{F}_q^n$ satisfying

$$M(\mathbf{x})^{-1} \cdot \begin{pmatrix} y_1 & \cdots & y_{m-r+1} \\ \vdots & \ddots & \vdots \\ y_r & \cdots & y_m \end{pmatrix} = \begin{pmatrix} u_1 \cdot k_1(\mathbf{x}) & \cdots & u_{m-r+1} \cdot k_{m-r+1}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ u_r \cdot k_r(\mathbf{x}) & \cdots & u_m \cdot k_m(\mathbf{x}) \end{pmatrix},$$
$$(k_{m+1}(\mathbf{x}), \ldots, k_{2m}(\mathbf{x})) = (u_1, \ldots, u_m).$$

The signature for the message $\mathbf{m}$ is $\mathbf{s} = S^{-1}(\mathbf{x})$.

Signature verification: Verify whether $F(\mathbf{s}) = \mathbf{m}$ holds.

Since M is generated such that the entries of $M(\mathbf{x})^{-1}$ are (constants or) linear polynomials, the signature generation requires only solving a system of n linear equations of n variables. The complexity of the signature generation is thus $O(n^3)$.

3 Key recovery attack on DTY signature scheme

We now propose our key recovery attack on Diene-Thabet-Yusuf's signature scheme.

Let $K : \mathbf{F}_q^n \to \mathbf{F}_q^n$ be the linear map with $K(\mathbf{x}) = (k_1(\mathbf{x}), \ldots, k_n(\mathbf{x}))$, $\tilde{P} : \mathbf{F}_q^n \to \mathbf{F}_q^m$ the quadratic polynomial map with

$$\tilde{P}(\mathbf{x}) = {}^t\!(p_1(\mathbf{x}), \ldots, p_m(\mathbf{x})) := {}^t\!(x_1 \cdot x_{m+1}, \ldots, x_m \cdot x_n)$$

and $\tilde{M}(\mathbf{x}) := \begin{pmatrix} M(\mathbf{x}) & & \\ & \ddots & \\ & & M(\mathbf{x}) \end{pmatrix}$. It is easy to see that

$$G(\mathbf{x}) = \tilde{M}(\mathbf{x})\tilde{P}(K(\mathbf{x})),$$

and then

$$F(\mathbf{x}) = (T\tilde{M}(\mathbf{x}))\tilde{P}((K(S(\mathbf{x}))).$$

Since T, K, S are affine maps and the entries of $\tilde{M}^{-1}$ are (constants or) linear polynomials of $\mathbf{x}$, there exist an $m \times m$ matrix $L = L(\mathbf{x})$ whose entries are (constants or) linear polynomials and quadratic polynomials $h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})$ such that

$$L(\mathbf{x})F(\mathbf{x}) = {}^t\!(h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})).$$

We can easily check that one can find such an L in polynomial time and the quadratic polynomials $h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})$ are linear sums of $p_1((K(S(\mathbf{x}))), \ldots,$ $p_m((K(S(\mathbf{x})))$. Then the coefficient matrices of $h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})$ are in the forms

$${}^t\!(KS) \begin{pmatrix} 0_m & * \\ * & 0_m \end{pmatrix} (KS).$$

This means that the polynomials $h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})$ are the balanced oil-vinegar type and then that Kipnis-Shamir's attack on the (balanced) oil-vinegar signature scheme [13, 8, 9] is available for $(h_1(\mathbf{x}), \ldots, h_m(\mathbf{x}))$. We can recover a linear map $S_1 : \mathbf{F}_q^n \to \mathbf{F}_q^n$ satisfying

$$(KS)S_1 = \begin{pmatrix} *_m & * \\ 0 & *_m \end{pmatrix}$$

in polynomial time. It is easy to see that the quadratic polynomials in $L(\mathbf{x})F(S_1(\mathbf{x}))$ are in the forms

$${}^t\mathbf{x} \begin{pmatrix} 0_m & * \\ * & *_m \end{pmatrix} \mathbf{x} + (\text{linear polynomial of } \mathbf{x}).$$

This is equivalent to the polynomials in the balanced oil-vinegar signature scheme [13]. We thus conclude that the attacker can generate dummy signatures for arbitrary messages feasibly and this signature scheme is not secure at all.

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WREATH DETERMINANTS, ZONAL SPHERICAL FUNCTIONS ON SYMMETRIC GROUPS AND THE ALON-TARSI CONJECTURE*

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Abstract

In the article, we give several formulas for a certain zonal spherical function on the symmetric group in terms of polynomial functions on matrices called the alpha-determinant and wreath determinant. We also explain the relation between these objects and the Alon-Tarsi conjecture on the enumeration of Latin squares. In particular, we give an alternative proofs of (i) Glynn's result on a special case of the Alon-Tarsi conjecture, and (ii) the result due to Kumar and Landsberg on the equivalence between a special case of Kumar's conjecture on plethysms and the Alon-Tarsi conjecture. Most of the results given here are already announced in the articles [8, 9].

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Key words and phrases. Symmetric groups, zonal spherical functions, alpha-determinants, wreath determinants, Latin squares, plethysms.

1 Introduction

For a given pair of positive integers n and k, let $\omega_{n,k}$ be the function on the symmetric group $\mathfrak{S}_{kn}$ of degree kn defined by

$$\omega_{n,k}(g) = \frac{1}{|\mathcal{K}|}\sum_{y\in\mathcal{K}} \chi^{(k^n)}(gy), \quad g \in \mathfrak{S}_{kn},$$

where $\mathcal{K} = \mathfrak{S}_{(k^n)}$ is a Young subgroup of $\mathfrak{S}_{kn}$ corresponding to the partition $(k^n) = (k,\dots,k) \vdash kn$, and $\chi^{(k^n)}$ is the irreducible character of $\mathfrak{S}_{kn}$ corresponding to the same partition (k^n). This function is biinvariant with respect to $\mathcal{K}$, that is,

$$\omega_{n,k}(ygy') = \omega_{n,k}(g), \qquad \forall g \in \mathfrak{S}_{kn},\ \forall y, y' \in \mathcal{K}.$$

We refer to the function $\omega_{n,k}$ as a zonal spherical on $\mathfrak{S}_{kn}$ with respect to $\mathcal{K}$. Note that in the case where $n = 2$, $\omega_{2,k}$ is indeed a zonal spherical function associated

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to the Gelfand pair $(\mathfrak{S}_{2k}, \mathfrak{S}_k \times \mathfrak{S}_k)$ in the ordinary sense (see, e.g. Macdonald [12, Chapter VII]).

The purpose of the article is to give several formulas for $\omega_{n,k}$ in terms of polynomial functions on matrices called the *alpha-determinant* [13, 14] (Theorem 4.1) and *wreath determinant* [10] (Theorem 4.6). The alpha-determinant is a parametric deformation of the ordinary determinant, which interpolates the determinant and permanent. The wreath-determinant wrdet_k is a polynomial function on the space $\mathrm{Mat}_{n,kn}$ consisting of n by kn matrices, which is defined via the alpha-determinant (see (3.1)), and it has a nice characterization in terms of a suitable $\mathrm{GL}_{kn} \times \mathcal{K}$-action (see (W1)–(W3) in §3). When $k = 1$, the 1-wreath determinant wrdet_1 on $\mathrm{Mat}_n = \mathrm{Mat}_{n,n}$ agrees with the usual determinant. In this sense, our result provides a 'quasi-determinantal' formula for the zonal spherical function $\omega_{n,k}$.

As an application of our formulas, we show that the values of $\omega_{n,k}$ do not vanish when k is equal to $p-1$ for a certain *odd prime number* p. In particular, we observe that the *Alon-Tarsi conjecture on the Latin squares* is true when the size of squares is $p-1$ for an odd prime p. This gives an alternative proof of Glynn's result [5]. We also look at a conjecture on certain *plethysms* due to Kumar and see that the conjecture in a special case is equivalent to the Alon-Tarsi conjecture, which is originally obtained in [11].

Most of the results given here are already announced in the articles [8, 9].

2 Preliminaries

2.1 General conventions

The symmetric group of degree n is denoted by $\mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, $P(\sigma) = (\delta_{i\sigma(j)})$ is the permutation matrix of σ. The set of m by n complex matrices is denoted by $\mathrm{Mat}_{m,n}$, and we write $\mathrm{Mat}_n = \mathrm{Mat}_{n,n}$ for short. The identity matrix of size n is I_n, and $\mathbf{1}_{m,n}$ is the m by n matrix all of whose entries are one. We write $\mathbf{1}_n$ to indicate $\mathbf{1}_{n,n}$. We denote by $A \otimes B$ the Kronecker product of matrices defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix} \in \mathrm{Mat}_{mp,nq}$$

for $A = (a_{ij}) \in \mathrm{Mat}_{m,n}$ and $B \in \mathrm{Mat}_{p,q}$. The general linear group of degree n is GL_n. We always work on the vector spaces and/or algebras over the complex number field $\mathbb{C}$. The cardinality of a set S is denoted by $|S|$.

Let x_{ij} $(1 \le i,j \le n)$ be independent commuting variables, and put $X = (x_{ij})_{1\le i,j\le n}$. For $M = (m_{ij}) \in \mathrm{Mat}_n$ such that $m_{ij} \in \mathbb{Z}_{\ge 0}$, define

$$x^M := \prod_{i,j} x_{ij}^{m_{ij}}.$$

By this notation, we have

$$\det X = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, x^{P(\sigma)}$$

for instance. When $p = p(x_{11}, \dots, x_{nn})$ is a polynomial in x_{ij}'s, we denote by $[p]_M$ the coefficient of the monomial x^M in p.

2.2 Double cosets

We fix a pair of positive integers n and k in what follows. Let $\Omega = (\Omega_1, \dots, \Omega_n)$ be a set partition of $\{1, 2, \dots, kn\}$ given by

$$\begin{aligned}\Omega_i &:= \left\{ m \in \mathbb{Z} \,\middle|\, \left\lceil \frac{m}{k} \right\rceil = i \right\} \\ &= \{(i-1)k + r \mid r = 1, 2, \dots, k\} \quad (i = 1, \dots, n)\end{aligned}$$

and define

$$\mathcal{K} := \{g \in \mathfrak{S}_{kn} \mid g\Omega_i = \Omega_i \ (i = 1, \dots, n)\}.$$

Notice that $\mathcal{K}$ is isomorphic to the direct product $\mathfrak{S}_k^n = \overbrace{\mathfrak{S}_k \times \cdots \times \mathfrak{S}_k}^{n}$ of the n copies of $\mathfrak{S}_k$. Put

$$m_{ij}(g) := |g\Omega_i \cap \Omega_j| \quad (1 \le i, j \le n), \qquad \mathsf{M}(g) := (m_{ij}(g))_{1 \le i,j \le n}$$

for $g \in \mathfrak{S}_{kn}$, that is, $m_{ij}(g)$ counts the number of elements in Ω_i which are sent into Ω_j by g. For $g, g' \in \mathfrak{S}_{kn}$, we see that

$$\mathcal{K}g\mathcal{K} = \mathcal{K}g'\mathcal{K} \iff \mathsf{M}(g) = \mathsf{M}(g')$$

and

$$|\mathcal{K}g\mathcal{K}| = \frac{|\mathcal{K}|^2}{\mathsf{M}(g)!},$$

where $\mathsf{M}(g)! = \prod_{i,j=1}^n m_{ij}(g)!$. Put

$$\mathcal{M}_{n,k} := \left\{ M = (m_{ij}) \in \mathrm{Mat}_n(\mathbb{Z}_{\ge 0}) \,\middle|\, \sum_{r=1}^n m_{ir} = \sum_{s=1}^n m_{sj} = k \ (1 \le i, j \le n) \right\}.$$

The map

$$\mathcal{K} \backslash \mathfrak{S}_{kn} / \mathcal{K} \ni \mathcal{K}g\mathcal{K} \mapsto \mathsf{M}(g) \in \mathcal{M}_{n,k}$$

is bijective. Thus $\mathcal{M}_{n,k}$ gives a 'coordinate system' for the set $\mathcal{K} \backslash \mathfrak{S}_{kn} / \mathcal{K}$ of double cosets.

2.3 Immanants and zonal spherical functions

For each $\lambda \vdash kn$, define

$$\omega_{\mathcal{K}}^{\lambda}(g) := \frac{1}{|\mathcal{K}|} \sum_{y \in \mathcal{K}} \chi^{\lambda}(gy) \quad (g \in \mathfrak{S}_{kn}), \tag{2.1}$$

where χ^λ is the irreducible character of $\mathfrak{S}_{kn}$ corresponding to λ. These are $\mathcal{K}$-biinvariant functions on $\mathfrak{S}_{kn}$, and hence we refer to these as zonal spherical functions.

Since χ^λ are $\mathbb{Z}$-valued, the functions $\omega_{\mathcal{K}}^\lambda$ are $\mathbb{Q}$-valued. Observe that $\omega_{n,k} = \omega_{\mathcal{K}}^{(k^n)}$. The function $\omega_{\mathcal{K}}^\lambda$ is identically zero unless $\lambda \geq (k^n)$ with respect to the dominance ordering

$$\lambda \geq \mu \iff \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \quad \forall i \geq 1$$

on partitions of the same size.

The *immanant* of a matrix $A = (a_{ij}) \in \mathrm{Mat}_N$ associated to $\lambda \vdash N \in \mathbb{Z}_{>0}$ is

$$\mathrm{Imm}^\lambda A = \sum_{\sigma \in \mathfrak{S}_N} \chi^\lambda(\sigma) \prod_{i=1}^{N} a_{i\sigma(i)}. \tag{2.2}$$

Notice that $\mathrm{Imm}^{(1^N)} A = \det A$ and $\mathrm{Imm}^{(N)} A = \mathrm{per}\, A$, where $\mathrm{per}\, A$ is the permanent of A. For later use, we give an expression of the value of $\omega_{\mathcal{K}}^\lambda$ in terms of immanants.

Lemma 2.1. *For any $A = (a_{ij}) \in \mathrm{Mat}_{n,kn}$, we have*

$$\mathrm{Imm}^\lambda(A \otimes \mathbf{1}_{k,1}) = \sum_{\tau \in \mathfrak{S}_{kn}} \omega_{\mathcal{K}}^\lambda(\tau) \prod_{j=1}^{kn} a'_{j\tau(j)}, \tag{2.3}$$

where $a'_{ij} = a_{\lceil i/k \rceil, j}$ is the (i,j)-entry of $A \otimes \mathbf{1}_{k,1}$.

Proof. Since $a'_{y(i)j} = a'_{ij}$ for any $y \in \mathcal{K}$, it follows that

$$\begin{aligned}
\mathrm{Imm}^\lambda(A \otimes \mathbf{1}_{k,1}) &= \sum_{\sigma \in \mathfrak{S}_{kn}} \chi^\lambda(\sigma) \prod_{i=1}^{kn} a'_{i\sigma(i)} = \frac{1}{|\mathcal{K}|} \sum_{y \in \mathcal{K}} \sum_{\sigma \in \mathfrak{S}_{kn}} \chi^\lambda(\sigma) \prod_{i=1}^{kn} a'_{y(i)\sigma(i)} \\
&= \frac{1}{|\mathcal{K}|} \sum_{y \in \mathcal{K}} \sum_{\tau \in \mathfrak{S}_{kn}} \chi^\lambda(\tau y) \prod_{j=1}^{kn} a'_{j\tau(j)} = \sum_{\tau \in \mathfrak{S}_{kn}} \omega_{\mathcal{K}}^\lambda(\tau) \prod_{j=1}^{kn} a'_{j\tau(j)}
\end{aligned}$$

as desired. □

Lemma 2.2. *Let $\lambda \vdash kn$.*

(i) *For $g \in \mathfrak{S}_{kn}$,*

$$\omega_{\mathcal{K}}^\lambda(g) = \frac{1}{|\mathcal{K}|} \mathrm{Imm}^\lambda((I_n \otimes \mathbf{1}_k) P(g)).$$

(ii) *It holds that*

$$\mathrm{Imm}^\lambda(X \otimes \mathbf{1}_k) = \sum_{\tau \in \mathfrak{S}_{kn}} \omega_{\mathcal{K}}^\lambda(\tau) x^{\mathrm{M}(\tau)}.$$

In particular,

$$\omega_{\mathcal{K}}^\lambda(g) = \frac{\mathrm{M}(g)!}{|\mathcal{K}|^2} \left[\mathrm{Imm}^\lambda(X \otimes \mathbf{1}_k)\right]_{\mathrm{M}(g)}$$

for $g \in \mathfrak{S}_{kn}$.

Proof. We get (i) if we set $A = (I_n \otimes \mathbf{1}_{1,k})P(g)$ with $g \in \mathfrak{S}_{kn}$ in (2.3). If we set $A = X \otimes \mathbf{1}_{1,k}$ in (2.3), then we have (ii) since $a'_{i\tau(i)} = x_{pq}$ when $i \in \Omega_p$ and $\tau(i) \in \Omega_q$ and

$$\sum_{\tau\in\mathfrak{S}_{kn}} \omega^\lambda_{\mathcal{K}}(\tau) x^{\mathsf{M}(\tau)} = \sum_{M\in\mathcal{M}_{n,k}} \sum_{\substack{\tau\in\mathfrak{S}_{kn}\\ \mathsf{M}(\tau)=M}} \omega^\lambda_{\mathcal{K}}(\tau) x^M = \sum_{M\in\mathcal{M}_{n,k}} \frac{|\mathcal{K}|^2}{M!} \omega^\lambda_{\mathcal{K}}(g_M) x^M,$$

where g_M is an arbitrarily chosen element in $\mathfrak{S}_{kn}$ such that $\mathsf{M}(g_M) = M$. □

3 The alpha-determinant and wreath determinant

We recall the definitions and basic facts on the alpha-determinant and wreath determinant. The alpha-determinant is first introduce by Vere-Jones [14] as α-permanent, whose definition is slightly different from ours; here we follow the convention in [13]. For the wreath determinant, see [10] for the detailed information.

First we define a class function $\nu(\cdot)$ on $\mathfrak{S}_N$ by

$$\nu(\sigma) := N - \sum_{i\geq 1} m_i(\sigma) = \sum_{i\geq 2} (i-1) m_i(\sigma)$$

for $\sigma \in \mathfrak{S}_N$ when the cycle type of σ is $1^{m_1(\sigma)} 2^{m_2(\sigma)} \dots N^{m_N(\sigma)}$. Notice that $\nu(\sigma\tau) = \nu(\sigma) + \nu(\tau)$ if σ and τ are disjoint.

Remark 3.1. For each $\sigma \in \mathfrak{S}_N$, $\nu(\sigma)$ is equal to the distance between the identity e and σ on the *Cayley graph* of $\mathfrak{S}_N$ whose generating set consists of all transpositions.

Remark 3.2. The value of $\nu(\sigma)$ for $\sigma \in \mathfrak{S}_N$ is invariant under the standard embedding $\mathfrak{S}_N \hookrightarrow \mathfrak{S}_{N'}$ ($N' > N$) which regards σ as an element in $\mathfrak{S}_{N'}$ leaving $N' - N$ letters $N+1, \dots, N'$ fixed. Namely, it would be natural to regard the function $\nu(\cdot)$ as a class function on the infinite symmetric group $\mathfrak{S}_\infty = \bigcup_{N\geq 1} \mathfrak{S}_N$.

The *alpha-determinant* of an N by N matrix $A = (a_{ij}) \in \mathrm{Mat}_N$ is

$$\det_\alpha A := \sum_{\sigma\in\mathfrak{S}_N} \alpha^{\nu(\sigma)} \prod_{i=1}^N a_{i\sigma(i)}.$$

Note that $\det_{-1} A = \det A$ and $\det_1 A = \operatorname{per} A$. The alpha-determinant is multilinear in rows and columns, is invariant under the transposition, and has Laplace expansion formula. We see that

$$\det_\alpha(AP(\sigma)) = \det_\alpha(P(\sigma)A)$$

for any $A \in \mathrm{Mat}_N$ and $\sigma \in \mathfrak{S}_N$ because $\nu(\cdot)$ is a class function on $\mathfrak{S}_N$, but the equation $\det_\alpha(AB) = \det_\alpha(BA)$ does not hold in general. We also note that we have

$$\det_\alpha \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \det_\alpha A \det_\alpha C$$

if A and C are square matrices.

Example 3.3. We have

$$\det_\alpha \mathbf{1}_N = \sum_{\sigma\in\mathfrak{S}_N} \alpha^{\nu(\sigma)} = \prod_{j=1}^{N-1}(1+j\alpha).$$

For an n by kn matrix $A=(a_{ij})\in \mathrm{Mat}_{n,kn}$, the *$k$-wreath determinant* of A is defined by

$$\operatorname{wrdet}_k A := \det_{-1/k}(A\otimes \mathbf{1}_{k,1}). \tag{3.1}$$

Note that the 1-wreath determinant wrdet_1 is the ordinary determinant. The wreath-determinant wrdet_k is characterized as a polynomial function on the space $\mathrm{Mat}_{n,kn}$ by the following three conditions up to a scalar multiple (see [10] for the proof):

(W1) wrdet_k is multilinear in columns.

(W2) $\operatorname{wrdet}_k(QA) = (\det Q)^k \operatorname{wrdet}_k(A)$ for $Q\in\mathrm{Mat}_n$ and $A\in\mathrm{Mat}_{n,kn}$.

(W3) $\operatorname{wrdet}_k(AP(\sigma)) = \operatorname{wrdet}_k(A)$ for $\sigma\in\mathcal{K}$ and $A\in\mathrm{Mat}_{n,kn}$. In other words, if $A_i\in\mathrm{Mat}_{n,k}$ $(i=1,2,\dots,n)$, then

$$\operatorname{wrdet}_k(A_1P(\sigma_1)\ A_2P(\sigma_2)\ \dots\ A_nP(\sigma_n)) = \operatorname{wrdet}_k(A_1\ A_2\ \dots\ A_n)$$

for any $\sigma_1,\dots,\sigma_n\in\mathfrak{S}_k$.

In fact, instead of (W3), the k-wreath determinant satisfies a slightly stronger relative invariance

(W3′) $\operatorname{wrdet}_k(AP(g)) = \chi_{n,k}(g)\operatorname{wrdet}_k(A)$ for $g\in\mathcal{K}\rtimes\mathfrak{S}_n = \mathfrak{S}_n\wr\mathfrak{S}_k < \mathfrak{S}_{kn}$ and $A\in\mathrm{Mat}_{n,kn}$, where $\chi_{n,k}$ is defined by

$$\chi_{n,k}(g) = (\operatorname{sgn}\tau)^k, \qquad g=(\sigma,\tau)\in\mathcal{K}\rtimes\mathfrak{S}_k. \tag{3.2}$$

(W3′) means that if $A_i\in\mathrm{Mat}_{n,k}$ $(i=1,2,\dots,n)$, then

$$\operatorname{wrdet}_k(A_{\tau(1)}\ A_{\tau(2)}\ \dots\ A_{\tau(n)}) = (\operatorname{sgn}\tau)^k\ \operatorname{wrdet}_k(A_1\ A_2\ \dots\ A_n)$$

for any $\tau\in\mathfrak{S}_n$. This readily follows from (W2) by taking $Q=I_k\otimes P(\tau)$. Here we regard the wreath product $\mathfrak{S}_n\wr\mathfrak{S}_k$ as a subgroup of $\mathfrak{S}_{kn}$ so that we have

$$P(g) = P(\sigma)\cdot(I_k\otimes P(\tau)), \qquad g=(\sigma,\tau)\in\mathfrak{S}_k\wr\mathfrak{S}_n.$$

Remark 3.4. The definition of the wreath determinant is a bit different from the original one in [10], where the k-wreath determinant is defined for the kn by n rectangular matrices.

Example 3.5. We have

$$\begin{aligned}\operatorname{wrdet}_k(I_n\otimes\mathbf{1}_{1,k}) = \det_{-1/k}(I_n\otimes\mathbf{1}_k) &= \det_{-1/k}\begin{pmatrix}\mathbf{1}_k & & & \\ & \mathbf{1}_k & & \\ & & \ddots & \\ & & & \mathbf{1}_k\end{pmatrix}\\ &= (\det_{-1/k}\mathbf{1}_k)^n = \Big(\frac{k!}{k^k}\Big)^n.\end{aligned} \tag{3.3}$$

More generally, for $A \in \mathrm{Mat}_n$, we have

$$\mathrm{wrdet}_k(A \otimes \mathbf{1}_{1,k}) = \mathrm{wrdet}_k(A \cdot (I_n \otimes \mathbf{1}_{1,k})) = \Big(\frac{k!}{k^k}\Big)^n (\det A)^k.$$

4 Formulas for zonal spherical functions

The alpha-determinant is written as a linear combination of immanants as

$$\det_\alpha A = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) \operatorname{Imm}^\lambda A, \tag{4.1}$$

where $f^\lambda = \chi^\lambda(e)$, e being the identity permutation, and

$$f_\lambda(\alpha) = \prod_{i=1}^{l(\lambda)} \prod_{j=1}^{\lambda_i} (1 + (j-i)\alpha)$$

is the modified content polynomial for λ. This is immediate from the well-known expansion formula

$$\alpha^{\nu(\cdot)} = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) \chi^\lambda. \tag{4.2}$$

Theorem 4.1. *For $g \in \mathfrak{S}_{kn}$, we have*

$$\begin{aligned}\omega_{n,k}(g) &= \frac{k^{kn}}{|\mathcal{K}|} \det_{-1/k}((I_n \otimes \mathbf{1}_k)P(g)) \\ &= \Big(\frac{k^k}{k!}\Big)^n \sum_{y \in \mathcal{K}} \Big(-\frac{1}{k}\Big)^{\nu(gy)}.\end{aligned}$$

Proof. By (4.1) and Lemma 2.2 (i), we have

$$\det_{-1/k}((I_n \otimes \mathbf{1}_k)P(g)) = \frac{|\mathcal{K}|}{(kn)!} \sum_{\lambda \vdash kn} f^\lambda f_\lambda(-1/k) \omega_{\mathcal{K}}^\lambda(g).$$

Since $f_\lambda(-1/k) = 0$ if $\lambda_1 > k$ and $\operatorname{Imm}^\lambda(A \otimes \mathbf{1}_{k,1}) = 0$ unless $\lambda \geq (k^n)$, only the term for $\lambda = (k^n)$ survives in the righthand side of the equation above. By the hook formula for f^λ and the definition of $f_\lambda(\alpha)$, we readily obtain

$$f^{(k^n)} f_{(k^n)}(-1/k) = \frac{(kn)!}{k^{kn}}.$$

This completes the proof of the first equality. The second equality is immediate by the definition of the alpha-determinant. $\square$

Using Theorem 4.1, we obtain the stability of $\omega_{n,k}$ with respect to n as well as the non-vanishingness of $\omega_{n,k}$ when $k+1$ is prime as follows.

Corollary 4.2. *If $m > n$, then $\omega_{m,k}(g) = \omega_{n,k}(g)$ for any $g \in \mathfrak{S}_{kn}$, where we regard $g \in \mathfrak{S}_{kn}$ as an element in $\mathfrak{S}_{km}$ by the standard embedding.*

Proof. We regard $\mathfrak{S}_k^m$ as a direct product $\mathfrak{S}_k^n \times \mathfrak{S}_k^{m-n}$. If $g \in \mathfrak{S}_{kn}$ and $(y_1, y_2) \in \mathfrak{S}_k^n \times \mathfrak{S}_k^{m-n}$, then gy_1 and y_2 are disjoint permutations, and hence it follows that $\nu(gy_1y_2) = \nu(gy_1) + \nu(y_2)$. Thus we have

$$\begin{aligned}\omega_{m,k}(g) &= \Big(\frac{k^k}{k!}\Big)^m \sum_{(y_1,y_2)\in\mathfrak{S}_k^n\times\mathfrak{S}_k^{m-n}} \Big(-\frac{1}{k}\Big)^{\nu(gy_1y_2)} \\ &= \Big(\frac{k^k}{k!}\Big)^m \sum_{y_1\in\mathfrak{S}_k^n} \Big(-\frac{1}{k}\Big)^{\nu(gy_1)} \sum_{y_2\in\mathfrak{S}_k^{m-n}} \Big(-\frac{1}{k}\Big)^{\nu(y_2)} \\ &= \Big(\frac{k^k}{k!}\Big)^n \sum_{y_1\in\mathfrak{S}_k^n} \Big(-\frac{1}{k}\Big)^{\nu(gy_1)} \\ &= \omega_{n,k}(g)\end{aligned}$$

as desired. □

Theorem 4.3. *Let p be an odd prime. The function $\omega_{n,k}$ does not vanish on $\mathfrak{S}_{kn}$ if $k = p-1$.*

Proof. By Theorem 4.1, we have

$$\omega_{n,k}(g) = \Big(\frac{(p-1)^{p-1}}{(p-1)!}\Big)^n \sum_{y\in\mathcal{K}} \Big(-\frac{1}{p-1}\Big)^{\nu(gy)} \equiv \frac{1}{|\mathcal{K}|} \sum_{y\in\mathcal{K}} 1 \equiv 1 \pmod p$$

for any $g \in \mathfrak{S}_{kn}$, which implies the desired nonvanishingness. □

Remark 4.4. In [7], the inverse of Theorem 4.3 is proved. In fact, the authors show that if $n \geq 3$ and $k+1$ is composite, then one can find $M \in \mathcal{M}_{n,k}$ such that $\big[(\det X)^k\big]_M = 0$.

We give a formula for the function $\omega_{n,k}$ in terms of the wreath determinant.

Lemma 4.5. *For $A \in \mathrm{Mat}_{n,kn}$, we have*

$$\mathrm{wrdet}_k A = \frac{1}{k^{kn}} \mathrm{Imm}^{(k^n)}(A \otimes \mathbf{1}_{k,1}).$$

Proof. By the definition of the wreath determinant and the formula (4.1), we have

$$\begin{aligned}\mathrm{wrdet}_k A &= \det_{-1/k}(A \otimes \mathbf{1}_{k,1}) \\ &= \frac{1}{(kn)!} \sum_{\lambda\vdash kn} f^\lambda f_\lambda(-1/k)\,\mathrm{Imm}^\lambda(A\otimes\mathbf{1}_{k,1}).\end{aligned}$$

The conclusion follows from a similar discussion as in the proof of Theorem 4.1. □

Theorem 4.6. *For $g \in \mathfrak{S}_{kn}$, we have*

$$\begin{aligned}\omega_{n,k}(g) &= \frac{\mathsf{M}(g)!}{|\mathcal{K}|}\big[(\det X)^k\big]_{\mathsf{M}(g)} \\ &= \frac{\mathrm{wrdet}_k((I_n\otimes\mathbf{1}_{1,k})P(g))}{\mathrm{wrdet}_k(I_n\otimes\mathbf{1}_{1,k})}.\end{aligned}$$

Proof. By Lemma 4.5, we see that

$$\operatorname{wrdet}_k(X \otimes \mathbf{1}_{1,k}) = \frac{1}{k^{kn}} \operatorname{Imm}^{(k^n)}(X \otimes \mathbf{1}_k).$$

On the other hand, by (W2) and (3.3), we have

$$\operatorname{wrdet}_k(X \otimes \mathbf{1}_{1,k}) = (\det X)^k \operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k}) = \Bigl(\frac{k!}{k^k}\Bigr)^n (\det X)^k.$$

Thus it follows that

$$(\det X)^k = \frac{1}{|\mathcal{K}|} \operatorname{Imm}^{(k^n)}(X \otimes \mathbf{1}_k).$$

Hence, by Lemma 2.2 (ii), we have the first equality. The second equality is obtained by Theorem 4.1 and the equation

$$\det_{-1/k}((I_n \otimes \mathbf{1}_k)P(g)) = \operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g)),$$

which follows from the definition of the wreath determinant. □

As a corollary, we see that the relative invariance of the function $\omega_{n,k}$ with respect to the wreath product $\mathfrak{S}_k \wr \mathfrak{S}_n$.

Corollary 4.7. *For any $g \in \mathfrak{S}_{kn}$ and $h, h' \in \mathfrak{S}_k \wr \mathfrak{S}_n$, we have*

$$\omega_{n,k}(hgh') = \chi_{n,k}(hh')\omega_{n,k}(g).$$

Here $\chi_{n,k}$ is the character of $\mathfrak{S}_k \wr \mathfrak{S}_n$ defined by (3.2). In particular, $\omega_{n,k}$ is $\mathfrak{S}_k \wr \mathfrak{S}_n$-biinvariant if k is even.

Proof. Let $h = (\sigma, \tau), h' = (\sigma', \tau') \in \mathcal{K} \rtimes \mathfrak{S}_n = \mathfrak{S}_k \wr \mathfrak{S}_n$. Since

$$(I_n \otimes \mathbf{1}_{1,k})P(h) = (I_n \otimes \mathbf{1}_{1,k})P(\sigma)(I_k \otimes P(\tau)) = P(\tau)(I_n \otimes \mathbf{1}_{1,k}),$$

we have

$$\begin{aligned}
\omega_{n,k}(hgh') &= \frac{\operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(hgh'))}{\operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})} \\
&= \frac{\operatorname{wrdet}_k(P(\tau)(I_n \otimes \mathbf{1}_{1,k})P(g)P(h'))}{\operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})} \\
&= \det P(\tau)^k \chi_{n,k}(h') \frac{\operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g))}{\operatorname{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})} \\
&= \chi_{n,k}(hh')\omega_{n,k}(g)
\end{aligned}$$

as desired. □

5 Applications

5.1 The Alon-Tarsi conjecture on Latin squares

A *Latin square* of degree n is an n by n matrix whose rows and columns are permutations of $1, 2, \dots, n$. The set of all Latin squares of degree n is denoted by $\mathrm{LS}(n)$.

Example 5.1. There are twelve Latin squares of degree 3:

$$\mathrm{LS}(3) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix},$$
$$\left. \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}.$$

For $L \in \mathrm{LS}(n)$, we associate $2n$ permutations $r_1, \dots, r_n, c_1, \dots, c_n \in \mathfrak{S}_n$ to it by

$$L = \begin{pmatrix} r_1(1) & \dots & r_1(n) \\ \vdots & \ddots & \vdots \\ r_n(1) & \dots & r_n(n) \end{pmatrix} = \begin{pmatrix} c_1(1) & \dots & c_n(1) \\ \vdots & \ddots & \vdots \\ c_1(n) & \dots & c_n(n) \end{pmatrix}.$$

Then we define

$$\operatorname{sgn} L := \prod_{i=1}^{n} \operatorname{sgn} r_i \prod_{i=1}^{n} \operatorname{sgn} c_i,$$

and we call L *even* (resp. *odd*) if $\operatorname{sgn} L = +1$ (resp. -1). We denote by $\mathrm{els}(n)$ and $\mathrm{ols}(n)$ the numbers of even and odd Latin squares of degree n respectively. Since the map $\mathrm{LS}(n) \ni L \mapsto P(\sigma)L \in \mathrm{LS}(n)$ for a given $\sigma \in \mathfrak{S}_n$ is a bijection and $\operatorname{sgn}(P(\sigma)L) = (\operatorname{sgn}\sigma)^n \operatorname{sgn} L$ for $L \in \mathrm{LS}(n)$, we have $\mathrm{els}(n) = \mathrm{ols}(n)$ when n is odd. When n is even, it is conjectured that the numbers of even and odd Latin squares are always different.

Conjecture 5.2 (Alon-Tarsi conjecture)**.** $\mathrm{els}(n) \neq \mathrm{ols}(n)$ *if n is even.*

This conjecture originally arose from the study of colorings of graphs. Indeed, if the Alon-Tarsi conjecture for even n is true, then we see that the Dinitz conjecture below for n follows [1].

Proposition 5.3 (Dinitz conjecture)**.** *The line graph of the biclique* (*or complete bipartite graph*) $K_{n,n}$ *is n-choosable.*

We remark that the Dinitz conjecture itself is already settled down by Galvin [4]. There are also various statements which are equivalent to or related with the Alon-Tarsi conjecture (see, e.g. [6, 11]). The Alon-Tarsi conjecture is proved to be true in the case where $n = p + 1$ by Drisko [2] and in the case where $n = p - 1$ by Glynn [5], where p is an odd prime; We also refer to [3].

We need another statement which is equivalent to the Alon-Tarsi conjecture. Define

$$\mathsf{L}(n) := \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathfrak{S}_n^n \mid P(\sigma_1) + \cdots + P(\sigma_n) = \mathbf{1}_n\}.$$

For $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathsf{L}(n)$, the matrix

$$L(\boldsymbol{\sigma}) := \sum_{i=1}^{n} i\, P(\sigma_i)$$

is a Latin square of degree n, and every Latin square is uniquely obtained in this way. A Latin square $L = L(\boldsymbol{\sigma})$ ($\boldsymbol{\sigma} \in \mathsf{L}(n)$) is called *symbol even* (resp. *symbol odd*) if

$$\operatorname{symsgn} L := \prod_{i=1}^{n} \operatorname{sgn} \sigma_i$$

is $+1$ (resp. -1). We denote by $\operatorname{sels}(n)$ and $\operatorname{sols}(n)$ the number of symbol even and symbol odd Latin squares of degree n respectively. It is known that

$$\operatorname{sels}(n) - \operatorname{sols}(n) = (-1)^{n(n-1)/2}(\operatorname{els}(n) - \operatorname{ols}(n))$$

for every n (see, e.g. [5]), so Conjecture 5.2 is equivalent to the

Conjecture 5.4. $\operatorname{sels}(n) \neq \operatorname{sols}(n)$ *if n is even.*

Since

$$\begin{aligned}
[(\det X)^n]_{\mathbf{1}_n} &= \sum_{\substack{\sigma_1, \ldots, \sigma_n \in \mathfrak{S}_n \\ P(\sigma_1) + \cdots + P(\sigma_n) = \mathbf{1}_n}} \prod_{i=1}^{n} (\operatorname{sgn} \sigma_i) \\
&= \sum_{\boldsymbol{\sigma} \in \mathsf{L}(n)} \operatorname{symsgn} L(\boldsymbol{\sigma}) \\
&= \sum_{L \in \mathrm{LS}(n)} \operatorname{symsgn} L = \operatorname{sels}(n) - \operatorname{sols}(n),
\end{aligned}$$

we obtain the following result by Theorem 4.6.

Theorem 5.5. *When n is even, the Alon-Tarsi conjecture on $\mathrm{LS}(n)$ is equivalent to the following assertions.*

(1) $[(\det X)^n]_{\mathbf{1}_n} \neq 0$.

(2) $\operatorname{wrdet}_n((I_n \otimes \mathbf{1}_{1,n})P(g_n)) = \operatorname{wrdet}_n(\overbrace{I_n \ \ldots \ I_n}^{n}) \neq 0$.

(3) $\omega_{n,n}(g_n) \neq 0$.

Here the permutation $g_n \in \mathfrak{S}_{n^2}$ is given by

$$g_n((i-1)n + j) = (j-1)n + i, \qquad 1 \le i, j \le n, \tag{5.1}$$

which is a product of $n(n-1)/2$ disjoint transpositions and $\mathsf{M}(g_n) = \mathbf{1}_n$.

Thus, Theorem 5.5 (3) together with Theorem 4.3 gives another proof of the

Corollary 5.6 (Glynn [5]). *The Alon-Tarsi conjecture for Latin squares of degree n is true if $n = p - 1$ for an odd prime p.*

5.2 A remark on Kumar's conjecture on plethysms

Let k and n be positive integers as heretofore, and V be a finite dimensional vector space over $\mathbb{C}$ such that $\dim V \geq n$. The symmetric group $\mathfrak{S}_m$ acts on $V^{\otimes m}$ from the right by

$$(v_1 \otimes \cdots \otimes v_m) \cdot \sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \quad (\sigma \in \mathfrak{S}_m).$$

This action linearly extends to that of the group algebra $\mathbb{C}\mathfrak{S}_m$. We understand that the symmetric tensor power $S^m(V)$ of V is a subspace of $V^{\otimes m}$ spanned by the vectors of the form

$$v_1 \cdots v_m := v_1 \otimes \cdots \otimes v_m \cdot \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}. \tag{5.2}$$

Set

$$\mathcal{H} = \mathcal{K} \rtimes \mathfrak{S}_n = \mathfrak{S}_k \wr \mathfrak{S}_n, \quad \mathcal{K}' = \mathfrak{S}_n^k, \quad \mathcal{H}' = \mathcal{K}' \rtimes \mathfrak{S}_k = \mathfrak{S}_n \wr \mathfrak{S}_k,$$

and

$$\boldsymbol{e}(G) = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}\mathfrak{S}_{kn}$$

for $G < \mathfrak{S}_{kn}$. We have then

$$S^n(S^k V) = V^{\otimes kn} \cdot \boldsymbol{e}(\mathcal{H}), \qquad S^k(S^n V) = V^{\otimes kn} \cdot \boldsymbol{e}(\mathcal{H}').$$

Define a linear transformation $\tau = \tau_{k,n}$ on $V^{\otimes kn}$ by

$$\tau\colon V^{\otimes kn} \ni \overbrace{\underbrace{v_1^1 \otimes \cdots \otimes v_k^1}_{k} \otimes \cdots \otimes \underbrace{v_1^n \otimes \cdots \otimes v_k^n}_{k}}^{n}$$

$$\longmapsto \overbrace{\underbrace{v_1^1 \otimes \cdots \otimes v_1^n}_{n} \otimes \cdots \otimes \underbrace{v_k^1 \otimes \cdots \otimes v_k^n}_{n}}^{k} \in V^{\otimes kn},$$

or equivalently,

$$\tau(v_1 \otimes v_2 \otimes \cdots \otimes v_{kn}) = (v_1 \otimes v_2 \otimes \cdots \otimes v_{kn}) \cdot g_{n,k},$$

where the permutation $g_{n,k} \in \mathfrak{S}_{kn}$ is defined by

$$g_{n,k}((i-1)n + j) = (j-1)k + i, \qquad 1 \leq i \leq k, \ 1 \leq j \leq n. \tag{5.3}$$

We notice that $g_{n,n}$ equals g_n defined in (5.1). Using this, we define a map $h_{n,k}$ by

$$h_{n,k} := p \circ \tau \circ i\colon S^n(S^k V) \overset{i}{\hookrightarrow} V^{\otimes kn} \overset{\tau}{\to} V^{\otimes kn} \overset{p}{\twoheadrightarrow} S^k(S^n V),$$

where i is the inclusion and p is the natural projection (i.e. multiplication by $\boldsymbol{e}(\mathcal{H}')$ from the right as in (5.2)). Notice that $h_{n,k}(v) = v \cdot g_{n,k}\boldsymbol{e}(\mathcal{H}')$ for $v \in S^n(S^k V)$. This map is clearly a $\mathrm{GL}(V)$-intertwiner between two left $\mathrm{GL}(V)$-modules $S^n(S^k V)$ and $S^k(S^n V)$.

Example 5.7.

$$
\begin{aligned}
h_{2,2}((v_1v_2)(v_3v_4)) &= (p\circ\tau)\left(\frac{v_1\otimes v_2+v_2\otimes v_1}{2}\otimes\frac{v_3\otimes v_4+v_4\otimes v_3}{2}\right)\\
&= \frac{1}{4}p\Bigg(v_1\otimes v_3\otimes v_2\otimes v_4+v_2\otimes v_3\otimes v_1\otimes v_4\\
&\qquad +v_1\otimes v_4\otimes v_2\otimes v_3+v_2\otimes v_4\otimes v_1\otimes v_3\Bigg)\\
&= \frac{(v_1v_3)(v_2v_4)+(v_2v_3)(v_1v_4)+(v_1v_4)(v_2v_3)+(v_2v_4)(v_1v_3)}{4}
\end{aligned}
$$

Motivated by the Hadamard-Howe conjecture on the maximality of $h_{n,k}$, it is conjectured by Kumar that $\ker h_{n,k}$ does not contain $\mathbf{E}_V^{(k^n)}$, the irreducible $\mathrm{GL}(V)$-module with highest weight $(k^n)=(k,\dots,k)$, if $n\le k$ and k is even (see [11, Conjecture 1.6]). We focus on this problem below.

By the Schur-Weyl duality

$$V^{\otimes kn}=\bigoplus_{\lambda\vdash kn}\mathbf{E}_V^{\lambda}\boxtimes\mathbf{M}_{kn}^{\lambda},$$

where $\mathbf{M}_{kn}^{\lambda}$ is the irreducible $\mathfrak{S}_{kn}$-module corresponding to λ, the multiplicity of $\mathbf{E}_V^{\lambda}$ in $S^n(S^kV)$ as a left $\mathrm{GL}(V)$-module is equal to $\dim(\mathbf{M}_{kn}^{\lambda}\cdot\boldsymbol{e}(\mathcal{H}))$, which is majorated by $\dim(\mathbf{M}_{kn}^{\lambda}\cdot\boldsymbol{e}(\mathcal{K}))=K_{\lambda(k^n)}$, the Kostka number.

Remark 5.8. Similarly, we see that the multiplicity of $\mathbf{E}_V^{\lambda}$ in $S^k(S^nV)$ is majorated by $K_{\lambda(n^k)}$. Especially, if $n>k$, then $S^k(S^nV)$ does not contain $\mathbf{E}_V^{(k^n)}$ since $K_{(k^n)(n^k)}=0$.

Lemma 5.9. *The multiplicity of $\mathbf{E}_V^{(k^n)}$ in $S^n(S^kV)$ is exactly one if k is even.*

Proof. Since we know that the multiplicity $\dim(\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{H}))$ of $\mathbf{E}_V^{(k^n)}$ in $S^n(S^kV)$ is at most one, we should show that it is at least one. Take a nonzero $\mathcal{K}$-invariant vector $w\in\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})$, which is unique up to constant multiple since $\dim\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})=K_{(k^n)(k^n)}=1$. We see that

$$w\cdot g=\omega_{n,k}(g)w+w^{\perp}(g) \tag{5.4}$$

for $g\in\mathfrak{S}_{kn}$ where $w^{\perp}(g)$ is a certain vector in the orthocomplement of $\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})$ in $\mathbf{M}_{kn}^{(k^n)}$ with respect to the invariant inner product on $\mathbf{M}_{kn}^{(k^n)}$. Since k is even, we see that $\omega_{n,k}(g)=1$ for $g\in\mathcal{H}$ by Corollary 4.7. Hence it follows that

$$w\cdot\boldsymbol{e}(\mathcal{H})=w\cdot\boldsymbol{e}(\mathcal{H})\boldsymbol{e}(\mathcal{K})=\frac{1}{|\mathcal{H}|}\sum_{g\in\mathcal{H}}(w\cdot g)\cdot\boldsymbol{e}(\mathcal{K})=w+\frac{1}{|\mathcal{H}|}\sum_{g\in\mathcal{H}}w^{\perp}(g)\cdot\boldsymbol{e}(\mathcal{K})=w.$$

Namely, we have $\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K})\subset\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{H})$. Thus we see that

$$\dim(\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{H}))\ge\dim(\mathbf{M}_{kn}^{(k^n)}\cdot\boldsymbol{e}(\mathcal{K}))=K_{(k^n)(k^n)}=1$$

as desired. □

Remark 5.10. If k is *odd*, then $w \cdot g = (\operatorname{sgn}\tau)w$ for $w \in \mathbf{M}_{kn}^{(k^n)}$ and $g = (\sigma,\tau) \in \mathcal{H}$. Thus, in this case, we have $\mathbf{M}_{kn}^{(k^n)} \cdot \boldsymbol{e}(\mathcal{H}) = 0$, and hence $S^n(S^kV)$ does not contain $\mathbf{E}_V^{(k^n)}$.

We restrict our attention on the special case where $k = n$ and n is even. We have $\mathcal{K} = \mathcal{K}'$ and $\mathcal{H} = \mathcal{H}'$ in this case. The map $h_{n,n}$ is then a $\mathrm{GL}(V)$-intertwiner from $S^n(S^nV)$ onto itself. Since the multiplicity of $\mathbf{E}_V^{(n^n)}$ in $S^n(S^nV)$ is one, the restriction of $h_{n,n}$ on $\mathbf{E}_V^{(n^n)}$ must be a scalar by Schur's lemma, and the scalar is given by $\omega_{n,n}(g_n)$ by (5.4) since $h_{n,n}(v) = v \cdot g_n\boldsymbol{e}(\mathcal{H})$. Therefore we obtain the

Theorem 5.11. *When n is even, we have*

$$h_{n,n}(v) = \omega_{n,n}(g_n)v$$

if $v \in S^n(S^nV)$ belongs to the (n^n)-isotypic component. In particular, $\ker h_{n,n} \supset \mathbf{E}_V^{(n^n)}$ *if and only if $\omega_{n,n}(g_n) = 0$.*

As a corollary, we obtain the

Corollary 5.12 ([11, Theorem 1.9 (b)])**.** *The Alon-Tarsi conjecture on* $\mathrm{LS}(n)$ *is equivalent to the assertion that* $\ker h_{n,n}$ *does not contain* $\mathbf{E}_V^{(n^n)}$*.*

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The Connes cyclic Hochschild cohomology theory for algebras involving derivations

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Abstract

This is nothing but a reviewing study based on the cyclic Hochschild cohomology part of Noncommutative Geometry invented by Connes, as the basic theory.

Cohomology, cyclic cohomology, noncommutative geometry
46L80, 46L87, 46L85

1 Introduction

Following Connes [13], with minor modification only, we would like to study the basic part of the cyclic Hochschild cohomology theory for algebras with derivations, as cHo-cho as butterfly look like. This is a sort of Yabu-Kogi (paving) gardening or studying in (such) a jungle bush like, to understand the contents to some extent by some considerable effort made.

Original notations are slightly changed by our taste.

It looks similar to the original contents, but not completely the same by our sense.

Our understanding might be shallow, narrow, and pointed like a pencil mightier than an apple.

We would like to figure it out the Connes theory by part by our interest.

Let us go to the symmetric world of cHo-cho.

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2 The cyclic Hochschild cohomology

Let A be an algebra over $\mathbb{C}$ the field of complex numbers. We denote by $L_c^n(A)$ the space of $(n+1)$-linear functionals on A such that

$$L_c^n(A) = \{\varphi : \Pi^{n+1} A \to \mathbb{C} \,|\, \varphi(a_1, \cdots, a_n, a_0) = (-1)^n \varphi(a_0, \cdots, a_n), \quad a_j \in A\}.$$

We consider the complex $(L_c^n(A), b)$ where $b : L_c^n(A) \to L_c^{n+1}(A)$ is the Hochschild coboundary map defined by that for $\varphi \in L_c^n(A)$,

$$\begin{aligned}(b\varphi)(a_0, \cdots, a_{n+1}) = &\sum_{j=0}^{n} (-1)^j \varphi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) \\ &+ (-1)^{n+1} \varphi(a_{n+1} a_0, \cdots, a_n).\end{aligned}$$

Definition 2.1. An (algebraic) cycle of dimension n is defined to be a triple $(\Omega, d, \int)$, where $\Omega = \oplus_{j=0}^n \Omega^j$ is a (differential) graded algebra over $\mathbb{C}$, with d a graded derivation of degree 1 such that $d^2 = 0$, and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace on Ω.

A cycle over an algebra A over $\mathbb{C}$ is defined by a cycle $(\Omega, d, \int)$ with a homomorphism $\rho : A \to \Omega^0$.

A cycle of dimension n over A is essentially determined by its character, which is the $(n+1)$-linear function τ defined by

$$\tau(a_0, \cdots, a_n) = \int \rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n)), \quad a_j \in A.$$

Those functionals are exactly the elements of $\ker(b) \cap L_c^n(A)$, to be proved later.

Given such two cycles $(\Omega, d, \int)$ and $(\Omega', d', \int')$ of the same dimension n, their sum cycle of dimension n is defined to be the direct sum $\Omega \oplus \Omega'$ of differential graded algebras Ω and Ω', with $d \oplus d'$ as a degree 1 graded derivation, and $\int \oplus \int'(\omega, \omega') = \int \omega + \int' \omega'$ for $\omega \in \Omega$ and $\omega' \in \Omega'$ as a graded trace.

Given two cycles $(\Omega, d, \int)$ and $(\Omega', d', \int')$ of dimension n and n' respectively, their tensor cycle of dimension $n+n'$ is defined to be the tensor product $\Omega \otimes \Omega'$ of differential graded algebras Ω and Ω', with $d \otimes d'$ as a degree 1 graded derivation, and $\int \otimes \int' \omega \otimes \omega' = (-1)^{nn'} \int \omega \int' \omega'$ for $\omega \in \Omega$ and $\omega' \in \Omega'$ as a graded trace.

Similarly, direct sums and tensor products of cycles over an algebra are defined.

Example 2.2. Let M be a smooth compact manifold. Let φ be a closed de Rham current of dimension q on M, with $q \leq \dim M = m$. Let $\Omega^j = C^\infty(M, \wedge^j T^* M)$ be the space of smooth differential forms on M of degree j, for $0 \leq j \leq q$. Then $\Omega = \oplus_{j=0}^q \Omega^j$ becomes a differential graded algebra with the usual product and (graded) differentiation, with $\int \omega = \varphi(\omega)$ for $\omega \in \Omega^q$ as a closed graded trace.

$\star$ May refer to [29]. Note that $\Omega^0 = C^\infty(M)$. For $f \in \Omega^0$, we have the derivation $df = \sum_{j=1}^m \frac{\partial f}{\partial x_j} dx_j \in \Omega^1 = C^\infty(M, T^*M)$ (locally). Any element ω of Ω^j has the local form $\sum_K f_K dx_K = \sum_{k_1,\cdots,k_j} f_{k_1,\cdots,k_j} dx_{k_1} \cdots dx_{k_j}$ with $f_{k_1,\cdots,k_j} \in C^\infty(M)$. The wedge product $\omega \wedge \omega'$ of forms $\omega = \sum_K f_K dx_K$ and $\omega' = \sum_{K'} f_{K'} dx_{K'}$ in Ω is the usual product

$$\omega \wedge \omega' = \omega\omega' = \sum_K f_K dx_K \sum_{K'} f_{K'} dx_{K'} = \sum_{K,K'} f_K f_{K'} dx_K dx_{K'},$$

so that $\Omega^j \Omega^k = \Omega^{j+k}$ for $0 \leq j+k \leq m$, and also $\Omega^{j+k} = \{0\}$ for $j+k \geq m+1$. Because $dx_j \wedge dx_j = 0$ and $dx_j \wedge dx_k = -dx_k \wedge dx_j$. It then follows that $\omega \wedge \omega' = (-1)^{jk} \omega' \omega$ for $\omega \in \Omega^j$, $\omega' \in \Omega^k$. For $\omega = \sum_K f_K dx_K \in \Omega^j$, we have the derivation $d\omega = \sum_K df_K dx_K$. For instance, in the case of $m = 2$, we have that for $f = f(x,y) \in \Omega^0$, with $f_x = \frac{\partial f}{\partial x}$,

$$\begin{aligned} \Omega^2 \ni d^2 f &= d(f_x dx + f_y dy) \\ &= (f_{xx} dx + f_{xy} dy) dx + (f_{yx} dx + f_{yy} dy) dy \\ &= f_{xx} dx dx + (-f_{xy} + f_{yx}) dx dy + f_{yy} dy dy = 0. \end{aligned}$$

Thus, $d^2 = 0$ on Ω^0. Also, $d^2\omega = 0$ for $\omega \in \Omega^1$ since $d^2\omega \in \Omega^3 = \{0\}$. Similarly, $d^2 = 0$ on Ω^2. $\square$

$\star$ We may have $q = m$. Or assume that $\Omega^{q+j} = \{0\}$ for $j \geq 1$. $\square$

$\star$ Define $\Omega_j M = \mathrm{Hom}(\Omega^j M, \mathbb{C}) = (\Omega^j M)^*$ with $\Omega^j M = \Omega^j$ the continuous linear dual of the space Ω^j of j-forms on M. Elements of $\Omega_j M$ are said to be de Rham j-currents on M. In particular, elements of $\Omega_0 M = C^\infty(M)^*$ are distributions on M as the usual integrals with the usual trace property. $\square$

$\star$ Note that

$$\int \omega \wedge \omega' = \varphi(\omega\omega') = \varphi((-1)^{\deg\omega \deg\omega'} \omega'\omega) = (-1)^{\deg\omega \deg\omega'} \int \omega' \wedge \omega. \quad \square$$

$\star$ Why do we need to have $dxdx = 0$ and $dxdy = -dydx$? We have the following answer by computing:

$$\begin{aligned} 0 &= d^2 \frac{x^2}{2} = d(xdx) = dxdx, \\ 0 &= d^2(xy) = d(ydx + xdy) = dydx + dxdy! \quad \square \end{aligned}$$

$\star$ Note also as given in [29] that

$$d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d\omega'. \quad \square$$

Example 2.3. Let M be a smooth oriented manifold. Let Γ be a discrete group acting on M by orientation preserving diffeomorphisms ψ_g, where the (right) action is written as $(x, g) \in M \times \Gamma \mapsto xg = \psi_g(x) \in M$.

$\star$ Note that for $g_1, g_2 \in \Gamma$, we have

$$\psi_{g_1 g_2}(x) = x(g_1 g_2) = (xg_1) g_2 = \psi_{g_2}(xg_1) = \psi_{g_2} \psi_{g_1} x.$$

Thus, $\rho_g = \psi_{g^{-1}}$ does make an action since $\rho_{g_1 g_1}(x) = x(g_1g_2)^{-1} = xg_2^{-1}g_1^{-1} = \rho_{g_1}\rho_{g_2}(x)$. $\square$

Denote by $F_c^*(M) = C_c^\infty(M, \wedge^* T_{\mathbb{C}}^*(M))$ the graded differential algebra of smooth differential forms on M with compact support. The group Γ acts on $F_c^*(M)$ by automorphisms, where the (left) action is defined by that for $\omega \in F_c^*(M)$ and $g \in \Gamma$,

$$g\omega = \psi_g^* \omega = \omega \circ \psi_{g^{-1}}.$$

$\star$ Note that for $g_1, g_1 \in \Gamma$, $x \in M$,

$$\begin{aligned} ((g_1g_2)\omega)(x) &= \omega(\psi_{(g_1g_2)^{-1}}(x)) = \omega(\psi_{g_2^{-1}}\psi_{g_1^{-1}}(x)) \\ &= g_2\omega(\psi_{g_1^{-1}}(x)) = g_1(g_2\omega)(x). \quad \square \end{aligned}$$

The algebraic crossed product $F_c^*(M) \rtimes_{\psi^*} \Gamma$ by the action ψ^* is a graded differential algebra Ω^* in the following sense. As a linear space, for $0 \leq p \leq \dim M$, Ω^p is the space $C_c^\infty(M \times \Gamma, \wedge^p T_{\mathbb{C}}^* M)$ of smooth forms with compact support on the disconnected manifold $M \times \Gamma$. Algebraically, such forms are written as finite sums $\sum_{g \in \Gamma} \omega_g u_g$ for $\omega_g \in F_c^*(M)$, where u_g as symbols are automorphisms of $F_c^*(M)$ such that the covariance relation $u_g \omega_k u_{g^{-1}} = \psi_g^* \omega_k$ for $g, k \in \Gamma$ holds. It then follows that

$$\sum_g \omega_g u_g \sum_k \omega'_k u_k = \sum_{g,k} (\omega_g \wedge \psi_g^* \omega'_k) u_{gk}$$

as a product rule. Moreover, the derivation of $F_c^*(M)$ extends to $F_c^*(M) \rtimes_{\psi^*} \Gamma$ by $d(\sum_g \omega_g u_g) = \sum_g (d\omega_g) u_g$. The graded trace on Ω^* is defined by $\int \sum_g \omega_g u_g = \int_M \omega_e$ with e the unit of Γ. It follows from the invariance under diffeomorphisms preserving the orientation of the integral of top-dimensional forms that the triple $(\Omega^*, d, \int)$ defines a cycle of dimension $n = \dim M$ over the crossed product algebra $C_c^\infty(M) \rtimes \Gamma$, with $\Omega_0 = C_c^\infty(M) \rtimes \Gamma$.

$\star$ Check that

$$\begin{aligned} d(\sum_g \omega_g u_g \sum_k \omega'_k u_k) &= \sum_{g,k} d(\omega_g \wedge \psi_g^* \omega'_k) u_{gk} \\ &= \sum_{g,k} d(\omega_g) \wedge \psi_g^* \omega'_k u_{gk} + \sum_{g,k} (-1)^{\deg \omega_g} \omega_g \wedge d(\psi_g^* \omega'_k) u_{gk} \\ &= d(\sum_g \omega_g u_g) \sum_k \omega'_k u_k + \sum_g (-1)^{\deg \omega_g} \omega_g u_g \left(\sum_k d(\psi_g^* \omega'_k) u_k \right) \end{aligned}$$

where it seems in a moment that the second term is not equal to

$$(-1)^{\deg \sum_g \omega_g u_g} (\sum_g \omega_g u_g) d(\sum_k \omega'_k u_k)$$

where $\deg \sum_g \omega_g u_g$ may not be defined if not homogeneous. That equation itself should be a graded derivation rule in this case. Moreover, it does not hold that

$$\sum_k d(\psi_g^* \omega'_k) u_k = \psi_g^* d(\sum_k \omega'_k u_k).$$

Indeed, as ω_k', for a simple form $f\omega$ with $f \in C_c^\infty(M)$ and some form $\omega \in \wedge^* T_{\mathbb{C}}^* M$,

$$d(\psi_g^*(f\omega)) = d((f \circ \psi_{g^{-1}})\omega) = d(f \circ \psi_{g^{-1}})\omega$$

and the chain rule implies that along a local direction,

$$\begin{aligned}\frac{\partial}{\partial x}(f \circ \psi_{g^{-1}}) &= \left(f' \circ \psi_{g^{-1}}\right) \frac{\partial \psi_{g^{-1}}}{\partial x} = \psi_g^* \left(f'\right) \frac{\partial \psi_{g^{-1}}}{\partial x} \\ &= \psi_g^*(f')\psi_g^* \left(\frac{\partial \psi_{g^{-1}}}{\partial x} \circ \psi_g\right) = \psi_g^* \left(f' \left(\frac{\partial \psi_{g^{-1}}}{\partial x} \circ \psi_g\right)\right)\end{aligned}$$

with f' the differential on M. $\square$

Example 2.4. Let Γ be a discrete group and $A = \mathbb{C}\Gamma$ the group ring of Γ over $\mathbb{C}$. Let $\Omega^*(\Gamma)$ be the graded differential algebra of the spaces $\Omega^n(\Gamma) = \Omega^n$ of finite linear combinations of symbols $g_0 dg_1 \cdots dg_n$ for $g_j \in \Gamma$, $0 \leq j \leq n$, and $n \geq 0$, with Ω^0 identified with A, where the product is given by

$$\begin{aligned}&(g_0 dg_1 \cdots dg_n)(g_{n+1} dg_{n+2} \cdots dg_m) = \\ &\sum_{j=1}^{n} (-1)^{n-j} g_0 dg_1 \cdots d(g_j g_{j+1}) \cdots dg_n dg_{n+1} \cdots dg_m + (-1)^n g_0 g_1 dg_2 \cdots dg_m,\end{aligned}$$

so that $\Omega^n \Omega^{m-n-1} = \Omega^{m-1}$, and the derivation $d : \Omega^n \to \Omega^{n+1}$ is defined by $d(g_0 dg_1 \cdots g_n) = dg_0 dg_1 \cdots dg_n$.

$\star$ $d(1) = 0$. Hence it follows that $d^2 = 0$. Because $d^2(g_0 dg_1 \cdots dg_n) = d(dg_0 \cdots dg_n) = d1 dg_0 \cdots dg_n = 0$. $\square$

$\star$ Note that $(g_0 dg_1)(g_2 dg_3) = g_0 d(g_1 g_2) dg_3 - g_0 g_1 dg_2 dg_3$, so that

$$\begin{aligned}&d((g_0 dg_1)(g_2 dg_3)) = dg_0 d(g_1 g_2) dg_3 - d(g_0 g_1) dg_2 dg_3 \\ &= g_2 dg_0 dg_1 dg_3 + g_1 dg_0 dg_2 dg_3 - g_1 dg_0 dg_2 dg_3 - g_0 dg_1 dg_2 dg_3 \\ &= d(g_0 dg_1) g_2 dg_3 + (-1)^{\deg g_0 dg_1} g_0 dg_1 d(g_2 dg_3)\end{aligned}$$

since $d(gg') = (dg)g' + (-1)^{\deg g} g dg' = g' dg + g dg'$. $\square$

Any normalized group cocycle $c \in Z^k(\Gamma, \mathbb{C})$ determines a k-dimensional cycle $(\Omega^*(\Gamma), d, \int)$ (cocycle and cycle corresponded!) with the following closed graded trace $\int$ on $\Omega^*(\Gamma)$ defined that $\int g_0 dg_1 \cdots dg_n = 0$ unless (if not) $n = k$ and $g_0 g_1 \cdots g_n = 1$, and

$$\int g_0 dg_1 \cdots dg_k = c(g_1, \cdots, g_k), \quad \text{if } g_0 \cdots g_k = 1.$$

Recall that the group cohomology $H^*(\Gamma, \mathbb{C})$ is by definition the cohomology of the classifying space $B\Gamma$. Equivalently, $H^*(\Gamma, \mathbb{C})$ is the cohomology of the complex (C^*, b) of the spaces $C^p = C^p(\Gamma)$ of all functions $\gamma : \Gamma^{p+1} \to \mathbb{C}$ such that $\gamma(gg_0, \cdots, gg_p) = \gamma(g_0, \cdots, g_p)$ for any $g, g_j \in \Gamma$, $0 \leq j \leq p$, with

$$(b\gamma)(g_0, \cdots, g_{p+1}) = \sum_{j=0}^{p+1} (-1)^j \gamma(g_0, \cdots, g_{j-1}, g_{j+1}, \cdots, g_{p+1}).$$

$\star$ If $\gamma \in C^0$, then $\gamma : \Gamma \to \mathbb{C}$ with $\gamma(gg_0) = \gamma(g_0)$ for any $g, g_0 \in \Gamma$. It follows that $\gamma(g) = \gamma(g^{-1}g) = \gamma(1)$ for any $g \in \Gamma$, which means that any $\gamma \in C^0$ is the constant function! Therefore, $C^0 \cong \mathbb{C}$. $\square$

$\star$ For $1 \in C^0$ as a constant function on Γ,

$$(b1)(g_0, g_1) = 1(g_1) - 1(g_0) = 1 - 1 = 0.$$

Thus, the map $b = 0$ on C^0. Hence $H^0(\Gamma, \mathbb{C}) = Z^0(\Gamma, \mathbb{C}) \cong \mathbb{C}$. $\square$

$\star$ If $\gamma \in C^1$, then $\gamma(gg_0, gg_1) = \gamma(g_0, g_1)$ for any $g, g_0, g_1 \in \Gamma$. Thus,

$$\gamma(g_0, g_1) = \gamma(g_0^{-1}g_0, g_0^{-1}g_1) = \gamma(1, g_0^{-1}g_1).$$

Therefore, the function $\gamma(g_1, g_2)$ may be identified with the function $\gamma(1, g)$ for $g \in \Gamma$ in this sense. $\square$

$\star$ For $\gamma \in C^1$, we have

$$(b\gamma)(g_0, g_1, g_2) = \gamma(g_1, g_2) - \gamma(g_0, g_2) + \gamma(g_0, g_1). \quad \square$$

$\star$ For $\gamma \in C^2$, we have

$$\begin{aligned}\gamma(g_0, g_1, g_2) &= \gamma(1, g_0^{-1}g_1, g_0^{-1}g_2) \quad (h_1 = g_0^{-1}g_1) \\ &= \gamma(1, h_1, h_1 g_1^{-1}g_2) = \gamma(1, h_1, h_1h_2) \quad (h_2 = g_1^{-1}g_2). \quad \square\end{aligned}$$

The group cocycle associated to $\gamma \in C^k$ with $b\gamma = 0$ is given by

$$c(g_1, \cdots, g_k) = \gamma(1, g_1, g_1g_2, \cdots, g_1g_2 \cdots g_k).$$

The normalization required above is that $c = 0$ if any $g_j = 1$ or if $g_1 \cdots g_k = 1$.

Any group cocycle can be normalized without changing its cohomology class. Because the above complex can be replaced, without altering its cohomology, by the subcomplex of skew-symmetric cochains, such that

$$\gamma^\sigma(g_0, \cdots, g_n) = \gamma(g_{\sigma(0)}, \cdots, g_{\sigma(p)}) = \mathrm{sign}(\sigma)\gamma(g_0, \cdots, g_p)$$

for $g_j \in \Gamma$ and $\sigma \in \mathfrak{S}_{p+1}$.

The differential algebra $\Omega^*(\Gamma)$ is independent of the choice of the cocycle c for $\int$.

The construction of $\Omega^*(\Gamma)$ starting from the group ring $A = \mathbb{C}\Gamma$ is a special case of the universal differential algebra $\Omega^*(A)$ associated to an algebra A (cf. [1], [23]). Briefly recall it as in the following.

Proposition 2.5. *Let A be an algebra over $\mathbb{C}$, not necessarily unital. Let $A^\sim = A \oplus \mathbb{C}1$ be the unital algebra obtained by adjoining a unit 1 to A. Let $\Omega^1(A) = A^\sim \otimes_{\mathbb{C}} A$ as a linear space. Then an A-bimodule structure on $\Omega^1(A)$ is defined by $x((a + \lambda 1) \otimes b) = (xa + \lambda x) \otimes b$, $((a + \lambda 1) \otimes b)y = (a + \lambda 1) \otimes by$, and*

$$x((a + \lambda 1) \otimes b)y = (xa + \lambda x) \otimes by - (xab + \lambda xb) \otimes y$$

for $a, b, x, y \in A$ and $\lambda \in \mathbb{C}$, and a derivation $d : A \to \Omega^1(A)$ is defined by $da = 1 \otimes a \in \Omega^1(A)$ for $a \in A$.

Let E be an A-bimodule and $\delta : A \to E$ a derivation. Then there exists a bimodule morphism $\rho : \Omega^1(A) \to E$ such that $\delta = \rho \circ d$, so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, equal] & \Omega^1(A) \arrow[d, "\rho"] \\ A \arrow[r, "\delta"] & E \end{tikzcd}$$

in the sense that $(\Omega^1(A), d)$ is the universal A-bimodule involving a derivation.

$\star$ Check that

$$\begin{aligned} &x_1(x_2((a + \lambda 1) \otimes b)y_1)y_2 = x_1((x_2a + \lambda x_2) \otimes by_1)y_2 - x_1((x_2ab + \lambda x_2b) \otimes y_1)y_2 \\ &= (x_1x_2a + \lambda x_1x_2) \otimes by_1y_2 - (x_1x_2a + \lambda x_1x_2)by_1 \otimes y_2 \\ &\quad - (x_1x_2ab + \lambda x_1x_2b) \otimes y_1y_2 + (x_1x_2ab + \lambda x_1x_2b)y_1 \otimes y_2 \\ &= (x_1x_2)((a + \lambda 1) \otimes b)(y_1y_2)! \quad \square \end{aligned}$$

$\star$ Note that

$$\begin{aligned} &d(a_1)a_2 + a_1d(a_2) = (1 \otimes a_1)a_2 + a_1(1 \otimes a_2) \\ &= 1 \otimes (a_1a_2) - (1a_1) \otimes a_2 + a_1 \otimes a_2 = 1 \otimes a_1a_2 = d(a_1a_2). \quad \square \end{aligned}$$

Let $\Omega^n(A) = \otimes_A^n \Omega^1(A)$ be the n-fold tensor product of the bimodule $\Omega^1(A) = A^\sim \otimes_{\mathbb{C}} A$. The universal graded differential algebra of A is defined to be $\Omega^*(A) = \oplus_{n=0}^\infty \Omega^n(A)$, with a square-zero graded derivation $d : \Omega^n(A) \to \Omega^{n+1}(A)$, which is extended uniquely from the differential $d : A \to \Omega^1(A)$, where $A = \Omega^0(A)$ may be assumed.

$\star$ Note that $d^2(a) = d(1 \otimes a) = d(1) \otimes d(a) = 0$ for $a \in A$, with $d(1) = 0$ assumed. $\square$

$\star$ For $(a_0 + \lambda 1) \otimes a_1 = a_0(1 \otimes a_1) + \lambda(1 \otimes a_1) \in \Omega^1(A)$, we have

$$d((a_0 + \lambda 1) \otimes a_1) = da_0 \otimes da_1 = da_0da_1 = d(a_0da_1). \quad \square$$

Remark that there is a natural linear space isomorphism J from $A^\sim \otimes (\otimes^n A)$ to $\Omega^n(A)$ defined by

$$J((a_0 + \lambda 1) \otimes a_1 \cdots \otimes a_n) = a_0da_1 \cdots da_n + \lambda da_1da_2 \cdots da_n$$

for $a_0, \cdots, a_n \in A$, $\lambda \in \mathbb{C}$.

Note that the cohomology $H\Omega^*(A)$ of the complex $(\Omega^*(A), d)$ in all dimensions are zero, including $H\Omega^0(A) = 0$ if we set $\Omega^0(A) = A$.

$\star$ Note that $d : A \to \Omega^1(A)$ is injective. Thus the kernel $\ker(d)$ is zero. Hence $H\Omega^0(A) = \ker(d) = 0$. The image $d(A) = 1 \otimes A$. Also, the kernel of $d : \Omega^1(A) \to \Omega^2(A)$ is $1 \otimes A$. Hence $H\Omega^1(A) = 0$. As well, the image

$d(\Omega^1(A)) = (1 \otimes_{\mathbb{C}} A) \otimes_A (1 \otimes_{\mathbb{C}} A)$, which is the kernel of d on $\Omega^2(A)$, so that $H\Omega^2(A) = 0$. $\square$

The product in $\Omega^*(A)$ given in a way analogus to that in $\Omega^*(\Gamma)$ is defined by

$$\begin{aligned}&(a_0da_1\cdots da_n)(a_{n+1}da_{n+2}\cdots da_m)\\&=\sum_{j=1}^{n}(-1)^{n-j}a_0da_1\cdots d(a_ja_{j+1})\cdots da_nda_{n+1}\cdots da_m+(-1)^na_0a_1da_2\cdots da_m.\end{aligned}$$

$\star$ Note that $(a_0da_1)(a_2da_3) = a_0d(a_1a_2)da_3 - a_0a_1da_2da_3$. Thus,

$$\begin{aligned}d((a_0da_1)(a_2da_3)) &= da_0d(a_1a_2)da_3 - d(a_0a_1)da_2da_3\\&= a_2da_0da_1da_3 + a_1da_0da_2da_3 - a_1da_0da_2da_3 - a_0da_1da_2da_3\\&= d(a_0da_1)a_2da_3 + (-1)^{\deg a_0da_1}a_0da_1d(a_2da_3). \quad \square\end{aligned}$$

Proposition 2.6. *Let $\tau : A^{n+1} \to \mathbb{C}$ be an $(n+1)$-dimensional functional on an algebra A over $\mathbb{C}$. The the following conditions are equivalent:*

(1) *There is an n-dimensional cycle $(\Omega, d, \int)$ and a homomorphism $\rho : A \to \Omega^0$, namely an n-cycle over A, such that*

$$\tau(a_0,\cdots,a_n) = \int \rho(a_0)d(\rho(a_1))\cdots d(\rho(a_n)), \quad a_0,\cdots,a_n \in A.$$

(2) *There exists a closed graded trace* tr *of dimension n on $\Omega^*(A)$ such that*

$$\tau(a_0,\cdots,a_n) = \mathrm{tr}(a_0da_1\cdots da_n), \quad a_0,\cdots,a_n \in A.$$

(3) *It holds that $\tau(a_1,\cdots,a_n,a_0) = (-1)^n\tau(a_0,\cdots,a_n)$ and*

$$\sum_{j=0}^{n}(-1)^j\tau(a_0,\cdots,a_ja_{j+1},\cdots,a_{n+1}) + (-1)^{n+1}\tau(a_{n+1}a_0,\cdots,a_n) = 0.$$

Proof. It follows from the universality of $\Omega^*(A)$ that (1) and (2) are equivalent.

$\star$ Note that there is the morphism ρ' by universality, for which the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega^1(A) \\ \| & & \downarrow \rho' \\ A & \xrightarrow{\rho} & \Omega^1 \end{array}$$

so that for the cycle $(\Omega, d, \int)$ associated to τ, the trace is defined by

$$\mathrm{tr}(a_0da_1\cdots da_n) = \int \rho'(da_0)d(\rho'(da_1))\cdots d(\rho'(da_n)),$$

which shows that (1) $\Rightarrow$ (2). Conversely, the triple $(\oplus_{j=0}^n\Omega^j(A), d, \tau)$ with τ as a closed graded trace of dimension n is an n-dimensional cycle over A. $\square$

Next show that (3) $\Rightarrow$ (2). Given any $(n+1)$-linear functional φ on A, define a linear functional $\varphi^\wedge$ on $\Omega^n(A)$ by

$$\begin{aligned}&(\varphi^\wedge \circ j)((a_0 + \lambda_0 1) \otimes a_1 \otimes \cdots \otimes a_n) = \varphi(a_0, a_1, \cdots, a_n)\\ &= \varphi^\wedge(a_0 da_1 \cdots da_n + \lambda_0 da_1 \cdots da_n)\end{aligned}$$

for any $\lambda_0 \in \mathbb{C}$.

$\star$ This is defined mod $\ker(d) = Z^n(A) \subset \Omega^n(A)$. Indeed, if $a_0 = 0$, then $\varphi^\wedge(\lambda_0 da_1 \cdots da_n) = \varphi(0, a_1, \cdots, a_n) = 0$ by linearity. $\square$

By construction, we have $\varphi^\wedge(d\omega) = 0$ for any $\omega \in \Omega^{n-1}(A)$, which means the closedness of $\varphi^\wedge$.

$\star$ Note that $d\omega \in B^n(A) \subset Z^n(A)$. $\square$

For τ satisfying (3), we show that $\tau^\wedge$ is a graded trace in the following sense that

$$\begin{aligned}&\tau^\wedge((a_0 da_1 \cdots da_k)(a_{k+1} da_{k+2} \cdots da_{n+1}))\\ &= (-1)^{k(n-k)} \tau^\wedge((a_{k+1} da_{k+2} \cdots da_{n+1})(a_0 da_1 \cdots da_k)).\end{aligned}$$

By the product rule in $\Omega^*(A)$, the left-hand side is equal to

$$\sum_{j=0}^{k} (-1)^{k-j} \tau(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1})$$

(with j from 0!), and the right-hand side is equal to

$$\sum_{j=0}^{n-k} (-1)^{k(n-k)+n-k-j} \tau(a_{k+1}, \cdots, a_{k+1+j} a_{k+1+j+1}, \cdots, a_0, a_1, \cdots, a_k),$$

where we let $a_{n+2} = a_0$ at the $(n-k)$-term. The cyclic permutation σ of $\sigma(l) = k+1+l$ has signature $\varepsilon(\sigma)$ equal to $(-1)^{n(k+1)}$, so that

$$\tau^\sigma(a_0, \cdots, a_n) = \tau(a_{\sigma(0)}, \cdots, a_{\sigma(n)}) = \varepsilon(\sigma)\tau(a_0, \cdots, a_n)$$

by cyclic hypothesis. Thus, the right-hand side is changed to

$$- \sum_{j=k+1}^{n} (-1)^{k-j} \tau(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{k-n} \tau(a_{n+1} a_0, a_1, \cdots, a_n)$$

with $k+1+j$ replaced with j' from $k+1$ to n for j from 0 to $n-k-1$, so that

$$\begin{aligned}&(-1)^{k(n-k)+n-k-(j'-k-1)}(-1)^{n(k+1)} = (-1)^{(k+1)(n-k)-j'+(k+1)+(k+1)n}\\ &= (-1)^{(k+1)(2n-k+1)-j'} = (-1)^{1-k^2-j'} = (-1)(-1)^{k-j'},\end{aligned}$$

since $(-1)^{\pm k^2} = (-1)^k$ because $k^2 = k \bmod 2$, and

$$(-1)^{k(n-k)+n-k-(n-k)}(-1)^{n(k+1)} = (-1)^{2nk+n-k^2} = (-1)^{k-n}$$

for the last term. Therefore, the above equality as a graded trace follows from the second sum equality hypothesis on τ, converted to the both-hand sides.

Show that (1) $\Rightarrow$ (3). We may assume that $A = \Omega^0$, so that ρ is the identity map. Then

$$\tau(a_0, a_1, \cdots, a_n) = \int (a_0 da_1) da_2 \cdots da_n = (-1)^{n-1} \int da_2 \cdots da_n (a_0 da_1)$$
$$= (-1)^n \int da_2 \cdots da_n (da_0) a_1 = (-1)^n \tau(a_1, a_2, \cdots a_n, a_0).$$

$\star$ Note that $a_0 da_1 = a_0(1 \otimes a_1) = a_0 \otimes a_1$ and $(da_0)a_1 = (1 \otimes a_0)a_1 = 1 \otimes a_0 a_1 - a_0 \otimes a_1$. Thus, $a_0 da_1 = -(da_0)a_1 + 1 \otimes a_0 a_1 = -(da_0)a_1 + d(a_0 a_1)$. It does hold $\int da_2 \cdots da_n d(a_0 a_1) = 0$ by closedness. $\square$

To prove the second equality in (3), we use the equality that $\int a\omega = \int \omega a$ for $\omega \in \Omega^n$ and $a \in A$.

$\star$ We have $(da_1)a_2 = d(a_1 a_2) - a_1 da_2$. Also,

$$\begin{aligned} & d(a_1)d(a_2)a_3 = d(a_1)d(a_2 a_3) - d(a_1)a_2 d(a_3) \\ & = d(a_1)d(a_2 a_3) - d(a_1 a_2)d(a_3) + a_1 d(a_2)d(a_3) \\ & = (-1)d(a_1 a_2)d(a_3) + (-1)^2 d(a_1)d(a_2 a_3) + (-1)^2 a_1 d(a_2)d(a_3). \quad \square \end{aligned}$$

It then follows as a general case that

$$\begin{aligned} & (da_1 \cdots da_n)a_{n+1} = \\ & \sum_{j=1}^{n} (-1)^{n-j} da_1 \cdots d(a_j a_{j+1}) \cdots da_{n+1} + (-1)^n a_1 da_2 \cdots da_{n+1}. \end{aligned}$$

Thus, the second equality in (3) follows from the above equality for $a_0(da_1 \cdots da_n)a_{n+1}$ as integrated with

$$\int a_0(da_1 \cdots da_n)a_{n+1} = \int (a_0 da_1 \cdots da_n)a_{n+1} = \int a_{n+1} a_0 da_1 \cdots da_n$$

and multipliying $(-1)^n$ on both sides. (The end of the proof.) $\square$

Recall now the definition of Hochschild cohomology groups $H^n(A, M)$ of an algebra A over $\mathbb{C}$ with coefficients in a bimodule M (cf. [8]). Let $A \otimes A^{\odot}$ be the tensor product of A with the opposite algebra $A^{\odot}$ of A with $\odot$ as product of $A^{\odot}$. Any bimodule M over A becomes a left $A \otimes A^{\odot}$-module.

$\star$ For $a \otimes b \in A \otimes A^{\odot}$, and $m \in M$, define $(a \otimes b)m = amb$. Then

$$(a_1 \otimes b_1)(a_2 \otimes b_2)m = (a_1 \otimes b_1)(a_2 m b_2) = a_1 a_2 m b_2 b_1 = (a_1 a_2 \otimes (b_1 \odot b_2))m.$$

$\square$

By definition, $H^n(A, M) = \mathrm{Ext}^n_{A \otimes A^{\odot}}(A, M)$ (What's this?) where A is viewed as a bimodule over A so that $a(m)b = amb$ for any $a, m, b \in A$.

Reformulate the definition of $H^n(A, M)$ using the standard resolution of the bimodule A (cf. [8]).

Define the complex $(C^n(A, M), b)$ as follows.

(a) Let $C^n(A, M)$ be the space of n-linear maps from $\oplus^n A$ to M.

(b) The boundary $bT \in C^{n+1}(A, M)$ for $T \in C^n(A, M)$ is defined by

$$\begin{aligned}&(bT)(a_1, \cdots, a_{n+1}) = a_1 T(a_2, \cdots, a_{n+1}) \\ &\quad + \sum_{j=1}^{n} (-1)^j T(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} T(a_1, \cdots, a_n) a_{n+1}.\end{aligned}$$

Definition 2.7. The Hochschild cohomology of A with coefficients in M is defined to be the cohomology $H^n(A, M)$ of the complex $(C^n(A, M), b)$.

$\star$ Note that $C^0(A, M) = M$. The boundary $bm \in C^1(A, M)$ for $m \in M$ is defined as that $(bm)(a) = ma - am$ for $a \in A$. Check that

$$\begin{aligned}&(b^2 m)(a_1, a_2) = a_1(bm)(a_2) - (bm)(a_1 a_2) + (bm)(a_1)a_2 \\ &= a_1(ma_2 - a_2 m) - (m(a_1 a_2) - (a_1 a_2)m) + (ma_1 - a_1 m)a_2 = 0.\end{aligned}$$

Therefore, $b^2 = 0$ on M. Moreover, $b^2 = 0$ on $C^n(A, M)$. □

$\star$ Since the boundary image $b(C^{n-1}(A, M)) = B^n(A, M) \subset C^n(A, M)$ is contained in the b-kernel $Z^n(A, M) \subset C^n(A, M)$, then $H^n(A, M)$ is defined to be the quotient $Z^n(A, M)/B^n(A, M)$. □

The dual space A^* of all linear functionals $\varphi : A \to \mathbb{C}$ is a bimodule over A in the sense that $(a\varphi b)(c) = \varphi(bca)$, for $a, b, c \in A$.

$\star$ Check that

$$(a_1(a_2\varphi b_1)b_2)(c) = (a_2\varphi b_1)(b_2 c a_1) = \varphi(b_1(b_2 c a_1)a_2) = ((a_1 a_2)\varphi(b_1 b_2))(c). \quad \square$$

Any n-cochain $T \in C^n(A, A^*)$ is considered as an $(n+1)$-linear functional $\tau : \oplus^{n+1} A \to \mathbb{C}$ by the following equality:

$$\tau(a_0, a_1, \cdots, a_n) = T(a_1, \cdots, a_n)(a_0).$$

The boundary $bT \in C^{n+1}(A, A^*)$ corresponds to the $(n+2)$-linear functional $b\tau : \oplus^{n+2} A \to \mathbb{C}$:

$$\begin{aligned}&(b\tau)(a_0, a_1, \cdots, a_{n+1}) = \\ &(bT)(a_1, \cdots, a_{n+1})(a_0) = a_1 T(a_2, \cdots, a_{n+1})(a_0) \\ &+ \sum_{j=1}^{n} (-1)^j T(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+1})(a_0) + (-1)^{n+1} T(a_1, \cdots, a_n)(a_0) a_{n+1} \\ &= \tau(a_0 a_1, a_2, \cdots, a_{n+1}) \\ &+ \sum_{j=1}^{n} (-1)^j \tau(a_0, a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} \tau(a_{n+1} a_0, a_1, \cdots, a_n).\end{aligned}$$

It then follows that the cyclic conditions of $\tau : \oplus^{n+1} A \to \mathbb{C}$ by the two equalities becomes that (a) $\tau^\gamma = \varepsilon(\gamma)\tau$ for any cyclic permutation γ of $\{0, 1, \cdots, n\}$ and (b) $b\tau = 0$.

The Hochshild coboundary b does not commute with cyclic permutations.

$\star$ For $T \in C^1(A, M)$, we have $T^\gamma = T$ with γ trivial, and for $\gamma' = (2, 1)$,

$$(bT)^{\gamma'}(a_1, a_2) = (bT)(a_2, a_1) = a_2 T(a_1) - T(a_2 a_1) + T(a_2)a_1,$$
$$(b(T^\gamma))(a_1, a_2) = a_1 T(a_2) - T(a_1 a_2) + T(a_1)a_2,$$

both of which look different obviously in general. $\square$

The Hochshild coboundary maps cochains satisfying (a) to cochains satisfying the same.

$\star$ For $T \in C^1(A, A^*)$, assume that for $\gamma = (1, 0)$ with $\varepsilon(\gamma) = -1$,

$$-\tau(a_0, a_1) = \tau^\gamma(a_0, a_1) = \tau(a_1, a_0) = T(a_0)(a_1).$$

Then, for $\gamma' = (1, 2, 0)$ with $\varepsilon(\gamma') = 1$,

$$\begin{aligned}
&(b\tau)^{\gamma'}(a_0, a_1, a_2) = (b\tau)(a_1, a_2, a_0) \\
&= \tau(a_1 a_2, a_0) - \tau(a_1, a_2 a_0) + \tau(a_0 a_1, a_2) \\
&= -\tau(a_0, a_1 a_2) + \tau(a_2 a_0, a_1) + \tau(a_0 a_1, a_2) = (b\tau)(a_0, a_1, a_2). (!) \quad \square
\end{aligned}$$

Define the linear map $P_c : C^n(A, A^*) \to C^n(A, A^*)$ by $P_c\varphi = \sum_{\gamma \in \mathfrak{P}_{c,n+1}} \varepsilon(\gamma)\varphi^\gamma$, where $\mathfrak{P}_{c,n+1}$ is the group of cyclic permutations of $\{0, 1, \cdots, n\}$.

The range $P_c(C^n(A, A^*))$ is the subspace $C^n_\lambda(A) = P^n_c(A)$ of $C^n(A, A^*)$ of cochains satisfying (a), denoted so by us.

$\star$ Note that

$$P_c(\varphi)^{\gamma'} = \sum_{\gamma \in \mathfrak{P}_{c,n+1}} \varepsilon(\gamma)(\varphi^\gamma)^{\gamma'} = \varepsilon(\gamma') \sum_{\gamma \in \mathfrak{P}_{c,n+1}} \varepsilon(\gamma'\gamma)\varphi^{\gamma'\gamma} = \varepsilon(\gamma')P_c(\varphi).$$

Also, if $\tau^\gamma = \varepsilon(\gamma)\tau$ for any cyclic permutations γ, then

$$P_c(\tau) = \sum_{\gamma \in \mathfrak{P}_{c,n+1}} \varepsilon(\gamma)\tau^\gamma = \left(\sum_{\gamma \in \mathfrak{P}_{c,n+1}} 1 \right) \tau. \quad \square$$

Lemma 2.8. *Define the operator $b' : C^n(A, A^*) \to C^{n+1}(A, A^*)$ by*

$$(b'\varphi)(x_0, \cdots, x_{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(x_0, \cdots, x_j x_{j+1}, \cdots, x_{n+1}),$$

so that the following diagram commutes

$$\begin{array}{ccc}
C^n(A, A^*) & \xrightarrow{b'} & C^{n+1}(A, A^*) \\
{\scriptstyle P_c}\downarrow & & \downarrow{\scriptstyle P_c} \\
C^n(A, A^*) & \xrightarrow{b} & C^{n+1}(A, A^*).
\end{array}$$

Proof. We have

$$((P_c \circ b')\varphi)(x_0, \cdots, x_{n+1}) =$$
$$\sum_{k=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+(n+1)k}\varphi(x_k, \cdots, x_{k+i}x_{k+i+1}, \cdots, x_{x-1})$$

with sub-indices mod $n+2 = 0$ as convention. In particular, $k+i$ is converted to j (mod n+2). Also,

$$((b \circ P_c)\varphi)(x_0, \cdots, x_{n+1}) =$$
$$\sum_{j=0}^{n}(-1)^j(P_c\varphi)(x_0, \cdots, x_jx_{j+1}, \cdots, x_{n+1}) + (-1)^{n+1}(P_c\varphi)(x_{n+1}x_0, \cdots, x_n).$$

For $0 \leq j \leq n$,

$$(P_c\varphi)(x_0, \cdots, x_jx_{j+1}, \cdots, x_{n+1}) = \sum_{k=0}^{j}(-1)^{nk}\varphi(x_k, \cdots, x_jx_{j+1}, \cdots, x_{k-1})$$
$$+ \sum_{k=j+2}^{n+1}(-1)^{n(k-1)}\varphi(x_k, \cdots, x_{n+1}, x_0, \cdots, x_jx_{j+1}, \cdots, x_{k-1}).$$

As well,

$$(P_c\varphi)(x_{n+1}x_0, \cdots, x_n) = \varphi(x_{n+1}x_0, \cdots, x_n)$$
$$+ \sum_{j=1}^{n}(-1)^{jn}\varphi(x_j, \cdots, x_n, x_{n+1}x_0, \cdots, x_{j-1}).$$

We need to check equal the signs as coefficients of corresponding terms such as $\varphi(x_k, \cdots, x_jx_{j+1}, \cdots, x_{k-1})$ of both sides of $P_cb'\varphi$ and $bP_c\varphi$. □

⋆ For $\tau \in C^1(A, A^*)$,

$$(P_cb'\tau)(x_0, x_1, x_2) = P_c(\tau(x_0x_1, x_2)) - P_c(\tau(x_0, x_1x_2)) =$$
$$\tau(x_0x_1, x_2) + \tau(x_1x_2, x_0) + \tau(x_2x_0, x_1)$$
$$- \tau(x_0, x_1x_2) - \tau(x_1, x_2x_0) - \tau(x_2, x_0x_1).$$

Also,

$$(bP_c\tau)(x_0, x_1, x_2) = (P_c\tau)(x_0x_1, x_2) - (P_c\tau)(x_0, x_1x_2) + (P_c\tau)(x_2x_0, x_1) =$$
$$= \tau(x_0x_1, x_2) - \tau(x_2, x_0x_1) - \tau(x_0, x_1x_2) + \tau(x_1x_2, x_0)$$
$$+ \tau(x_2x_0, x_1) - \tau(x_1, x_2x_0).$$

Both of which shows that $P_c \circ b'$ is just equal to $b \circ P_c$ on $C^1(A, A^*)$. □

Corollary 2.9. *$(P_c^n(A), b)$ becomes a subcomplex of the Hochschild complex $(C^n(A, A^*), b)$.*

Proof. The above lemma implies that $P_c^n(A) = P_c(C^n(A, A^*))$ is mapped into $P_c^{n+1}(A) = P_c(C^{n+1}(A, A^*))$ under the boundary map b. $\square$

We may denote by $cH^n(A)$ the n-th cohomology group of the complex $(P_c^n(A), b)$, which is called the n-th cycic cohomology group of an algebra A.

In particular, $cH^0(A) = P_c(Z^0(A, A^*))$ is exactly the linear space of traces on A.

$\star$ Note that $C^0(A, A^*) = A^*$. For $\varphi \in A^*$, we have $(b\varphi)(a) = \varphi a - a\varphi$ for $a \in A$. If $b\varphi = 0$, then $(\varphi a)(c) = \varphi(ac)$ is equal to $(a\varphi)(c) = \varphi(ca)$ for any $a, c \in A$, which means that φ is a trace on A. It then follows that

$$Z^0(A, A^*) = \ker(b \text{ on } A^*) = \mathrm{Tr}(A)$$

which is the space of traces of A. Hence $P_c(Z^0(A, A^*)) = cZ^0(A) = \mathrm{Tr}(A)$. $\square$

Example 2.10. Let $A = \mathbb{C}$. Then for $n \geq 0$, $cH^{2n+1}(\mathbb{C}) = 0$, but $cH^{2n}(\mathbb{C}) = \mathbb{C}$.

$\star$ Indeed, $\mathbb{C}^* \cong \mathbb{C}$ since any $\varphi \in \mathbb{C}^*$ is identified with the multiplication operator by a $p \in \mathbb{C}$, so that $\varphi(z) = pz$ for $z \in \mathbb{C}$. Certainly, any $\varphi \in \mathbb{C}^*$ is a trace on $\mathbb{C}$. Therefore, $cH^0(\mathbb{C}) = \mathrm{Tr}(\mathbb{C}) = \mathbb{C}^* \cong \mathbb{C}$.

Moreover, $C^n(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}$ since $\otimes^n\mathbb{C} \cong \mathbb{C}$. Also, the boundary map b_{2n} is the zero map on $\mathbb{C}$, and b_{2n+1} is the isomorphism on $\mathbb{C}$, so that

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "b_{2n-1}=\mathrm{id}"] \arrow[d, "P_c"', "0"] & C^{2n}(\mathbb{C}, \mathbb{C}^*) \arrow[r, "b_{2n}=0"] \arrow[d, "P_c"', "\mathrm{id}"] & C^{2n+1}(\mathbb{C}, \mathbb{C}^*) \arrow[r, "b_{2n+1}=\mathrm{id}"] \arrow[d, "P_c"', "0"] & \mathbb{C} \arrow[d, "P_c"', "\mathrm{id}"] \\
\mathbb{C} \arrow[r, "b_{2n-1}=\mathrm{id}"] & P_c^{2n}(\mathbb{C}) \arrow[r, "b_{2n}=0"] & P_c^{2n+1}(\mathbb{C}) \arrow[r, "b_{2n+1}=\mathrm{id}"] & \mathbb{C}
\end{tikzcd}$$

Because the orders of the groups $\mathfrak{P}_{c,2n}$ and $\mathfrak{P}_{c,2n+1}$ of cyclic permutations of $\{0, 1, \cdots, 2n\}$ and $\{0, 1, \cdots, 2n+1\}$ are odd and even respectively. Therefore,

$$\begin{aligned} cH^{2n}(\mathbb{C}) &= \ker(b_{2n})/\mathrm{im}(b_{2n-1}) = \mathbb{C}/\{0\} \cong \mathbb{C}, \\ cH^{2n+1}(\mathbb{C}) &= \ker(b_{2n+1})/\mathrm{im}(b_{2n}) = \{0\}/\{0\} \cong \{0\}. \quad \square \end{aligned}$$

It is so checked by [29] that $H^0(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}$ and $H^n(\mathbb{C}, \mathbb{C}^*) \cong 0$ for any $n \geq 1$.

It then follows that the subcomplex $(P_c^n(A), b)$ is not a retraction of the complex $(C^n(A, A^*), b)$.

Any homomorphism $\rho : A \to B$ of algebras induces a morphism $\rho^* : P_c^n(B) \to P_c^n(A)$ of complexes defined by

$$(\rho^*\varphi)(a_0, \cdots, a_n) = \varphi(\rho(a_0), \cdots, \rho(a_n)), \quad a_0, \cdots, a_n \in A.$$

As well, the map $\rho^* : cH^n(A) \to cH^n(A)$ is induced.

$\star$ Consider the following diagram:

$$\begin{tikzcd}
P_c^{n-1}(B) \arrow[r, "b_{n-1}"] \arrow[d, "\rho^*"'] & P_c^n(B) \arrow[r, "b_n"] \arrow[d, "\rho^*"'] & P_c^{n+1}(B) \arrow[d, "\rho^*"'] \\
P_c^{n-1}(A) \arrow[r, "b_{n-1}"] & P_c^n(A) \arrow[r, "b_n"] & P_c^{n+1}(A).
\end{tikzcd}$$

This diagram commutes, because for $\tau \in P_c^n(B)$,

$$(\rho^* b_n \tau)(a_0, \cdots, a_{n+1}) = \tau(\rho(a_0 a_1), \rho(a_2), \cdots, \rho(a_{n+1}))+$$
$$\sum_{j=1}^{n}(-1)^j \tau(\rho(a_0), \cdots, \rho(a_j a_{j+1}), \cdots, \rho(a_{n+1})) + (-1)^{n+1}\tau(\rho(a_{n+1}a_0), \cdots, \rho(a_n))$$
$$= \tau(\rho(a_0)\rho(a_1), \rho(a_2), \cdots, \rho(a_{n+1}))+$$
$$\sum_{j=1}^{n}(-1)^j \tau(\rho(a_0), \cdots, \rho(a_j)\rho(a_{j+1}), \cdots, \rho(a_{n+1}))$$
$$+ (-1)^{n+1}\tau(\rho(a_{n+1})\rho(a_0), \cdots, \rho(a_n))$$
$$= b_n(\rho^*\tau)(a_0, \cdots, a_{n+1}).$$

Therefore, $cZ^n(B) = \ker(b_n)$ is mapped into $cZ^n(A)$. As well, $cB^n(B) = \mathrm{im}(b_{n-1})$ is mapped into $cB^n(A)$. Hence any class $[\tau] = \tau + cB^n(B) \in cH^n(B)$ is mapped to $\rho^*[\tau] = \rho^*\tau + cB^n(A) \in cH^n(A)$. □

For an homomorphism $\rho : A \to A$, the induced map $\rho^* : cH^n(A) \to cH^n(A)$ depends only on the class of ρ modulo inner automorphisms of A, as shown below.

Proposition 2.11. *Let A be a unital algebra (which is stable under taking tensor product with $M_2(\mathbb{C})$). Let u be an invertible element of A. Define the inner automorphism $\mathrm{Ad}(u)$ of A by u to be $\mathrm{Ad}(u)(x) = uxu^{-1}$ for $x \in A$. Then the induced map $\mathrm{Ad}(u)^* : cH^*(A) \to cH^*(A)$ is the identity map.*

Proof. Let $t \in A$ and let ${}_t\delta$ be the inner derivation of A by t defined by ${}_t\delta(x) = tx - xt = [t, x]$ for $x \in A$. Given $\varphi \in cZ^n(A)$, a coboundary $\psi \in cB^n(A)$ is defined by

$$\psi(a_0, \cdots, a_n) = \sum_{j=0}^{n} \varphi(a_0, \cdots, {}_t\delta(a_j), \cdots, a_n),$$

as checked in the following. Let $\psi_t(a_0, \cdots, a_{n-1}) = \varphi(a_0, \cdots, a_{n-1}, t)$ so that $\psi_t \in C^{n-1}(A, A^*)$. Compute $bP_c\psi_t$ which is the equal to $P_c b'\psi_t$ so that

$$(b'\psi_t)(a_0, \cdots, a_n) = \sum_{j=0}^{n-1}(-1)^j \varphi(a_0, \cdots, a_j a_{j+1}, \cdots, a_n, t) =$$
$$(b\varphi)(a_0, \cdots, a_n, t) - (-1)^n \varphi(a_0, \cdots, a_{n-1}, a_n t) + (-1)^n \varphi(t a_0, \cdots, a_{n-1}, a_n).$$

Since $b\varphi = 0$,

$$P_c b'\psi_t = (-1)^n \left\{ \sum_{k=0}^{n}(-1)^{nk}\varphi(ta_k, \cdots, a_{k-1}) - \sum_{k=0}^{n}(-1)^{nk}\varphi(a_k, \cdots, a_{n+k}t) \right\}$$
$$= (-1)^n \left\{ \sum_{k=0}^{n}\varphi(a_0, \cdots, a_{k-1}, ta_k, \cdots, a_n) - \sum_{k'=0}^{n}\varphi(a_0 \cdots, a_{k'}t, \cdots, a_n) \right\}$$

with $n+k=k-1=k'$ and $k=k'+1$. Therefore, $P_c b'\psi_t = (-1)^n\psi$, so that $\psi = (-1)^n P_c b'\psi_t$, which is equal to $b((-1)^n P_c\psi_t) \in cB^n(A)$.

To prove that $\varphi \in cZ^n(A)$ and $\varphi \circ \mathrm{Ad}(u)$ for an invertible element u of A are in the same cohomology class, we may replace A by the 2×2 matrix algebra $M_2(A)$ over A, u by the corresponding invertible matrix

$$u^{\sim} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in GL_2(A),$$

and φ by $\varphi^{\sim} \in cZ^n(M_2(A))$ defined by

$$\varphi^{\sim}(a_0\otimes b_0, a_1\otimes b_1, \cdots, a_n\otimes b_n) = \varphi(a_0,\cdots,a_n)\mathrm{tr}(b_0\cdots b_n)$$

for $a_0,\cdots,a_n \in A$ and $b_0,\cdots,b_n \in M_2(\mathbb{C})$. Now we have

$$\begin{aligned} v_1v_2 &= \left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 & -u \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -u \\ u^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = u^{\sim}. \end{aligned}$$

Moreover, $v_j = \exp\frac{\pi}{2}v_j$ for $j=1,2$. The result follows from the above discussion (cf. [24] for a purely algebraic proof). $\square$

$\star$ The assumption of tensor product stability by matrix algebras over $\mathbb{C}$ is satisfied if A is a stable C^*-algebra in the sense that $A \cong A\otimes\mathbb{K}$, where $\mathbb{K}$ is the C^*-algebra of all compact operators on an infinite dimensional Hilbert space. If the cyclic cohomology theory is stable invariant in such a sense, then such a replacement is allowed.

$\star$ Check that

$$\begin{aligned} \exp\begin{pmatrix} 0 & \frac{\pi}{2} \\ -\frac{\pi}{2} & 0 \end{pmatrix} &= \begin{pmatrix} 1-\frac{1}{2}\frac{\pi^2}{2^2}+\cdots & \frac{\pi}{2}-\frac{1}{3!}\frac{\pi^3}{2^3}+\cdots \\ -\frac{\pi}{2}+\frac{1}{3!}\frac{\pi^3}{2^3}-\cdots & 1-\frac{1}{2}\frac{\pi^2}{2^2}+\cdots \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = v_2. \end{aligned}$$

As well,

$$\begin{aligned} \exp\frac{\pi}{2}v_1 &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\exp\begin{pmatrix} 0 & -\frac{\pi}{2} \\ \frac{\pi}{2} & 0 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} = v_1. \end{aligned}$$

$\star$ It then follows that $v = v_1v_2$ (as well as v^{-1}) and the identity matrix of $M_2(A)$ is connected continuously by a continuous path p defined by $p(t) = \exp t\frac{\pi}{2}v_1 \exp t\frac{\pi}{2}v_2 \in GL_2(A)$ for $t\in[0,1]$, as a homotopy. If the cyclic cohomology theory is homotopy invariant, then the cohomology class equivalence desired now follows.

$\star$ $M_2(\mathbb{C})^*$ is identified with $M_2(\mathbb{C})$ since any $\varphi \in M_2(\mathbb{C})^*$ can be defined by $\varphi(x) = \mathrm{tr}(xp)$ for any $x \in M_2(\mathbb{C})$ and some $p \in M_2(\mathbb{C})$.

$\star$ For $\varphi_1, \varphi_2 \in M_2(\mathbb{C})^*$ given, $\varphi_1 \otimes \varphi_2 \in (\otimes^2 M_2(\mathbb{C}))^*$ can be defined by $(\varphi_1 \otimes \varphi_2)(x_1 \otimes x_2) = \varphi_1(x_1)\varphi_2(x_2)$.

Let us now characterize the coboundaries as the cyclic cocycles which extend to cyclic cocycles on arbitrary algebras containing an algebra A. In fact, extendibility to a certain tensor product algebra $C \otimes_{\mathbb{C}} A$ would be enough.

Following Karoubi [20], [21], let us assume that C is the algebra of infinite complex matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with $a_{ij} \in \mathbb{C}$ such that the set of complex number entries a_{ij} is finite, and the number of nonzero a_{ij} per line or column is bounded.

For any (unital) algebra A, the algebra $CA = C \otimes_{\mathbb{C}} A$ is algebraically contractible in the sense that it verifies the hypothesis of the following lemma, so that it has trivial cyclic cohomology.

Lemma 2.12. *Let A be a unital algebra. Assume that there exists a homomorphism $\rho : A \to A$ and an invertible element x of $M_2(A)$ such that*

$$x\alpha(a)x^{-1} = x \begin{pmatrix} a & 0 \\ 0 & \rho(a) \end{pmatrix} x^{-1} = \beta(a) = \begin{pmatrix} 0 & 0 \\ 0 & \rho(a) \end{pmatrix}, \quad a \in A.$$

It then follows that $cH^n(A) = 0$ for all $n \geq 0$.

Proof. Let $\varphi \in cZ^n(A)$ and $\varphi^\sim = \varphi \times \mathrm{tr} \in cZ^n(M_2(A))$. By definition, $\alpha, \beta : A \to M_2(A)$ are homomorphisms. Since α and β are similar by an invertible element of $GL_2(A)$, then $\varphi^\sim \circ \alpha$ and $\varphi^\sim \circ \beta$ are in the same cohomology class. Since

$$\alpha(a) = a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \rho(a) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} &\varphi^\sim(\alpha(a_0), \cdots, \alpha(a_n)) = \varphi^\sim(\alpha(a_0), 0, \cdots, 0) + \cdots + \varphi^\sim(0, \cdots, 0, \alpha(a_n)) \\ &= \varphi(a_0, 0, \cdots, 0)\mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \varphi(\rho(a), 0, \cdots, 0)\mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \cdots \\ &\quad + \varphi(0, \cdots, 0, a_n)\mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \varphi(0, \cdots, 0, \rho(a_n))\mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \varphi(a_0, \cdots, a_n) + \varphi(\rho(a_0), \cdots, \rho(a_n)). \end{aligned}$$

Similarly,

$$\varphi^\sim(\beta(a_0), \cdots, \beta(a_n)) = \varphi(\rho(a_0), \cdots, \rho(a_n)).$$

Therefore, in $cH^n(A)$,

$$[\varphi^\sim \circ \alpha] = [\varphi] + [\varphi \circ \rho] = [\varphi^\sim \circ \beta] = [\varphi \circ \rho].$$

Hence $[\varphi]$ is zero. $\square$

Definition 2.13. We say that a cycle vanishes, or it is a vanishing cycle when the algebra Ω^0 satisfies the assumption of the above lemma.

For an n-dimensional cycle $(\Omega, d, \int)$ and a homomorphism $\rho : A \to \Omega^0$, the $(n+1)$-linear functional τ on A as its character is defined by

$$\tau(a_0, \cdots, a_n) = \int \rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n)).$$

Proposition 2.14. *For $\tau : A^{n+1} \to \mathbb{C}$ an $(n+1)$-linear functional on A,*
(1) *$\tau \in cZ^n(A)$ if and only if τ is the character of a cycle.*
(2) *$\tau \in cB^n(A)$ if and only if τ is the character of a vanishing cycle.*

Proof. (1) It is shown in Proposition above that τ is the character of a cycle if and only if $\tau \in P_c^n(A) = cC^n(A)$ and $b\tau = 0$.

(2) For $(\Omega, d, \int)$ a vanishing cycle, we have $cH^n(\Omega^0) = 0$ for all n. It says that $cZ^n(A) = cB^n(A)$. Therefore, $\tau \in cB^n(A)$ as a coboundary.

Conversely, if $\tau \in cB^n(A)$, then $\tau = b\psi$ for some $\psi \in cC^{n-1}(A)$. Extend τ to $CA = C \otimes A$ as an n-linear functional $\psi^\sim$ on CA so that

$$\psi^\sim(1 \otimes a_0, \cdots, 1 \otimes a_{n-1}) = \psi(a_0, \cdots, a_{n-1}), \quad a_0, \cdots, a_{n-1} \in A$$

and that $(\psi^\sim)^\lambda = \varepsilon(\lambda)\psi^\sim$ for any cyclic permutation λ of $\{0, \cdots, n-1\}$. Let $\rho : A \to CA$ be the homomorphism defined by $\rho(a) = 1 \otimes a$. Then $b(\psi^\sim)$ is an n-cocycle on CA and $\rho^* b(\psi^\sim) = \tau$. Since $b(\psi^\sim) \in cZ^n(CA)$, it is the character of a cycle $(\Omega^*(CA), d, \int)$ with $\Omega^0 = CA$, so vanishing. Therefore,

$$\begin{aligned}\tau(a_0, \cdots, a_n) &= b(\psi^\sim)(\rho(a_0), \cdots, \rho(a_n)) \\ &= \int \rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n)),\end{aligned}$$

Namely, τ is the character of such a vanishing cycle. □

$\star$ Since $\psi^\sim \in cC^{n-1}(CA)$, then $b(\psi^\sim) \in cB^n(CA) \subset cZ^n(CA)$. Since $\rho^* : cC^n(CA) \to cC^n(A)$, then $\rho^* b(\psi^\sim) \in cB^n(A)$. Moreover, $b(\psi^\sim) = b\psi = \tau$,

$$\begin{aligned}\rho^* b(\psi^\sim)(a_0, \cdots, a_n) &= b(\psi^\sim)(\rho(a_0), \cdots, \rho(a_n)) \\ &= b\psi(a_0, \cdots, a_n) = \tau(a_0, \cdots, a_n).\end{aligned}$$

For differential graded algebras $\Omega^*(A)$ and $\Omega^*(B)$ of algebras A and B respectively, the graded tensor product $\Omega^*(A) \otimes \Omega^*(B)$ is defined.

The universal property of $\Omega^*(A \otimes B)$ of the tensor product algebra $A \otimes B$ implies that there is a natural homomorphism $\pi : \Omega^*(A \otimes B) \to \Omega^*(A) \otimes \Omega^*(B)$.

In general, $\Omega^*(A \otimes B)$ is not equal to $\Omega^*(A) \otimes \Omega^*(B)$.

$\star$ By the universality, the following diagram commutes:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{d \otimes d} & \Omega^1(A) \otimes \Omega^1(B) \\ \Vert & & \uparrow \pi \\ A \otimes B & \xrightarrow{d} & \Omega^1(A \otimes B). \end{array}$$

Moreover, $d \otimes d$ to $\Omega^1(A) \otimes \Omega^1(B)$ can be replaced with $d \otimes \mathrm{id}$ to $\Omega^1(A) \otimes B$ and $\mathrm{id} \otimes d$ to $A \otimes \Omega^1(B)$ as well.

For cochains $\varphi \in C^n(A, A^*)$ and $\psi \in C^m(B, \mathfrak{B}^*)$, define the cup product $\#$ of φ and ψ by the associated equality

$$(\varphi\#\psi)^\wedge = (\varphi^\wedge \otimes \psi^\wedge) \circ \pi$$

as a graded trace on $\Omega^*(A \otimes B)$, with $\varphi^\wedge$ and $\psi^\wedge$ on $\Omega^*(A)$ and $\Omega^*(B)$ respectively.

Theorem 2.15. *The cup product defined so above induces a homomorphism from $cH^n(A) \otimes cH^m(B)$ into $cH^{n+m}(A \otimes B)$.*

The character of the tensor product of two cycles is the cup product of the characters of the cycles.

Proof. Let $\varphi \in cZ^n(A)$ and $\psi \in cZ^m(B)$. Let $\varphi^\wedge$ be the closed graded trace on $\Omega^*(A)$ associated to φ, and $\psi^\wedge$ on $\Omega^*(B)$ to ψ. Then $\varphi^\wedge \otimes \psi^\wedge$ is a closed graded trace on $\Omega^*(A) \otimes \Omega^*(B)$, and $(\varphi\#\psi)^\wedge$ on $\Omega^*(A \otimes B)$. Hence $\varphi\#\psi \in cZ^{n+m}(A \otimes B)$.

Next, given cycles $(\Omega, d, \int)$ and $(\Omega', d', \int')$ and homomorphisms $\rho : A \to \Omega$ and $\rho' : B \to \Omega'$, there is a commutative triangle to doubled square:

$$\begin{array}{ccccc}
\Omega^*(A \otimes B) & \xrightarrow{\pi} & \Omega^*(A) \otimes \Omega^*(B) & \xrightarrow{\rho^\sim \otimes (\rho')^\sim} & \Omega \otimes \Omega' \\
\| & & \downarrow{\scriptstyle \rho^\sim \otimes (\rho')^\sim} & & \downarrow{\scriptstyle \int \otimes \int'} \\
\Omega^*(A \otimes B) & \xrightarrow{(\rho\otimes\rho')^\sim} & \Omega \otimes \Omega'` & \xrightarrow{\int^\sim} & \mathbb{C}
\end{array}$$

so that the character $\int^\sim$ of the tensor product cycle $\Omega \otimes \Omega'$ with $\rho \otimes \rho' : A \otimes B \to \Omega \otimes \Omega'$ is given by the cup product of the characters:

$$\int^\sim (\rho \otimes \rho')^\sim = (\int \rho^\sim \otimes \int' (\rho')^\sim)\pi = \int \rho^\sim \# \int' (\rho')^\sim.$$

It remains to show that if $\varphi \in cB^n(A)$ as a coboundary, then $\varphi\#\psi$ is a coboundary as $\varphi\#\psi \in cB^{n+m}(A \otimes B)$. This follows from the above Proposition and the trivial fact that the tensor product of any cycle with a vanishing cycle is vanishing. $\square$

$\star$ Note that as a possible sense,

$$\begin{aligned}
&(\varphi\#\psi)^\wedge((a_0 \otimes b_0)d(a_1 \otimes b_1) \cdots d(a_n \otimes b_n)d(a_{n+1} \otimes b_{n+1}) \cdots d(a_{n+m} \otimes b_{n+m})) \\
&= \varphi^\wedge(a_0 da_1 \cdots d(a_n \cdots a_{n+m}))\psi^\wedge(b_0 d(b_1 \cdots b_{n+1}) \cdots db_{n+m}) + \cdots \\
&= \varphi(a_0, a_1, \cdots, a_n \cdots a_{n+m})\psi(b_0, b_1 \cdots b_{n+1}, \cdots, b_{n+m}) + \cdots \\
&= (\varphi\#\psi)(a_0 \otimes b_0, a_1 \otimes b_1, \cdots, a_n \otimes b_n, a_{n+1} \otimes b_{n+1}, \cdots, a_{n+m} \otimes b_{n+m}).
\end{aligned}$$

$\star$ Note that $M_2(A) \otimes B \cong M_2(\mathbb{C}) \otimes A \otimes B \cong M_2(A \otimes B)$. If there is a homomorphism $\rho : A \to A$ such that $x(\mathrm{id}_A \oplus \rho)x^{-1} = 0 \oplus \rho$ as a diagonal sum for some $x \in GL_2(A)$, then $\rho \otimes \mathrm{id}_B : A \otimes B \to A \otimes B$ such that

$$\begin{aligned}(x(\mathrm{id}_A \oplus \rho)x^{-1}) \otimes \mathrm{id}_B = (0 \oplus \rho) \otimes \mathrm{id}_B \\ = x((\mathrm{id}_A \otimes \mathrm{id}_B) \oplus (\rho \otimes \mathrm{id}_B))x^{-1} = 0 \oplus (\rho \otimes \mathrm{id}_B).\end{aligned}$$

Is this correct?

Corollary 2.16. *The cyclic cohomology $cH^*(\mathbb{C})$ is identified with a polynomial ring with one generator σ of degree 2.*

Each cyclic cohomology $cH^*(A)$ of an algebra A is a (left or right) module over the ring $cH^*(\mathbb{C})$.

Proof. It is checked that $cH^{2n+1}(\mathbb{C}) = 0$ and $cH^{2n}(\mathbb{C}) = \mathbb{C}$ for $n \geq 0$. Let 1 be the unit of $\mathbb{C}$. Any $\varphi \in cZ^n(\mathbb{C})$ is characterized by $\varphi(1, \cdots, 1)$, up to multiplications by complex numbers, so that

$$\varphi(z_0, z_1, \cdots, z_n) = z_0 z_1 \cdots z_n \varphi(1, \cdots, 1), \quad z_0, \cdots, z_n \in \mathbb{C}.$$

For $\varphi \in cZ^{2m}(\mathbb{C})$ and $\psi \in cZ^{2m'}(\mathbb{C})$, we would like to compute the cup product $\varphi \# \psi$. Since $1 = 1^2$ is an idempotent, we have

$$\begin{aligned}d1 = d(1^2) = 1d1 + (d1)1, \\ 1(d1)1 = 1(d1)1 + 1(d1)1, \quad 1(d1)1 = 0,\end{aligned}$$

and $1(d1)^2 = (d1)^2 1$ over $\mathbb{C}$. Similar identities hold for $1 \otimes 1$ and $\pi(1 \otimes 1) \in \Omega^*(\mathbb{C}) \otimes \Omega^*(\mathbb{C})$.

Namely,

$$\begin{aligned}d(1 \otimes 1) = (1 \otimes 1)d(1 \otimes 1) + d(1 \otimes 1)(1 \otimes 1), \\ 0 = (1 \otimes 1)d(1 \otimes 1)(1 \otimes 1).\end{aligned}$$

Then

$$\pi((1 \otimes 1)d(1 \otimes 1)d(1 \otimes 1)) = (1d1d1) \otimes 1 + 1 \otimes (1d1d1) \in \Omega^2 \otimes \Omega^0 + \Omega^0 \otimes \Omega^2$$

with $2 = 2 \cdot 1$ and $\binom{1+1}{1} = \frac{2!}{1!1!} = 2$. Thus,

$$(\varphi \# \psi)(1 \otimes 1, \cdots, 1 \otimes 1) = \frac{(m+m')!}{m!m'!} \varphi(1, \cdots, 1)\psi(1, \cdots, 1).$$

$\star$ We have $\binom{m+m'}{m} = \frac{(m+m')!}{m!m'!}$. That's it! Note that

$$\begin{aligned}(\varphi \# \psi)(1 \otimes 1, \cdots, 1 \otimes 1) = (\varphi \# \psi)^\wedge((1 \otimes 1)d(1 \otimes 1) \cdots d(1 \otimes 1)) \\ = \varphi^\wedge(1d1 \cdots d1)\psi^\wedge(1d1 \cdots d1) + \cdots = \varphi(1, \cdots, 1)\psi(1, \cdots, 1) + \cdots\end{aligned}$$

as a possible computation. In particular, $\varphi \# \varphi$ is identified with φ since $\varphi(1, \cdots, 1) \in \mathbb{C}$. It then follows that $cH^*(\mathbb{C})$ as an algebra by the cup product is isomorphic to the polynomial ring $\mathbb{C} + \mathbb{C}\sigma = \mathbb{C}[\sigma]$ with $\sigma^2 = \sigma$, although $cH^*(\mathbb{C}) = \oplus_{n=0}^{\infty} cH^{2n}(\mathbb{C}) = \oplus_{n=0}^{\infty} \mathbb{C}$ is infinite-dimensional as a vector space over $\mathbb{C}$. $\square$

Now let $\varphi \in cZ^n(A)$. Let σ be the generator of $cH^2(\mathbb{C}) = \mathbb{C}$, where σ is the cyclic 2-cocycle of $\mathbb{C}$ with $\sigma(1,1,1) = 1 \in \mathbb{C}$, identified with its class $[\sigma] \in cH^2(\mathbb{C})$. The cup product implies the module action as the S-map (beyond the night sky) by

$$S : cH^2(\mathbb{C}) \otimes cH^n(A) \xrightarrow{\#} cH^{n+2}(\mathbb{C} \otimes A) \cong cH^{n+2}(A).$$

This formula also holds when the power 2 is replaced with 0.

Check that $\sigma\#\varphi = \varphi\#\sigma$ holds as follows.

$\star$ We check this that

$$\begin{aligned}
&(\varphi\#\sigma)(a_0, \cdots, a_n, a_{n+1}, a_{n+2}) = (\varphi^\wedge \otimes \sigma^\wedge)((a_0 \otimes 1)d(a_1 \otimes 1) \cdots d(a_{n+2} \otimes 1)) \\
&= \varphi^\wedge(a_0 da_1 \cdots d(a_n a_{n+1} a_{n+2}))\sigma^\wedge(1d1d1) + \cdots, \\
&(\sigma\#\varphi)(a_0, \cdots, a_n, a_{n+1}, a_{n+2}) = (\sigma^\wedge \otimes \varphi^\wedge)((1 \otimes a_0)d(1 \otimes a_1) \cdots d(1 \otimes a_{n+2})) \\
&= \sigma^\wedge(1d1d1)\varphi^\wedge(a_0 da_1 \cdots d(a_n a_{n+1} a_{n+2})) + \cdots \quad \square
\end{aligned}$$

For $\varphi \in cZ^n(A)$, define $S\varphi = \sigma\#\varphi = \varphi\#\sigma \in cZ^{n+2}(A)$. Then we have $S(cB^n(A)) \subset sB^{n+2}(A)$. (The end of the proof.) $\square$

We do have a definition of S as a morphism of cochain complexes as follows.

Lemma 2.17. *For any cochain $\varphi \in cC^n(A)$, define $S\varphi \in cC^{n+2}(A)$ by $S\varphi = \frac{1}{n+3}P_c(\sigma\#\varphi)$. Then $S\varphi = \sigma\#\varphi$ for $\varphi \in cZ^n(A)$, so that this map S for cyclic cochains extends the S-map for cyclic cocycles of A.*

Also, $bS\varphi = \frac{n+1}{n+3}Sb\varphi$ for $\varphi \in cC^n(A)$.

Proof. If $\varphi \in cZ^n(A)$, then $(\sigma\#\varphi)^\lambda = \varepsilon(\lambda)(\sigma\#\varphi)$ for any cyclic permutation λ of $\{0, 1, \cdots, n+2\}$.

$\star$ Thus,

$$P_c(\sigma\#\varphi) = \sum_{\gamma \in \mathfrak{P}_{c,n+3}} \varepsilon(\gamma)(\sigma\#\varphi)^\gamma = (n+3)(\sigma\#\varphi).$$

We have

$$(bP_c\varphi)\#\psi = bP_c(\varphi\#\psi).$$

$\star$ If so, it follows that if $\psi = \sigma$, then

$$\begin{aligned}
bS\varphi &= \frac{1}{n+3}bP_c(\varphi\#\sigma) = \frac{1}{n+3}(bP_c\varphi)\#\sigma \\
&= \frac{n+4}{n+3}\frac{1}{n+4}P_c(b'\varphi)\#\sigma = \frac{n+4}{n+3}\frac{1}{n+4}P_c((b'\varphi)\#\sigma) = \frac{n+\mathbf{4}}{n+3}S(b'\varphi).
\end{aligned}$$

There is the one term difference between $b'\varphi$ and $b\varphi$. $\square$

For Hochschild cocycles $\varphi \in Z^n(A, A^*)$ and $\psi \in Z^m(B, B^*)$, we have the Hochschild cocycle $\varphi\#\psi \in Z^{n+m}(A \otimes B, A^* \otimes B^*)$, and the corresponding product $[\varphi\#\psi]$ of cohomology classes is related to the other product $\vee$, so that $[\varphi\#\psi] = \frac{(n+m)!}{n!m!}([\varphi] \vee [\psi])$ (cf. [8]).

Proposition 2.18. *For any cocycle $\varphi \in cZ^n(A)$, the $S\varphi$ is a Hochschild coboundary $b\psi \in B^{n+2}(A)$, where $\psi \in C^{n+1}(A)$ is given by*

$$\psi(a_0, \cdots, a_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j-1} \varphi^\wedge(a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_{n+1})).$$

Proof. It is checked that the coboundary of the j-th term in the sum above is equal to

$$\varphi^\wedge(a_0(da_1 \cdots da_{j-1})a_j a_{j+1}(da_{j+2} \cdots da_{n+2})).$$

□

A chain of dimension $n+1$ means a quadruple $(\Omega, \partial\Omega, d, \int)$ where Ω and $\partial\Omega$ are differential graded algebras of dimensions $n+1$ and n, with a surjective morphism $r : \Omega \to \partial\Omega$ of degree 0, and $\int : \Omega^{n+1} \to \mathbb{C}$ is a graded trace such that $\int d\omega = 0$ for any $\omega \in \Omega^n$ such that $r(\omega) = 0$.

$$\begin{array}{ccccc} \Omega^n & \xrightarrow{d} & \Omega^{n+1} & \xrightarrow{\int} & \mathbb{C} \\ {\scriptstyle r}\downarrow & & & & \uparrow \\ (\partial\Omega)^n & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

The boundary of such a chain means the cycle $(\partial\Omega, d, \int')$, where the following diagram commutes

$$\begin{array}{ccc} \Omega^n & \xrightarrow{r} & (\partial\Omega)^n \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \int'} \\ \Omega^{n+1} & \xrightarrow{\int} & \mathbb{C} \end{array}$$

so that $\int' \omega' = \int d\omega$ for $\omega \in \Omega^n$ with $r(w) = w' \in (\partial\Omega)^n$. The surjectivity of r implies that $\int'$ is a graded trace on $\partial\Omega$, which is closed by construction.

$\star$ If $\omega' \in (\partial\Omega)^{n-1}$, then there is $\omega \in \Omega^{n-1}$ such that $r(\omega) = \omega'$ by surjectivity of r. Then $r(d\omega) = d(\omega')$ since r is a morphism. Hence

$$\int' d(\omega') = \int d(d\omega) = \int 0 = 0,$$

which shows that $\int'$ is closed. Graded traceness of $\int$ implies that of $\int'$ by definition.

Definition 2.19. Let A be an algebra, and let $\rho : A \to \Omega$ and $\rho' : A \to \Omega'$ be two cycles over A. We say that Ω and Ω' are cobordant over A if there exists a chain Ω'' with boundary $\Omega \oplus (\Omega')^\sim$, and a homomorphism $\rho'' : A \to \Omega''$ such that $r \circ \rho'' = (\rho, \rho')$, where $(\Omega')^\sim = \Omega$ with $\int' = -\int$ the sign changed.

⋆ Note that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\rho''"'] & A \arrow[d, "(\rho,\rho')"] \\ \Omega'' \arrow[r, "r"] & \partial\Omega'' = \Omega \oplus (\Omega')^{\sim} \end{tikzcd}$$

In this case, we may write $\Omega \sim_{cob} \Omega'$.

Using a fiber product of algebras, it is checked that the relation of cobordism is transitive. Namely, $\Omega \sim_{cob} \Omega'$ and $\Omega' \sim_{cob} \Omega''$ implies that $\Omega \sim_{cob} \Omega''$.

It is symmetric. Namely, $\Omega \sim_{cob} \Omega'$ implies $\Omega' \sim_{cob} \Omega$.

It is checked that any cycle Ω over A is cobordant to itself. Namely, $\Omega \sim_{cob} \Omega$.

Example 2.20. Let $\Omega^0 = C^\infty([0,1]) = A$. Let $\Omega^1 = C^\infty([0,1], T^*[0,1])$ be the space of smooth 1-forms on $[0,1]$. Let d be the usual differential, so that $df = \frac{df}{dx}dx$ for $f \in A$. Set $\partial\Omega = \mathbb{C} \oplus \mathbb{C}$. Let $\int$ be the usual integral, so that

$$\int df = [f(x)]_{x=0}^1 = f(1) - f(0) \in \mathbb{C}.$$

Let r be the restriction of functions of A to the boundary set $\{0,1\}$, so that $r(f) = (f(0), f(1)) \in \mathbb{C}^2$ for $f \in A$. Then $\Omega = \Omega^0 \oplus \Omega^1$ becomes a chain of dimension with boundary $\partial\Omega = (\mathbb{C} \oplus \mathbb{C}, d, \varphi)$, where $\varphi(a,b) = a - b$.

⋆ Note that the following diagram commutes:

$$\begin{tikzcd} \Omega^0 \arrow[r, "d"] \arrow[d, "r"'] & \Omega^1 \arrow[d, "\int"] \\ \partial\Omega = \mathbb{C} \oplus \mathbb{C} \arrow[r, "\varphi"] & \mathbb{C} \end{tikzcd}$$

so that $\int \circ d = -\varphi \circ r$.

Tensoring a given cycle Ω' over an algebra A' with the above chain $(\Omega, \partial\Omega)$ gives the desired cobordism.

⋆ It says that

$$\begin{tikzcd} A' \arrow[r, equal] \arrow[d, "\rho''"'] & A' \arrow[d, "(\rho',\rho')"] \\ \Omega'' = \Omega \otimes \Omega' \arrow[r, "r"] & \partial\Omega'' = \partial\Omega \otimes \Omega' = \mathbb{C}^2 \otimes \Omega' = \Omega' \oplus \Omega'. \end{tikzcd}$$

Equivalently, smooth functions in the above chain may be replaced by polynomial functions.

It follows that cobordism is an equivalence relation.

Lemma 2.21. *Let τ_1, τ_2 be the characters of two cobordant cycles Ω_1 and Ω_2 over a unital algebra A by a chain Ω with $\rho : A \to \Omega$ and $\int : \Omega^{n+1} \to \mathbb{C}$. Then there exists a Hochschild cocycle $\varphi \in Z^{n+1}(A, A^*)$ such that $\tau_1 - \tau_2 = B_0\varphi$, where*

$$(B_0\varphi)(a_0, \cdots, a_n) = \varphi(1, a_0, \cdots, a_n) - (-1)^{n+1}\varphi(a_0, \cdots, a_n, 1).$$

Proof. For $a_0, \cdots, a_{n+1} \in A$, let

$$\varphi(a_0, \cdots, a_{n+1}) = \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_{n+1}).$$

Let $\omega = \rho(a_0) d\rho(a_1) \cdots \rho(a_n) \in \Omega^n$. By hypothesis we have

$$(\tau_1 - \tau_2)(a_0, a_1, \cdots, a_n) = \int d\omega.$$

Since $\rho(1)\rho(a_0) = \rho(1a_0) = \rho(a_0)$, we have

$$\begin{aligned} d\omega = &(d\rho(1))\rho(a_0) d\rho(a_1) \cdots d\rho(a_n) \\ &+ \rho(1) d\rho(a_0) \cdots d\rho(a_n). \end{aligned}$$

The tracial property of $\int$ implies

$$\int d\omega = (-1)^n \varphi(a_0, a_1, \cdots, a_n, 1) + \varphi(1, a_0, \cdots, a_n).$$

It is checked as checked before that φ defined in that form above is a Hochschild cocycle so that $b\varphi = 0$, using the tracial property of $\int$. □

Lemma 2.22. *Let $\tau_1, \tau_2 \in cZ^n(A)$ such that $\tau_1 - \tau_2 = B_0\varphi$ for some $\varphi \in Z^{n+1}(A, A^*)$. Then any two cycles over A with characters τ_1 and τ_2 are cobordant.*

Proof. Let $\rho : A \to \Omega$ be a cycle Ω over A with character τ. It is shown that the cycle is cobordant with $(\Omega^*(A), \tau^\wedge)$. In the cobordism of Ω with itself above, with restriction maps r_0 and r_1 both to Ω, considered is the subalgebra

$$\{\omega \in \Omega \otimes C^\infty([0,1], \mathbb{C}^2) \,|\, r_1(\omega) \in \Omega'\}$$

where Ω' is the graded differential subalgebra of Ω generated by $\rho(A)$. This defines a cobordism of Ω with Ω'.

Now the homomorphism $\rho^\sim : \Omega^*(A) \to \Omega'$ is surjective, and it follows that $(\rho^\sim)^* \int = \int \circ \rho^\sim = \tau^\wedge$, so that

$$\begin{array}{ccc} \Omega^*(A) & \xrightarrow{\tau^\wedge} & \mathbb{C} \\ \big\downarrow \rho^\sim & & \big\| \\ \Omega \supset \Omega' & \xrightarrow{\int} & \mathbb{C}. \end{array}$$

Modifying the restriction map in the canonical cobordism of $(\Omega^*(A), \tau^\wedge)$ with itself we obtain a cobordism of $(\Omega^*(A), \tau^\wedge)$ with Ω'.

It is shown that $(\Omega^*(A), \tau_1^\wedge)$ and $(\Omega^*(A), \tau_2^\wedge)$ are cobordant. Let μ be the linear functional on $\Omega^{n+1}(A)$ defined by

$$\begin{cases} \mu(a_0 da_1 \cdots da_{n+1}) = \varphi(a_0, \cdots, a_{n+1}), \\ \mu(da_1 \cdots da_{n+1}) = (B_0\varphi)(a_1, \cdots, a_{n+1}). \end{cases}$$

It is checked that μ is a graded trace on $\Omega^*(A)$. By the Hochschild cocycle property of φ, as the tracial character of a cycle, it is obtained that

$$\mu(a(b\omega)) = \mu((b\omega)a), \quad a, b \in A, \omega \in \Omega^{n+1}.$$

It is checked that

$$\mu(a\omega) = \mu(\omega a), \quad \omega = da_1 \cdots da_{n+1}.$$

The right hand side is computed as

$$\begin{aligned}
\mu(\omega a) &= \mu((da_1 \cdots da_{n+1})a) \\
&= \mu(\sum_{j=1}^{n} (-1)^{n+1-j} da_1 \cdots d(a_j da_{j+1}) \cdots da_{n+1} da) \\
&\quad + \mu(da_1 da_2 \cdots da_n d(a_{n+1}a)) + (-1)^{n+1}\mu(a_1 da_2 \cdots da_{n+1} da) \\
&= \sum_{j=1}^{n+1-j} (B_0\varphi)(a_1, \cdots, a_j a_{j+1}, \cdots, a_{n+1}, a) \\
&\quad + (B_0\varphi)(a_1, a_2, \cdots, a_{n+1}a) + (-1)^{n+1}\varphi(a_1, a_2, \cdots, a_{n+1}, a) \\
&= (-1)^n((b' B_0\varphi) - \varphi)(a_1, a_2, \cdots, a_{n+1}, a).
\end{aligned}$$

It is checked that for an arbitrary cochain $\varphi \in C^{n+1}(A, A^*)$,

$$B_0 b\varphi + b' B_0 \varphi = \varphi - (-1)^{n+1}\varphi^\lambda,$$

where λ is the cyclic permutation defined by $\lambda(j) = j - 1$. If φ is a cocycle, then $b\varphi = 0$, and then $b' B_0\varphi - \varphi = (-1)^n \varphi^\lambda$, so that

$$\mu(\omega a) = \varphi^\lambda(a_1, a_2, \cdots, a_{n+1}, a) = \varphi(a, a_1, \cdots a_n) = \mu(a\omega).$$

It remains to check that for any $a \in A$ and $\omega \in \Omega^n$,

$$\mu((da)\omega) = (-1)^n \mu(\omega da).$$

If $\omega \in d\Omega^{n-1}$, then this follows from the fact that $B_0\varphi = \tau_1 - \tau_2 \in cC^n$ with $\mu = B_0\varphi$ on ω, and characters as signed traces. If $\omega = a_0 da_1 \cdots da_n$, then that is a consequence of the cocycle property of $B_0\varphi$. Indeed, we have $bB_0\varphi = 0$. Hence

$$b' B_0\varphi(a_0, a_1, \cdots, a_n, a) = (-1)^n B_0\varphi(aa_0, a_1, \cdots, a_n),$$

and since $b' B_0\varphi = \varphi - (-1)^{n+1}\varphi^\lambda$, then

$$\varphi(a_0, \cdots, a_n, a) - (-1)^{n+1}\varphi(a, a_0, \cdots, a_n) = (-1)^n (B_0\varphi)(aa_0, a_1, \cdots, a_n),$$

and therefore,

$$\begin{aligned}
\mu((da)\omega) &= \mu((da)a_0 da_1 \cdots da_n) \\
&= \mu((d(aa_0) - ada_0)da_1 \cdots da_n) \\
&= \mu(d(aa_0)da_1 \cdots ad_n) - \mu(ada_0 da_1 \cdots da_n) \\
&= (B_0\varphi)(aa_0, a_1, \cdots, a_n) - \varphi(a, a_0, a_1, \cdots, a_n) \\
&= (-1)^n \varphi(a_0, \cdots, a_n, a) = (-1)^n \mu(a_0 da_1 \cdots da_n da) = (-1)^n \mu(\omega da).
\end{aligned}$$

To end the proof of this lemma, modified is the natural cobordism between $(\Omega^*(A), \tau_1^\wedge)$ and itself, given by the tensor product of $\Omega^*(A)$ with the algebra of differential forms on $[0,1]$, by adding to the integral the term $\mu \circ r_1$, with r_1 the restriction map to the point $\{1\}$ in $[0,1]$. $\square$

Corollary 2.23. *Two cocycles $\tau_1, \tau_2 \in cZ^n(A)$ as characters correspond to cobordant cycles over A if and only if $\tau_1 - \tau_2$ belongs to the subspace $cZ^n(A) \cap B_0(Z^{n+1}(A, A^*))$.*

Since we have $P_c\tau = (n+1)\tau$ as cyclic anti-symmetrisation for any $\tau \in cC^n(A)$, the subspace above is contained in $B(Z^{n+1}(A, A^*))$, where $B = P_c \circ B_0 : C^{n+1} \to C^n$.

Lemma 2.24. (a) *We have $bB = -Bb$.*
(b) *It then follows that*

$$cZ^n(A) \cap B_0(Z^{n+1}(A, A^*)) = B(Z^{n+1}(A, A^*)).$$

Proof. For any cochain $\varphi \in C^{n+1}(A, A^*)$, we have

$$B_0 b\varphi + b' B_0 \varphi = \varphi - (-1)^{n+1}\varphi^\lambda$$

where λ is the cyclic permutation defined by $\lambda(j) = j - 1 \pmod{n+2}$. It then follows that $P_c B_0 b\varphi + P_c b' B_0 \varphi = 0$, with $P_c B_0 = B$ and $P_c b' = bP_c$, so that $P_c b' B_0 = bB$.

$\star$ Note that

$$\begin{aligned}
&B_0 b\varphi(a_0, \cdots, a_{n+1}) \\
&= (b\varphi)(1, a_0, \cdots, a_{n+1}) - (-1)^{n+2}(b\varphi)(a_0, \cdots, a_{n+1}, 1) \\
&= \varphi(1a_0, \cdots, a_{n+1}) + \sum_{i=1}^{n+1}(-1)^i \varphi(1, \cdots, a_{i-1}a_i, \cdots, a_{n+1}) + (-1)^{n+2}\varphi(a_{n+1}1, \cdots, a_n) \\
&- (-1)^n[\varphi(a_0 a_1, \cdots, 1) + \sum_{i=1}^{n}(-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, 1) \\
&\quad + (-1)^{n+1}\varphi(a_0, \cdots, a_{n+1}1) + (-1)^{n+2}\varphi(1a_0, a_1, \cdots, a_{n+1})].
\end{aligned}$$

Also,

$$\begin{aligned}
&b' B_0 \varphi(a_0, \cdots, a_{n+1}) \\
&= \sum_{j=0}^{n}(-1)^j (B_0\varphi)(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) \\
&= \sum_{j=0}^{n}(-1)^j \{\varphi(1, a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) - (-1)^{n+1}\varphi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}, 1)\} \\
&= \sum_{j=1}^{n+1}(-1)^{j-1} \{\varphi(1, a_0, \cdots, a_{j-1} a_j, \cdots, a_{n+1}) - (-1)^{n+1}\varphi(a_0, \cdots, a_{j-1} a_j, \cdots, a_{n+1}, 1)\}.
\end{aligned}$$

It then follows that

$$
\begin{aligned}
&B_0 b\varphi(a_0, \cdots, a_{n+1}) - b' B_0 \varphi(a_0, \cdots, a_{n+1}) \\
&= \varphi(1a_0, \cdots, a_{n+1}) + (-1)^{n+2}\varphi(a_{n+1}1, \cdots, a_n) \\
&\quad - (-1)^n[(-1)^{n+1}\varphi(a_0, \cdots, a_{n+1}1) + (-1)^{n+2}\varphi(1a_0, a_1, \cdots, a_{n+1})] \\
&= \varphi(a_0, \cdots, a_{n+1}) + (-1)^{n+2}\varphi(a_{n+1}, \cdots, a_n) \\
&\quad + \varphi(a_0, \cdots, a_{n+1}) - \varphi(a_0, a_1, \cdots, a_{n+1}) \\
&= (\varphi - (-1)^{n+1}\varphi^\lambda)(a_0, \cdots, a_{n+1})
\end{aligned}
$$

checking completed!

(b) By (a) we have $BZ^{n+1}(A, A^*) \subset cZ^n(A)$.

$\star$ Note that $B = P_c B_0 : C^{n+1} \to C^n$, so that the image is in cC^n. Thus, $bB = -Bb : Z^{n+1} \to -B\{0\} = \{0\}$. Hence BZ^{n+1} is contained in $cZ^n(A)$.

It is shown that $BZ^{n+1}(A, A^*) \subset B_0 Z^{n+1}(A, A^*)$. Let $\beta \in BZ^{n+1}(A, A^*)$, so that $\beta = B\varphi$ for some $\varphi \in Z^{n+1}(A, A^*)$.

Constructed in a canonical way is a cochain $\psi \in C^n(A, A^*)$ such that $\frac{1}{n+1}\beta = B_0(\varphi - b\psi)$.

Let $\theta = B_0\varphi - \frac{1}{n+1}\beta$. Then $P_c\theta = 0$.

$\star$ $P_c\theta = B\varphi - \frac{1}{n+1}P_c\beta = \beta - \frac{1}{n+1}(n+1)\beta = 0$.

It then follows that there exists a canonical ψ such that $\psi - \varepsilon(\lambda)\psi^\lambda = \theta$, where λ is the cyclic permutation generator of $\{0, 1, \cdots, n\}$ defined as $\lambda(i) = i - 1$.

$\star$ It seems to have that $P_c(\psi - \varepsilon(\lambda)\psi^\lambda) = 0$. The kernel of P_c may have such elements canonically. The converse is also true?

It then holds that $B_0 b\psi = \theta$. We use the equality $B_0 b\psi - b' B_0 \psi = \psi - \varepsilon\lambda\psi^\lambda$, so that we need to show that $b' B_0 \psi = 0$. We have

$$
\begin{aligned}
&B_0\psi(a_0, \cdots, a_{n-1}) \\
&= \psi(1, a_0, \cdots, a_{n-1}) - (-1)^n\psi(a_0, \cdots, a_{n-1}, 1) \\
&= (-1)^{n-1}(\psi - \varepsilon(\lambda)\psi^\lambda)(a_0, \cdots, a_{n-1}, 1) \\
&= (-1)^{n-1}\theta(a_0, \cdots, a_{n-1}, 1) \\
&= (-1)^{n-1}\{\varphi(1, a_0, \cdots, a_{n-1}, 1) - (-1)^{n+1}\varphi(a_0, \cdots, a_{n-1}, 1, 1)\} \\
&\quad + \frac{1}{n+1}\beta(a_0, \cdots, a_{n-1}, 1).
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&b'B_0\psi(a_0,\cdots,a_n)\\
&=(-1)^{n-1}\sum_{j=0}^{n-1}(-1)^j\{\varphi(1,a_0,\cdots,a_ja_{j+1},\cdots,a_n,1)+(-1)^n\varphi(a_0,\cdots,a_ja_{j+1},\cdots,a_n,1,1)\}\\
&\quad+\frac{1}{n+1}(-1)^n\sum_{j=0}^{n-1}(-1)^j\beta(a_0,\cdots,a_ja_{j+1},\cdots,a_n,1)\\
&=(-1)^n\{b\varphi(1,a_0,\cdots,a_n,1)-\varphi(a_0,\cdots,a_n,1)\}\\
&\quad-\{b\varphi(a_0,\cdots,a_n,1,1)-(-1)^n\varphi(a_0,\cdots,a_n,1)\}\\
&\quad+\frac{1}{n+1}(-1)^nb\beta(a_0,\cdots,a_n,1)=0
\end{aligned}
$$

since $b\varphi=0$, cancellation, and $b\beta=bB\varphi=-Bb\varphi=0$. □

Corollary 2.25. (1) *The image of* $B:C^{n+1}\to C^n$ *is equal to* $cC^n=P_cC^n$.
(2) $cB^n(A)\subset B_0Z^{n+1}(A,A^*)$.

Proof. (1). Let $\varphi\in cC^n$. There is a linear functional $\varphi_0:A\to\mathbb{C}$ with $\varphi_0(1)=1$. Let

$$
\begin{aligned}
&\psi(a_0,\cdots,a_{n+1})=\\
&\varphi_0(a_0)\varphi(a_1,\cdots,a_{n+1})+(-1)^n\varphi((a_0-\varphi_0(a_0)1),a_1,\cdots,a_n)\varphi_0(a_{n+1}).
\end{aligned}
$$

Then we have $\psi(1,a_0,\cdots,a_n)=\varphi(a_0,\cdots,a_n)$ and

$$
\begin{aligned}
&\psi(a_0,\cdots,a_n,1)=\varphi_0(a_0)\varphi(a_1,\cdots,a_n,1)\\
&+(-1)^n\varphi(a_0,a_1,\cdots,a_n)+(-1)^{n+1}\varphi_0(a_0)\varphi(1,a_1,\cdots,a_n)\\
&=(-1)^n\varphi(a_0,\cdots,a_n)
\end{aligned}
$$

by the cyclic property of φ and cancellation. Thus,

$$
\begin{aligned}
&B_0\psi(a_0,\cdots,a_n)=\psi(1,a_0,\cdots,a_n)-(-1)^{n+1}\psi(a_0,\cdots,a_n,1)\\
&=\varphi(a_0,\cdots,a_n)-(-1)^{n+1}(-1)^n\varphi(a_0,\cdots,a_n)=2\varphi(a_0,\cdots,a_n).
\end{aligned}
$$

Thus, $B\frac{1}{2(n+1)}\psi=P_c\frac{1}{2(n+1)}B_0\psi=P_c\frac{1}{n+1}\varphi=\varphi$.

(2). By (1), $b\varphi$ for $\varphi\in cC^n$ has the form $b(B\psi)=-Bb\psi$ for some $\psi\in C^{n+1}$. Then $-Bb\psi\in BZ^{n+2}(A,A^*)$, which is equal to $cZ^{n+1}(A)\cap B_0Z^{n+2}(A,A^*)$ by the lemma above. It then follows that $cB^{n+1}(A)$ is contained in $B_0Z^{n+2}(A,A^*)$ (certainly corrected slightly). Hence, $cB^n(A)\subset B_0Z^{n+1}(A,A^*)$ obtained. □

Corollary 2.26. *We have a well-defined map* B *from the Hochschild cohomology group* $H^{n+1}(A,A^*)$ *to* $cH^n(A)$.

Proof. Note that $H^{n+1}(A.A^*) = Z^{n+1}/B^{n+1}$ with $B^{n+1} = bC^n$. For any $[\varphi] = \varphi + bC^n \in H^{n+1}$, we have

$$B([\varphi]) = B(\varphi + bC^n) = B\varphi + BbC^n = B\varphi + bBC^n \subset B\varphi + b(cC^{n-1}),$$

with $B\varphi \in cZ^n(A)$. Therefore, $B([\varphi]) = [B\varphi] \in cH^n(A)$. $\square$

Theorem 2.27. *Two cycles over a unital algebra A are cobordant if and only if the classes $[\tau_1], [\tau_2] \in cH^n(A)$ of their characters differ by a class of the image $BH^{n+1}(A, A^*) \subset cH^n(A)$.*

Proof. It is shown above that such characters τ_1 and τ_2 have the difference $\tau_1 - \tau_2 = B_0\varphi$ for some $\varphi \in Z^{n+1}(A, A^*)$. Thus, $P_c\tau_1 - P_c\tau_2 = P_cB_0\varphi = B\varphi$. Then

$$[P_c\tau_1] - [P_c\tau_2] = [B\varphi] = B([\varphi]) \in cH^n(A)$$

with $[P_c\tau_j] = [\tau_j]$ for $j = 1, 2$. $\square$

The direct sum $\Omega_1 \oplus \Omega_2$ of two cycles over A is a cycle over A. Cobordism classes of cycles over A make a group $M^*(A)$ by the direct sum.

$\star$ Note that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "(\rho_1'', \rho_2'')"'] & A \arrow[d, "(\rho_1, \rho_1', \rho_2, \rho_2')"] \\ \Omega_1'' \oplus \Omega_2'' \arrow[r, "r"] & \partial\Omega_1'' \oplus \partial\Omega_2'' = \Omega_1 \oplus (\Omega_1')^{\sim} \oplus \Omega_2 \oplus (\Omega_2')^{\sim}. \end{tikzcd}$$

The tensor product $\Omega_1 \otimes \Omega_2$ of cycles over A and B gives a natural map from $M^*(A) \times M^*(B)$ to $M^*(A \otimes B)$.

$\star$ Note that the following diagram commutes:

$$\begin{tikzcd} A \otimes B \arrow[r, equal] \arrow[d, "\rho_1'' \otimes \rho_2''"'] & A \otimes B \arrow[d, "(\rho_1, \rho_1') \otimes (\rho_2, \rho_2')"] \\ \Omega_1'' \otimes \Omega_2'' \arrow[r, "r"] & \partial\Omega_1'' \otimes \partial\Omega_2'' = (\Omega_1 \oplus (\Omega_1')^{\sim}) \otimes (\Omega_2 \oplus (\Omega_2')^{\sim}). \end{tikzcd}$$

Furthermore, more restriction to be continued seems to be needed.

The group $M^*(\mathbb{C})$ is identified with $cH^*(\mathbb{C}) = \mathbb{C}[\sigma]$ as a ring.

$\star$ Because the map B is zero in this case.

Thus, the groups $M^*(A)$ are $M^*(\mathbb{C})$-modules, $\mathbb{C}[\sigma]$-modules, and so vector spaces. It follows from the theorem above that the vector space $M^*(A)$ is isomorphic to the quotient space $cH^*(A)/\mathrm{im}(B)$.

Moreover, the group $M^*(A)$ has the following interpretation related:

Theorem 2.28. *There is an isomorphism of $cH^n(A)/\mathrm{im}(B)$ with the quotient of the space of closed graded traces of degree n on the differential algebra $\Omega^*(A)$ by those of $d^t\mu = \mu \circ d$ for μ graded traces on $\Omega^*(A)$ of degree $n+1$, where d^t denotes the natural differential induced on graded traces.*

Proof. It is shown that for $\tau \in cZ^n(A)$ given, we have $\tau^\wedge = d^t\mu$ for some a graded trace μ if and only if τ belongs to the image of B containing cB^n. Assume that $\tau^\wedge = d^t\mu$. Then we have $\tau = B_0\varphi$, where $\varphi \in Z^{n+1}(A, A^*)$ is the Hochschild cocylce defined by

$$\varphi(a_0, a_1, \cdots, a_{n+1}) = \mu(a_0 da_1 \cdots da_{n+1})$$

as shown that $\tau_1 - \tau_2 = \int \circ d = B_0\varphi$. Thus, $\tau = \frac{1}{n+1}P_c B_0\varphi = \frac{1}{n+1}B\varphi \in \mathrm{im}(B)$.

Conversely, if $\tau \in \mathrm{im}(B)$, then we have $\tau = B_0\varphi$ for some $\varphi \in Z^{n+1}(A, A^*)$ since BZ^{n+1} is equal to $B_0 Z^{n+1} \cap cZ^n$. Define the linear functional μ on $\Omega^{n+1}(A)$ by

$$\mu(a_0 da_1 \cdots da_{n+1}) = \varphi(a_0, \cdots, a_{n+1}), \quad \mu(da_1 \cdots da_{n+1}) = (B_0\varphi)(a_1, \cdots, a_{n+1})$$

as before. Then, obtained is a graded trace such that

$$\mu(da_0 da_1 \cdots da_n) = \tau(a_0, \cdots, a_n)$$

so that $\mu(d\omega) = d^t\mu(\omega) = \tau^\wedge(\omega)$ for $\omega \in \Omega^n(A)$. □

It follows that $M^*(A)$ is the homology of the complex of graded traces on $\Omega^*(A)$ with the differential $d^t = \circ d$. This theory is dual to the theory obtained as the cohomology of the quotient of the complex $(\Omega^*(A), d)$ by the subcomplex of commutators. It also appears independently in the work of M. Karoubi [22] as a natural range for the higher Chern character defined on all the Quillen algebraic K-theory groups. Therefore, the theorem above and the analogous dual statement imply that the pairing with K-theory group of degree zero and one is extended to all the algebraic K-groups by applying results of Karoubi, and as well, the cohomology of the complex $(\Omega^*(A)/[*, *], d)$ is computed by applying results below.

The Connes complex $(cC^n(A), b)$ is by construction a subcomplex of the Hochschild complex $(C^n(A, A^*), b)$ with the indentity inclusion map I as a morphism of complexes, so that there is an exact sequence of complexes

$$0 \to cC^n \xrightarrow{\quad I \quad} C^n \xrightarrow{\quad q \quad} C^n/cC^n = Q_n \to 0.$$

There corresponds to this a long exact sequence of cohomology groups.

It is proved that the cohomology $H^n(C/cC) = H^n(Q_n)$ of the complex C/cC is equal to $H^{n-1}(cC) = cH^{n-1}(A)$.

Therefore, the long exact sequence takes the following form:

$$\begin{array}{lllll} 0 \to cC^0(A) \xrightarrow{\quad I \quad} & H^0(A, A^*) \xrightarrow{\quad q \quad} & H^0(Q_0) = cH^{-1}(A) & \xrightarrow{\quad \partial \quad} & cH^1(A) \\ \xrightarrow{\quad I \quad} & H^1(A, A^*) \xrightarrow{\quad q \quad} & H^1(Q_1) = cH^0(A) & \xrightarrow{\quad \partial \quad} & cH^2(A) \\ \xrightarrow{\quad I \quad} & \cdots \xrightarrow{\quad q \quad} & \cdots & \xrightarrow{\quad \partial \quad} & cH^n(A) \\ \xrightarrow{\quad I \quad} & H^n(A, A^*) \xrightarrow{\quad q \quad} & H^n(Q_n) = cH^{n-1}(A) & \xrightarrow{\quad \partial \quad} & cH^{n+1}(A) \quad \cdots \end{array}$$

On the other hand, constructed above are morphisms S and B of cochain complexes inducing morphisms of cyclic cohomology and H cohomology

$$cH^{n-1}(A) \xrightarrow{S} cH^{n+1}(A), \quad H^n(A,A^*) \xrightarrow{B} cH^{n-1}(A).$$

It is proved that the long exact sequence above is changed into the following form by B and S:

$$\begin{array}{llllllll} \cdots \xrightarrow{I} & H^{n-1}(A,A^*) & \xrightarrow{B} & cH^n(A) & \xrightarrow{S} & cH^n(A) & & \\ \xrightarrow{I} & H^n(A,A^*) & \xrightarrow{B} & cH^{n-1}(A) & \xrightarrow{S} & cH^{n+1}(A) & \cdots \end{array}$$

There corresponds to the pair (b,B) of morphisms of complexes a double complex defined by $C^{n,m} = C^{n-m}(A,A^*)$, where the first differential is given by the Hochschild coboundary b

$$b : C^{n,m} = C^{n-m} \to C^{n-m+1} = C^{n+1-m} = C^{n+1,m}$$

with $b^2 = 0$, and the second by the operator B

$$B : C^{n,m} = C^{n-m} \to C^{n-m-1} = C^{n-(m+1)} = C^{n,m+1}$$

with $B^2 = 0$.

$\star$ Because the image of $C^{n,m} = C^{n-m}$ by B is cC^{n-m-1}, so that $B_0 cC^{n-m-1} = \{0\}$.

Thus, since we have $bB = -Bb$, the graded commutative exact diagram is obtained so that for $n > m$, with (n,m) as the coordinate in the plane,

$$\begin{array}{ccccc} & & C^{n,n} = C^0 & & \\ & & \vdots & & \\ & & B\uparrow & & \\ & & C^{n,m+1} = C^{n-m-1} & \xrightarrow{b} & C^{n+1.m+1} = C^{n-m} \\ & & B\uparrow & & \uparrow B \\ C^{m,m} = C^0 \cdots & \xrightarrow{b} & C^{n,m} = C^{n-m} & \xrightarrow{b} & C^{n+1,m} = C^{n+1-m} \end{array}$$

By construction, the cohomology of this double complex depends only on the parity of n even or odd. It is proved that the sum of the even and odd groups is canonically isomorphic to

$$cH^*(A) \otimes_{cH^*(\mathbb{C})} \mathbb{C} = H^*(A),$$

where $cH^*(\mathbb{C})$ acts on $\mathbb{C}$ by evaluation at $\sigma = 1$.

$\star$ Note that even and odd groups are located on the anti-diagonal from top left to down right.

$$\begin{tikzcd}
C^{2n-1} \arrow[r, "b"] & C^{2n} & \\
C^{2n} \arrow[u, "B"] \arrow[r, "b"] & C^{2n+1} \arrow[u, "B"'] \arrow[r, "b"] & C^{2n+2} \\
& C^{2n+2} \arrow[u, "B"] \arrow[r, "b"] & C^{2n+3} \arrow[u, "B"]
\end{tikzcd}$$

The second filtration of that double complex by

$$F^q = \sum_{m \geq q} C^{n,m} = \sum_{m \geq q} C^{n-m} = C^{n-q} \oplus C^{n-(q+1)} \oplus \cdots \oplus (C^{n-n} = C^0)$$

yields the same filtration of $H^*(A)$ as the filtration by dimensions of cycles.

The associated spectral sequence is convergent, and coincides with the spectral sequence coming from the above exact couple.

These results are based on the next two lemmas.

Lemma 2.29. *Let $\psi \in C^n(A, A^*)$ such that $b\psi \in cC^{n+1}(A)$. Then $B\psi \in cZ^{n-1}(A)$ and $SB\psi = n(n+1)b\psi$ in $cH^{n+1}(A)$.*

$\star$ Note that it says that

$$\begin{tikzcd}
C^n(A, A^*) \arrow[r, "b", "(assumed)"'] \arrow[d, "B"'] & cC^{n+1}(A) \arrow[d, "n(n+1)"] \\
cZ^{n-1}(A) \arrow[r, "S"] & cH^{n+1}(A)
\end{tikzcd}$$

Proof. We have $B\psi = P_c B_0 \psi \in cC^{n-1}$. Since $b\psi \in cC^{n+1}(A)$, we have

$$b(B\psi) = -Bb\psi = -P_c B_0 b\psi = -P_c 0 = 0.$$

Thus, $B\psi \in cZ^{n-1}$. Also, $b\psi \in cZ^{n+1}$ since $b(b\psi) = 0$.

Let $\varphi = B\psi \in cZ^{n-1}$. Then we have $S\varphi = b\psi' \in B^{n+1}$ as shown above, where

$$\psi'(a_0, \cdots, a_n) = \sum_{j=1}^{n} (-1)^{j-1} \varphi^\wedge(a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_n)).$$

It is shown that there exists $\psi'' \in C^n$ such that $\psi'' - \psi \in B^n$ and

$$\psi' - \varepsilon(\lambda)(\psi')^\lambda = n(n+1)(\psi'' - \varepsilon(\lambda)(\psi'')^\lambda)$$

where $\lambda(i) = i - 1$ for $i \in \{1, \cdots, n\}$ and $\lambda(0) = n$.

It is checked that

$$(\psi' - \varepsilon(\lambda)(\psi')^\lambda)(a_0, \cdots, a_n) = (-1)^{n-1}(n+1)\varphi(a_n a_0, a_1, \cdots, a_{n-1}).$$

We have

$$(\psi')^\lambda(a_0, \cdots, a_n) = \sum_{j=1}^{n}(-1)^{j-1}\varphi^\wedge(a_n(da_0 \cdots da_{j-2})a_{j-1}(da_j \cdots da_{n-1}))$$
$$= \sum_{j=1}^{n}(-1)^{j-1+0(n-2)}\varphi^\wedge((da_0 \cdots da_{j-2})a_{j-1}(da_j \cdots da_{n-1})a_n) \quad (j-1=j')$$
$$= \sum_{j'=0}^{n-1}(-1)^{j'}\varphi^\wedge((da_0 \cdots da_{j'-1})a_{j'}(da_{j'+1} \cdots da_{n-1})a_n).$$

Let $\omega_j = a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_{n-1})a_n$. Then

$$\begin{aligned} d\omega_j &= (da_0 \cdots da_{j-1})a_j(da_{j+1} \cdots da_{n-1})a_n \\ &+ (-1)^{j-1}a_0(da_1 \cdots da_j \cdots da_{n-1})a_n \\ &+ (-1)^n a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_n). \end{aligned}$$

Thus, for $j \in \{1, \cdots, n-1\}$, by closedness we have

$$\begin{aligned} 0 = \varphi^\wedge(d\omega_j) &= \varphi^\wedge((da_0 \cdots da_{j-1})a_j(da_{j+1} \cdots da_{n-1})a_n) \\ &+ (-1)^{j-1}\varphi^\wedge(a_0(da_1 \cdots da_j \cdots da_{n-1})a_n) \\ &+ (-1)^n\varphi^\wedge(a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_n)). \end{aligned}$$

Therefore, multiplying $\varepsilon(\lambda)(-1)^{j-1} = (-1)^n(-1)^{j-1}$

$$\begin{aligned} 0 &= \varepsilon(\lambda)(-1)^{j-1}\varphi^\wedge((da_0 \cdots da_{j-1})a_j(da_{j+1} \cdots da_{n-1})a_n) \\ &+ (-1)^n\varphi^\wedge(a_0(da_1 \cdots da_j \cdots da_{n-1})a_n) \\ &+ (-1)^{j-1}\varphi^\wedge(a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_n)). \end{aligned}$$

It then follows that

$$\begin{aligned} &(-1)^{n-1}(-1)^{n(n-n)}\varphi^\wedge(a_n a_0(da_1 \cdots da_j \cdots da_{n-1})) \\ &= (-1)^{j-1}\varphi^\wedge(a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_n)) \\ &- \varepsilon(\lambda)(-1)^j\varphi^\wedge((da_0 \cdots da_{j-1})a_j(da_{j+1} \cdots da_{n-1})a_n) \end{aligned}$$

as a part of $\psi' - \varepsilon(\lambda)(\psi')^\lambda$. Taking into account the cases $j = 0$ for $(\psi')^\lambda$ and $j = n$ for ψ' gives checking completed.

Now determined is the desired ψ'' such that $\psi'' - \psi \in B^n(A, A^*)$ and

$$(\psi'' - \varepsilon(\lambda)(\psi'')^\lambda)(a_0, \cdots, a_n) = \frac{(-1)^{n-1}}{n}\varphi(a_n a_0, \cdots, a_{n-1}).$$

Let $\theta = B_0\psi$. Write $\theta = \theta_1 + \theta_2$ with $P_c\theta_1 = 0$, $\theta_2 \in cC^{n-1}(A)$ so that $\theta_2 = \frac{1}{n}\varphi$.

$\star$ Let $\theta_2 = \frac{1}{n}P_c\theta = \frac{1}{n}B\psi = \frac{1}{n}\varphi \in cC^{n-1}(A)$ and let $\theta_1 = \theta - \theta_2$. Then $\theta = \theta_1 + \theta_2$ with $P_c\theta_1 = P_c\theta - P_c\theta_2 = n\theta_2 - n\theta_2 = 0$.

Since $P_c\theta_1 = 0$, there exists a canonical $\psi_1 \in C^{n-1}$ such that $\theta_1 = D\psi_1$, where $D\psi_1 = \psi_1 - \varepsilon(\lambda)(\psi_1)^\lambda$, which is certainly constructed and obtained above.

As shown in the proof of $b \circ P_c = P_c \circ b'$, checked is that $D \circ b = b' \circ D$.

$\star$ Check indeed that for $f \in C^1(A, A^*)$, with $\varepsilon(\lambda(0,1,2)) = 1$,

$$\begin{aligned}
&(D \circ b)(f)(a_0, a_1, a_2) = (bf - \varepsilon(\lambda)(bf)^\lambda)(a_0, a_1, a_2) \\
&= f(a_0a_1, a_2) - f(a_0, a_1a_2) + f(a_2a_0, a_1) - (bf)(a_2, a_0, a_1) \\
&= f(a_0a_1, a_2) - f(a_0, a_1a_2) + f(a_2a_0, a_1) \\
&\quad - f(a_2a_0, a_1) + f(a_2, a_0a_1) - f(a_1a_2, a_0) \\
&= f(a_0a_1, a_2) - f(a_0, a_1a_2) + f(a_2, a_0a_1) - f(a_1a_2, a_0).
\end{aligned}$$

On the other hand, with $\varepsilon(\lambda(0,1)) = -1$,

$$\begin{aligned}
&(b' \circ D)(f)(a_0, a_1, a_2) = (Df)(a_0a_1, a_2) - (Df)(a_0, a_1a_2) \\
&= f(a_0a_1, a_2) + f(a_2, a_0a_1) - f(a_0, a_1a_2) - f(a_1a_2, a_0).
\end{aligned}$$

Therefore, $D \circ b = b' \circ D$ in this case.

It then follows that $D(b\psi_1) = b'(D\psi_1) = b'\theta_1$.

Let $\psi'' = \psi - b\psi_1$. It then follows that $\psi'' - \psi = -b\psi_1 \in B^n$.

As checked before, we have $D = B_0b + b'B_0$. Hence

$$D\psi = B_0b\psi + b'B_0\psi = b'B_0\psi = b'\theta = b'\theta_1 + b'\theta_2$$

since $b\psi \in cC^{n+1}(A)$ so that $B_0b\psi = 0$. Therefore,

$$D\psi'' = D\psi - Db\psi_1 = b'\theta_1 + b'\theta_2 - b'\theta_1 = b'\theta_2 = \frac{1}{n}b'\varphi.$$

Since $b\varphi = b(B\psi) = 0$, we have

$$b'\varphi = (-1)^{n-1}\varphi(a_na_0, a_1, \cdots, a_{n-1})$$

since

$$(b\varphi)(a_0, \cdots, a_n) = (b'\varphi)(a_0, \cdots, a_n) + (-1)^n\varphi(a_na_0, a_1 \cdots, a_n).$$

Thus,

$$D\psi'' = (-1)^{n-1}\frac{1}{n}\varphi(a_na_0, a_1, \cdots, a_{n-1}).$$

Summing up we obtain

$$\begin{aligned}
[\psi] &= [\psi + (\psi'' - \psi)] = [\psi''], \\
2[b\psi'] &= [b(\psi' - \varepsilon(\lambda)(\psi')^\lambda)] \\
&= n(n+1)[b(\psi'' - \varepsilon(\lambda)(\psi'')^\lambda)] \\
&= 2n(n+1)[b\psi''] = 2n(n+1)[b\psi]
\end{aligned}$$

(possibly in this sense). Therefore, it follows that

$$[SB\psi] = [S\varphi] = [b\psi'] = [n(n+1)b\psi] \in cH^{n+1}(A).$$

$\square$

⋆ Note that

$$\begin{aligned}
&(b\tau)^\lambda(a_0, a_1, a_2) = (b\tau)(a_2, a_0, a_1) \\
&= \tau(a_2a_0, a_1) - \tau(a_2, a_0a_1) + \tau(a_1a_2, a_0), \\
&- b(\tau)^\lambda(a_0, a_1, a_2) = -\tau^\lambda(a_0a_1, a_2) + \tau^\lambda(a_0, a_1a_2) - \tau^\lambda(a_2a_0, a_1) \\
&= -\tau(a_2, a_0a_1) + \tau(a_1a_2, a_0) + \tau(a_2a_0, a_1)
\end{aligned}$$

if τ has graded traceness as does ψ'.

Corollary 2.30. *The image of the map $S : cH^{n-1}(A) \to cH^{n+1}(A)$ is the kernel of the map $I : cH^{n+1}(A) \to cH^{n+1}(A, A^*)$.*

Proof. Since $BC^n = cC^{n-1}$, we have $S[BC^n] = S[cC^{n-1}]$ the image of S, which is equal to $b[C^n]$ by the lemma above. By exactness of the long exact sequence with respect to $C^*(A, A^*)$, we have $b[C^n]$ zero in $H^{n+1}(A, A^*)$, which is viewed as $[bC^n]$ in $cH^{n+1}(A)$ mapped to zero by the map I, namely the kernel of I. $\square$

Lemma 2.31. *There is the natural bijective map*

$$\frac{\mathrm{im}(B) \cap \ker(b)}{b(\mathrm{im}(B))} \to \frac{\ker(B) \cap \ker(b)}{b(\ker(B))}$$

at $cZ^{n+1}(A)$ as well as $Z^{n+1}(A, A^*)$, as $cH^{n+1}(A)$.

⋆ Note that

$$\begin{array}{ccc}
C^{n+2}(A, A^*) & \xrightarrow{B} & cC^{n+1}(A) \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\
C^{n+3}(A, A^*) & \xrightarrow{B} & cC^{n+2}(A)
\end{array}$$

Proof. Let $\varphi \in \mathrm{im}(B) \cap \ker(b)$. Let $\varphi \in cZ^{n+1}(A)$. Assume $\varphi \in b(\ker(B))$, say $\varphi = b\psi$ with $\psi \in \ker(B)$. Since $n(n+1)b\psi = SB\psi = S(0) = 0$ in cH^*, then $\varphi \in cB^{n+1}(A)$, so that $\varphi \in b(\mathrm{im}(B))$, with this $B : C^{n+1} \to cC^n$ onto. Thus, the injectivity is shown.

Let $\varphi \in Z^{n+1}(A, A^*)$, $B\varphi = 0$ and a canonical $\psi \in C^n(A, A^*)$, with $\psi - \varepsilon(\lambda)\psi^\lambda = B_0\varphi$. As above, we have $B_0b\psi = B_0\varphi$ since $\beta = B\varphi = 0$. It follows that $\varphi' = \varphi - b\psi \in cZ^{n+1}(A)$ since $b\varphi' = 0 - 0 = 0$ and

$$D\varphi' = B_0b\varphi' + b'B_0\varphi' = B_00 + b'(B_0\varphi - B_0b\varphi) = 0$$

which implies cyclic property.

It is shown that $B\psi \in b(cC^{n-2})$. Since

$$D\psi = \psi - \varepsilon(\lambda)\psi^\lambda = B_0\varphi = B_0 b\psi,$$

with $D = B_0 b + b' B_0$, we have $b' B_0 \psi = 0$.

It is checked that $(b')^2 = 0$ and that the cohomology on $C^n(A, A^*)$ by b' is trivial.

If $b'\varphi_1 = 0$, namely, $\varphi_1 \in Z^n$, then $b'\varphi_1(a_0, \cdots, a_n, 1) = 0$, that is, $\varphi_1 = b'\varphi_2$, namely $\varphi_1 \in B^n$, where

$$\varphi_2(a_0, \cdots, a_{n-1}) = (-1)^{n-1}\varphi_1(a_0, \cdots, a_{n-1}, 1).$$

$\star$ Note that

$$\begin{aligned}0 = b'\varphi_1(a_0, \cdots, a_n, 1) &= \sum_{j=0}^{n}(-1)^j \varphi_1(a_0, \cdots, a_j a_{j+1}, \cdots, 1)\\ &= \varphi_1(a_0 a_1, \cdots, 1) - \cdots + (-1)^n \varphi_1(a_0, \cdots, a_n, 1).\end{aligned}$$

On the other hand,

$$\begin{aligned}b'\varphi_2(a_0, \cdots, a_n) &= \sum_{j=0}^{n-1}(-1)^j \varphi_2(a_0, \cdots, a_j a_{j+1}, \cdots, a_n)\\ &= \sum_{j=0}^{n-1}(-1)^j(-1)^{n-1}\varphi_1(a_0, \cdots, a_j a_{j+1}, \cdots, a_n, 1)\\ &= (-1)^{n-1}\varphi_1(a_0 a_1, \cdots, a_n, 1) + \cdots + \varphi_1(a_0, \cdots, a_{n-1}a_n, 1).\end{aligned}$$

It then follows that

$$\varphi_1(a_0, \cdots, a_n, 1) = b'\varphi_2(a_0, \cdots, a_n).$$

Thus, since $b'(B_0\psi) = 0$, then $B_0\psi = b'\theta$ for some $\theta \in C^{n-2}$ by b' cohomology triviality, and

$$B\psi = P_c B_0 \psi = P_c b'\theta = bP_c\theta \in b(cC^{n-2}).$$

Since $cC^{n-2} = \mathrm{im}(B)$, we have $B\psi = bB\theta_1$ for some $\theta_1 \in C^{n-1}$. Hence,

$$B(\psi + b\theta_1) = B\psi + Bb\theta_1 = B\psi - bB\theta_1 = 0.$$

Thus, $\psi + b\theta_1 \in \ker(B)$, so that $b\psi \in b(\ker(B))$. As $\varphi - b\psi \in cZ^{n+1}$, this shows the end of the proof for surjectivity. $\square$

Corollary 2.32. *We have $S = n(n+1)bB^{-1}$ as the map from $cH^{n-1}(A)$ to $cH^{n+1}(A)$. Namely, $S(B\psi) = b\psi$ up to constant.*

Proof. Given $\varphi \in cZ^{n-1}(A)$, we have $\varphi \in \mathrm{im}(B)$, and thus $\varphi = B\psi$ for some ψ. This uniquely determines

$$b\psi \in \frac{\ker(b) \cap \ker(B)}{b(\ker(B)} = cH^{n+1}(A).$$

Moreover, $b\psi$ is equal to $\frac{1}{n(n+1)}S\varphi$ by choosing ψ as

$$\psi(a_0, \cdots, a_n) = \frac{1}{n(n+1)} \sum_{j=1}^{n} (-1)^{j-1} \varphi^\wedge(a_0(da_1 \cdots da_{j-1})a_j(da_{j+1} \cdots da_n)).$$

Hence,

$$S\varphi = SB\psi = n(n+1)b\psi = n(n+1)bB^{-1}\varphi.$$

□

Theorem 2.33. *There is the long exact sequence involving B, S, and I*

$$\begin{array}{llllll} \cdots \xrightarrow{\quad B \quad} & cH^{n-1}(A) & \xrightarrow{\quad S \quad} & cH^{n+1}(A) & \xrightarrow{\quad I \quad} & H^{n+1}(A, A^*) \\ \xrightarrow{\quad B \quad} & cH^{n}(A) & \xrightarrow{\quad S \quad} & cH^{n+2}(A) & \xrightarrow{\quad I \quad} & H^{n+2}(A, A^*) \end{array}$$

Proof. It is shown above that $\mathrm{im}(S)$ is equal to $\ker(I)$ at cH^k. As well, $\ker(S)$ is equal to $\mathrm{im}(B)$ at cH^{k-2}.

$\star$ If $b\psi = 0$, then $\psi \in Z^n$, and then $B\psi \in \mathrm{im}(B)$ and $S(B\psi) = 0 = b\psi$.

Next, $B \circ I = 0$ since B is zero on cC. Thus $\mathrm{im}(I)$ is contained in $\ker(B)$. If $\varphi \in Z^n(A, A^*)$ and $B\varphi \in cB^{n-1}$, then $B\varphi = bB\theta$ for some $\theta \in C^{n-1}$, so that

$$\varphi + b\theta \in \ker(B) \cap \ker(b) \subset \mathrm{im}(I) + b(\ker(B))$$

by the above lemma.

$\star$ We have

$$B(\varphi + b\theta) = B\varphi + Bb\theta = B\varphi - bB\theta = 0.$$

As well, $b(\varphi + b\theta) = 0 + 0 = 0$.

Note that $\mathrm{im}(I)$ is identified with cZ^n, which may be identified with $\mathrm{im}(B)$.

□

Corollary 2.34. *For*

$$0 \to cC \xrightarrow{\quad i \quad} C \xrightarrow{\quad q \quad} C/cC \to 0.$$

the exact sequence of complexes, the morphism $\partial : C/cC \to C$ of complexes induces an isomorphism from $H^n(C/cC)$ to $cH^{n-1}(A)$ and identifies the long exact sequence derived by the exact sequence of complexes with that by B-S-I.

Proof. This follows from the Five Lemma applied to

$$\begin{tikzcd}
H^n(cC) \arrow[r, "i"] \arrow[d, equal] & H^n(C) \arrow[r, "q"] \arrow[d, equal] & H^n(C/cC) \arrow[r, "\partial", "+1"'] \arrow[d, "\partial^{\sim}", "-1"'] & H^{n+1}(cC) \arrow[r, "i"] \arrow[d, equal] & H^{n+1}(C) \arrow[d, equal] \\
cH^n(A) \arrow[r, "I"] & H^n(A, A^*) \arrow[r, "B"] & cH^{n-1}(A) \arrow[r, "S"] \arrow[d, equal] & cH^{n+1}(A) \arrow[r, "I"] & H^{n+1}(A, A^*) \\
& & H^{n-1}(cC) & &
\end{tikzcd}$$

with a consideration as for $\partial^{\sim}$ as the boundary map of degree -1. □

Corollary 2.35. (a) *Two cycles with characters τ_1 and τ_2 are cobordant if and only if $S\tau_1 = S\tau_2$ in $cH^*(A)$.*

(b) *We have a canonical isomorphism $M^*(A) \otimes_{M^*(\mathbb{C})} \mathbb{C} \cong H^*(A)$.*

(c) *The canonical filtration $F^nH^*(A)$ corresponds under that isomorphism to the filtration of the left side by the dimension of the cycles.*

Proof. (a) $\star$ We have

$$H^k(A, A^*) \xrightarrow{B} cH^{k-1}(A) \xrightarrow{S} cH^{k+1}(A).$$

It is shown by cobordism that $[\tau_1] - [\tau_2] \in BH^k(A, A^*) \subset cH^{k-1}$. Hence,

$$S[\tau_1] - S[\tau_2] = [S\tau_1] - [S\tau_2] = [0] \in SBH^k(A, A^*) \subset cH^{k+1}(A).$$

(b) Both sides are identical with the inductive limit of the system $(cH^n(A), S)$.

$\star$ We have $M^*(A)$ is isomorphic to $cH^*(A)/\mathrm{im}(B)$, as well as $M^*(\mathbb{C})$ is equal to $cH^*(\mathbb{C}) = \mathbb{C}[\sigma]$. Since $S = n(n+1)bB^{-1}$ as σ, then it follows.

(c) $\star$ Note that

$$F^qH^{n-m}(A) = \sum_{m \geq q} H^{n-m}(A) = H^{n-q} \oplus H^{n-(q+1)} \oplus \cdots (H^{n-n} = H^0).$$

Also,

$$F^qM^{n-m}(A) = \sum_{m \geq q} M^{n-m}(A) = M^{n-q} \oplus M^{n-(q+1)} \oplus \cdots (M^{n-n} = M^0).$$

□

We may normalize the differentials b and B (b-B) to the following differentials d_1 and d_2 of the doubled complex $C^{*,*}$ so that the S map is given by $d_1d_2^{-1}$ simply. Define

(a) $C^{n,m} = C^{n-m}(A, A^*)$ for $n, m \in \mathbb{Z}$, with $C^{-k} = \{0\}$ k positive.

(b) $d_1\varphi = (n - m + 1)b\varphi \in C^{n+1,m}$ for $\varphi \in C^{n,m}$.

(c) $d_2\varphi = \frac{1}{n-m}B\varphi \in C^{n,m+1}$, with as zero if $n = m$.

Note that $d_1d_2 = -d_2d_1$ follows from $Bb = -bB$.

$\star$ We check that for $\varphi \in C^{n,m}$ with $B\varphi \in C^{n,m+1}$, $b\varphi \in C^{n+1,m}$,

$$\begin{aligned} d_1 d_2 \varphi &= d_1 \frac{1}{n-m} B\varphi = \frac{1}{n-m}(n-(m+1)+1) bB\varphi = bB\varphi, \\ d_2 d_1 \varphi &= d_2 (n-m+1) b\varphi = (n-m+1) \frac{1}{n+1-m} Bb\varphi = Bb\varphi. \end{aligned}$$

Therefore, $d_1 d_2 + d_2 d_1 = 0$ follows. As well, for $B\varphi = \psi \in C^{n,m+1} = C^{n-m-1} = C^{k-1}$,

$$\begin{aligned} d_1 d_2^{-1} \psi &= d_1 (n-m) \varphi = (n-m)(n-m+1) b\varphi \\ &= (k+1)(k+2) bB^{-1} \psi = S\psi. \end{aligned}$$

Theorem 2.36. (*a*) *The initial term E_2 of the spectral sequence associated to the filtration $F_p C = \sum_{n \geq p} C^{n,m}$ in the first variable n by b is equal to zero.*

(*b*) *For the second filtration $F^q C = \sum_{m \geq q} C^{n,m}$ by B, we have $H^p(F^q C) = cH^n(A)$ for $n = p - 2q$.*

(*c*) *The cohomology of the double complex $C = C^{*,*}$ is given by*

$$H^n(C) = \begin{cases} H^{\mathrm{even}}(A) = H^{\mathrm{ev}}(A), & \text{if } n \text{ is even,} \\ H^{\mathrm{odd}}(A) = H^{\mathrm{od}}(A), & \text{if } n \text{ is odd.} \end{cases}$$

(*d*) *The spectral sequence associated to the second filtration is convergent, and it converges to the associated graded*

$$\sum F^q H^*(A) / F^{q+1} H^*(A)$$

and it coincides with the spectral sequence associated with the exact couple by B, S, I. In particular, its initial term E_2 is given by

$$\ker(I \circ B) / \mathrm{im}(I \circ B).$$

Proof. (a) We consider the exact sequence of complexes of cochains

$$0 \to \mathrm{im}(B) \longrightarrow \ker(B) \longrightarrow \ker(B)/\mathrm{im}(B) \to 0$$

where the coboundary is given by Hochschild b. As shown above, the first map from $\mathrm{im}(B)$ to $\ker(B)$ induces an isomorphism in cohomology. Thus, the b cohomology of the complex $\ker(B)/\mathrm{im}(B)$ is zero.

(b) Let $\varphi \in (F^q C)^p = \sum_{m \geq q, n+m=p} C^{n,m}$, satisfy $d\varphi = 0$, where $d = d_1 + d_2$. Then it is cohomologus in $F^q C$ to an element ψ of $C^{p-q,q} = C^{p-2q}$, by (a). Then $d\psi = 0$ means $\psi \in \ker(b) \cap \ker(B)$, and $\psi \in \mathrm{im}(d)$ means $\psi \in b(\ker(B))$. Thus,

$$(\ker(b) \cap \ker(B))/b(\ker(B)) \cong (\mathrm{im}(B) \cap \ker(b))/b(\mathrm{im}(B)) = cH^{p-2q}(A).$$

(c) By the computation of S as $d_1 d_2^{-1}$, the map from $H^p(F^q C)$ to $H^p(F^{q-1} C)$ with $p - 2(q-1) = p - 2q + 2$ is given by the map $-S$ from $cH^{p-2q}(A)$ to $cH^{p-2q+2}(A)$.

$\star$ For $n = p - 2q$, if p is even, then so is n with q any, and if p is odd, then so is n with q any.

(d) The convergence of the spectral sequence by the second variable is obvious, because $C^{n,m} = C^{n-m} = 0$ for $m > n$, with $n - m < 0$. The filtration of $H^n(C)$ given by $H^n(F^qC)$ coincides with the natural filtration of $H^*(A)$, as given in (c). Therefore, the limit of the spectral sequence is the associated graded

$$\begin{cases} \sum_q F^q H^{\mathrm{ev}}(A)/F^{q+1}H^{\mathrm{ev}}(A), & \text{for } n \text{ even}, \\ \sum_q F^q H^{\mathrm{od}}(A)/F^{q+1}H^{\mathrm{od}}(A), & \text{for } n \text{ odd}. \end{cases}$$

$\star$ Note that $F^qC/F^{q+1}C = C^{n,q} = C^{n-q}$.

$\star$ As well, that means $F^q \sum_{l\geq q} H^{n-2l}/F^{q+1}\sum_{l\geq q+1} H^{n-2l} = H^{n-2q}$ in this sense (possibly).

The initial term E_2 by B is given by $\ker(I\circ B)/\mathrm{im}(I\circ B)$. It is checked that it coincides with the spectral sequence of the exact couple by B-S-I.

$\star$ Note that $H^{\mathrm{ev}}(A)$ and $H^{\mathrm{od}}(A)$ are invariant under S, so that $I\circ S\circ B$ can be shorten as $I \circ B$. □

Remark 2.37. For $n = p - 2q$, the cyclic cohomology $cH^n(A)$ is identified with $H^p(F^qC)$ in the (d_1, d_2) bicomplex by the following sign convention by S or $-S$ that $\varphi \in cZ^n(A)$ corresponds to $(-1)^{[\frac{n}{2}]}\varphi \in C^{p-q,q} = C^{p-2q}$. This sign is used in comparing the expressions for the pairing of cyclic cohomology with K-theory.

There is the following product of Cartan-Eilenberg [8]

$$\vee : H^n(A, M_1) \otimes H^m(A, M_2) \to H^{n+m}(A, M_1 \otimes_A M_2).$$

It then follows that $H^*(A, A)$ becomes a graded commutative algebra by using $A \otimes_A A = A$ as an A-bimodule, and it acts on $H^*(A, A^*)$ since $A \otimes_A A^* = A^*$. In particular, any derivation δ of A defines an element of $H^1(A, A)$, denoted as $[\delta]$. The explicit formula for the product $\vee$ of [8] gives

$$(\varphi \vee \delta)(a_0, a_1, \cdots, a_{n+1}) = \varphi(\delta(a_{n+1})a_0, a_1, \cdots, a_n), \quad \varphi \in Z^n(A, A^*)$$

at the cochain level. It is checked that the class of $\varphi \vee \delta$ coincides with that of

$$\begin{aligned}&(\delta\#\varphi)(a_0, a_1, \cdots, a_{n+1})\\ &= \frac{1}{n+1}\sum_{j=1}^{n+1}(-1)^j \varphi^\wedge(a_0(da_1\cdots ad_{j-1})\delta(a_j)(da_{j+1}\cdots da_{n+1}))\end{aligned}$$

at the level of cohomology. It then follows that

$$\begin{aligned}\delta^*\varphi(a_0, \cdots, a_n) &= \sum_{i=1}^n \varphi(a_0, \cdots, \delta(a_i), \cdots, a_n) \quad a_i \in A\\ &= (I\circ B)(\delta\#\varphi) + \delta\#((I\circ B)\varphi) \in H^{n+1}(A, A^*).\end{aligned}$$

This is the natural extension of the basic formula in differential geometry such that $\partial_X = di_X + i_X d$, expressing the Lie derivative with respect to a vector field X on a manifold.

Let A be a unital algebra, B a locally convex topological algebra, and $\varphi \in cZ^n(B)$ a continuous cocycle. Let $(\rho_t)_{t\in[0,1]}$ be a family of homomorphisms $\rho_t : A \to B$ such that the function $\rho_t(a)$ for $t \in [0,1]$ and $a \in A$, defined as $t \mapsto \rho_t(a)$, belongs to the algebra $C^1([0,1], B)$ of C^1-maps from $[0,1]$ to B. Then the cocycles $\rho_0^*\varphi = \varphi \circ \rho_0$ and $\rho_1^*\varphi = \varphi \circ \rho_1$ has the images by S coincide.

To prove this, the Hochschild cocycle $\varphi\#\psi$ on $B \otimes C^1([0,1])$ giving the cobordism of φ with itself, where $\psi(f_0, f_1) = \int_0^1 f_0 df_1$ for $f_0, f_1 \in C^1([0,1])$ is extended (identically) to a Hochshild cocycle on $C^1([0,1], B)$. Then the map $\rho : A \to C^1([0,1], B)$ defined as $(\rho(a))(t) = \rho_t(a)$ defines a chain over A giving a cobordism of $\rho_0^*\varphi$ with $\rho_1^*\varphi$, and thus, the same mod the image of B, and hence the same class by S, since $S \circ B$ is zero, with $B = P_c B_0$. It then follows that $\rho_0^* = \rho_1^* : H^*(B) \to H^*(A)$ if restricted to continuous cocycles.

The cyclic cohomology theory $cH^*(A)$ is defined as the cohomology of a complex (cC^n, b). Also, for pairs of algebras A and B with a surjective homomorphism $\pi : A \to B$, a relative cH^* theory $cH^*(A, B)$ can be developed. To the exact sequence of complexes:

$$0 \to cC^n(B) \xrightarrow[(\cdot)\circ\pi]{\pi^*} cC^n(A) \xrightarrow{q} cC^n(A)/cC^n(B) = cC^n(A, B) \to 0$$

corresponds a long exact sequence of cohomology groups. Using the Five Lemma, the statements on the absolute cohomology groups can be extended to the relative cohomology groups, provided that the Hochschild theory $H^*(A, A^*)$ is extended up to the relative version.

Characterized by M. Wodzicki [30] are non-unital algebras which satisfy excision in Hochschild homology and cyclic homology by the property H-U.

An algebra over $\mathbb{C}$ is defined to be H-unital if and only if the b' complex is acyclic.

Also, it is shown by J. Cuntz and D. Quillen [14] that excision holds in full generality in periodic cyclic cohomology.

3 Examples

The first example (α)

Let V be a compact smooth manifold and $A = C^\infty(V)$ the algebra of smooth functions on V, endowed with the natural Fréchet space topology. The topology of A is given by the semi-norms defined by $p_n(f) = \sup_{|\alpha|\leq n} |\partial^\alpha f|$ using local charts in V.

We consider only continuous multilinear forms on A with respect to the topology. As a locally convex vector space, $C^\infty(V)$ is nuclear, namely, its topological tensor products are uniquely defined. Therefore, the n-fold topological tensor product $\otimes^n C^\infty(V)$ is isomorphic to $C^\infty(V^n)$.

In particular, the algebra $B = A \otimes A^{\mathrm{opp}}$ is isomorphic to $C^\infty(V^2)$ by commutativity. Thus, A viewed as an A-bimodule corresponds to the$B$-module given by the diagonal inclusion $\Delta : V \to V \times V$, $\Delta(p) = (p, p)$, $p \in V$. A projective resolution of the diagonal in $V \times V$ is used to obtain that [10]

Proposition 3.1. *Let $A = C^\infty(V)$ be the locally convex topological algebra of smooth functions on a smooth compact manifold V.*

Then there is a canonical isomorphism of the continuous Hochschild cohomology group $H^k(A, A^*)$ of classes $[\varphi]$ with the space D_k of k-dimensional de Rham currents C_φ on V, defined by

$$\langle C_\varphi, f_0 df_1 \wedge \cdots \wedge df_k \rangle = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \varphi(f_0, f_{\sigma(1)}, \cdots, f_{\sigma(k)})$$

for $f_0, \cdots, f_k \in C^\infty(V)$.

Moreover, the operator

$$I \circ B : H^k(A, A^*) \xrightarrow{\ B\ } cH(A)^{k-1} \xrightarrow{\ I\ } H^{k-1}(A, A^*)$$

is given by under the isomorphism the de Rham boundary $d^t = (\cdot) \circ d$ for currents k times multiplied.

The analogous statement for the algebra of polynomials on an affine variety is due to Hochschild-Kostant-Rosenberg [19].

The de Rham complex of the manifold V is recovered from that proposition.

If A is replaced with the matrix algebra $M_q(A) = M_q(\mathbb{C}) \otimes A$ of matrices over A, then commutativity as well as the exterior algebra are lost, but the cohomology groups $H^k(A, A^*)$ make sense, using the Morita invariance of Hochschild cohomology to yield the same result, as for $k = 1$. In this case, the Hochschild cocycle class associated to a current $C \in D_k$ is given by the following formula:

$$\begin{aligned}&\varphi_C(f_0 \otimes \mu_0, f_1 \otimes \mu_1, \cdots, f_k \otimes \mu_k) \\ &= \langle C, f_0 df_1 \wedge \cdots \wedge df_k \rangle \mathrm{tr}(\mu_0 \cdots \mu_k), \quad f_j \in C^\infty(V), \mu_j \in M_q(\mathbb{C}).\end{aligned}$$

Theorem 3.2. [10]. *For $A = C^\infty(V)$ the locally convex topological algebra, we have*

(1) *For each k,*

$$cH^k(A) \cong \ker(b) \oplus H_{k-2}(V, \mathbb{C}) \oplus H_{k-4}(V, \mathbb{C}) \oplus \cdots \oplus H_0(V, \mathbb{C})$$

where $H_*(V, \mathbb{C})$ is the usual de Rham homology of V, and b is the de Rham boundary, with $\ker(b) \subset D_k$.

(2) *With filtration by dimension,*

$$H^*(A) \cong H_*(V, \mathbb{C}).$$

Proof. Let $\varphi \in cH^k(A)$. Then the current $C = C_{I(\varphi)}$ associated to $I(\varphi) \in H^k(A, A^*)$ is given by

$$\langle C, f_0 df_1 \wedge \cdots \wedge df_k \rangle = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varphi(f_0, f_{\sigma(1)}, \cdots, f_{\sigma(k)}).$$

$\star$ Because $\varphi = P_c \varphi$ as well as cyclic property to do.

That is closed since $B(I(\varphi)) = 0$. Thus, the cochain $\varphi^\sim$ defined by

$$\varphi^\sim(f_0, f_1, \cdots, f_k) = \langle C, f_0 df_1 \wedge \cdots \wedge df_k \rangle$$

belongs to $cZ^k(A)$. The class of $\varphi - \varphi^\sim$ in $cH^k(A)$ determined is in the kernel of I by construction. Then there exists by exactness $\psi \in cH^{k-2}(A)$ such that $S\psi = \varphi - \varphi^\sim$, and ψ is unique modulo the image of B. Thus, the homology class of the closed current $C(I(\psi))$ is well determined. Moreover, the class of $\psi - \psi^\sim$ in $cH^{k-2}(A)$ is determined, similarly. Repeating this process we obtain the sequence of homology classes $\omega_j \in H_{k-2j}(V, \mathbb{C})$ for $j \geq 1$. By construction, φ is in the same class in $cH^k(A)$ as $C^\sim + \sum_{j \geq 1} \omega_j^\sim$ (with $S^j \omega_j^\sim = C^\sim$ corrected),

$$\omega_j^\sim(f_0, f_1, \cdots, f_{k-2j}) = \langle \omega_j, f_0 df_1 \wedge \cdots \wedge df_{k-2j} \rangle$$

for any closed current ω_j in the class. This shows that the map constructed so is injection of $cH^k(A)$ to the direct sum as in the statement.

Surjectiveness of the map is obvious.

$\star$ For $\omega = C + \sum_{j \geq 1} \omega_j$ in the direct sum, we define

$$\varphi_\omega = \varphi_{C + \sum_{j \geq 1} \omega_j} = \varphi_C + \sum_{j \geq 1} \varphi_{\omega_j}$$

where each term may have cyclic property by composing with P_c (Pc-zation) if necessary, and as well have extended trivially to by the domain of C.

By the construction of the isomorphism for (1), the map $S : cH^k(A) \to cH^{k+2}(A)$ associates to each $C \in \ker(b)$ its homology class. Thus the inclusion for (2) follows. In this example, the spectral sequence by B-S-I is degenerate. Its E_2 term is the de Rham homology of V. $\square$

It follows from the theorem above that the periodic cyclic cohomology of $C^\infty(V)$ with natural filtration is the de Rham homology of the manifold V.

Let $[S^1]$ be the fundamental class of the circle S^1. The image $S^k[S^1] \in cH^{2k+1}(C^\infty(S^1))$ under the peridicity operator extends to the algebra $C^\alpha(S^1)$ of Hölder continuous functions f of exponent $\alpha > \frac{1}{2k+1}$, so that

$$|f(x) - f(y)| \leq C d(x, y)^\alpha$$

where d is the usual metric on S^1.

Proposition 3.3. *Let $C_c^\alpha(\mathbb{R})$ be the algebra of Hölder continuous functions of exponent $\alpha > \frac{1}{2k+1}$, with compact support.*

(a) The following cyclic cocycle $\tau_k \in cH^{2k+1}(C_c^\alpha(\mathbb{R}))$ is defined:

$$\tau_k(f_0, f_1, \cdots, f_{2k+1})$$
$$= \int f_0(x_0)\frac{f_1(x_1)-f_1(x_0)}{x_1-x_0}\frac{f_2(x_2)-f_2(x_1)}{x_2-x_1}\cdots\frac{f_{2k}(x_{2k})-f_{2k}(x_{2k-1})}{x_{2k}-x_{2k-1}}$$
$$\frac{f_{2k+1}(x_0)-f_{2k+1}(x_{2k})}{x_0-x_{2k}}\Pi_{i=0}^{2k}dx_i.$$

(b) The restriction of τ_k to $C_c^\infty(\mathbb{R})$ is equal to $c_k S^k \tau_0$ in cH^{2k+1}, where τ_0 is the homology fundamental class of $\mathbb{R}$ defined by $\tau_0(f_0, f_1) = \int f_0 df_1$, and

$$c_k = \frac{(2\pi i)^{2k}}{2^k(2k+1)(2k-1)\cdots 3\cdot 1} = \frac{(-2)^k\pi^{2k}}{(2k+1)!!}.$$

The multiple integral in the statement above makes sense since each of the gradient terms has the large O form

$$\frac{f_j(x_j)-f_j(x_{j-1})}{x_j-x_{j-1}} = O(|x_j-x_{j-1}|^{\alpha-1}).$$

$\star$ Note that

$$\left|\frac{f_j(x_j)-f_j(x_{j-1})}{x_j-x_{j-1}}\right| \leq \frac{C_j|x_j-x_{j-1}|^\alpha}{|x_j-x_{j-1}|} = C_j|x_j-x_{j-1}|^{\alpha-1}.$$

Also,

$$\left|\int f_0(x_0)C_1|x_1-x_0|^{\alpha-1}C_{2k+1}|x_0-x_{2k}|^{\alpha-1}dx_0\right|$$
$$\leq \int_{\mathrm{supp}(f_0)} M_0C_1C_{2k+1}|x_1-x_0|^{\alpha-1}|x_0-x_{2k}|^{\alpha-1}dx_0.$$

It seems that the integral certainly exists when $\alpha \geq 1$. As well,

$$\int C_1|x_1-x_0|^{\alpha-1}C_2|x_2-x_1|^{\alpha-1}dx_1$$
$$\leq \int C_1C_2|x_1^2-(x_0+x_2)x_1+x_0x_2|^{\alpha-1}dx_1.$$

It is checked that τ_k as in (a) is a cyclic cocycle, satisfying (b).

A similar formula for $S^k[S^1] \in cH^{2k+1}(C^\alpha(S^1))$ is also obtained.

Using this formula, the following explicit formula for the push-forward $\psi_*[S^1]$ by the map $\psi : S^1 \to V$, Hölder continuous of exponent $\alpha > \frac{1}{2k+1}$:

$$\psi_*[S^1](f_0, \cdots, f_{2k+1}) = \tau_k(\psi^* f_0, \psi^* f_1, \cdots, \psi^* f_{2k+1})$$

for $f_0, \cdots, f_{2k+1} \in C^\infty(V)$, where $\psi^* f_j = f_j \circ \psi$.

This gives a cyclic cocycle $\psi_*[S^1] \in cH^{2k+1}(C^\infty(V))$.

The formula $\int_{S^1} \psi^* f_0 d(\psi^* f_1)$ may make sense if the functions $g_j = \psi^* f_j$ on S^1 belong to the Sobolev space $W^{\frac{1}{2}}$ by the finiteness of $\sum n|g^\wedge(n)|^2$, where $g^\wedge$ is the Fourier transform of g. Thus, it may not for $\psi^* f$, for $f \in C^\infty(V)$, only Höler continuous of exponent $\alpha < \frac{1}{2}$.

In the periodic cyclic cohomology of $C^\infty(V)$, the above cocycle coincides with $S^k \psi'_*[S^1]$, where ψ' is smooth and homotopic to ψ.

Remark 3.4. Let $A = C^\infty(V)$ and assume that the Euler characteristic of V vanishes, to simplify. Let $T_{\mathbb{C}}(V)$ be the complexified tangent bundle on V. Let $p_2 : V \times V \to V$ be the second projection and $p_2^* T_{\mathbb{C}}(V)$ the pull-back of $T_{\mathbb{C}}(V)$ by p_2. Let X be a non-vanishing section of the vector bundle on $V \times V$, such that $X(a, b)$ coincides with the real tangent vector $\exp_b^{-1}(a)$ for $(a, b) \in V \times V$ close enough to the diagonal of $V \times V$, where $\exp_b : T_b(V) \to V$ is the exponential map associated to a given Riemann metric on V.

Let E_k be the complex vector bundle on $V \times V$ defined as the pull-back of the exterior power $\wedge^k T_{\mathbb{C}}^*(V)$ by the second projection p_2. The contraction i_X by the section X gives a well-defined complex of $C^\infty(V \times V)$-modules, as $C^\infty(V)$-bimodules, such that

$$\begin{aligned} C^\infty(V) \xleftarrow[(\cdot)\circ\Delta]{\Delta^*} C^\infty(V \times V) &\xleftarrow[\langle X, \cdot \rangle]{i_X} C^\infty(V \times V, E_1) \\ \longleftarrow \quad \cdots \quad &\xleftarrow[\langle X, \cdots \rangle]{} C^\infty(V \times V, E_n) \leftarrow 0, \end{aligned}$$

where $n = \dim V$ and $\Delta : V \to V \times V$ the diagonal map.

This gives an explicit projective resolution M' of the A-bimodule A, and as well a proof for the first proposition ([10]).

Let M be the standard resolution of the bimodule A, by $M_k = (A \otimes A^{\odot}) \otimes (\otimes^k A)$, with the boundary $b_k : M_k \to M_{k-1}$ given by

$$\begin{aligned} b_k(1 \otimes a_1 \otimes \cdots \otimes a_k) &= (a_1 \otimes 1) \otimes (a_2 \otimes \cdots \otimes a_k) \\ &+ \sum_{j=1}^{k-1} (-1)^j (1 \otimes 1) \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k \\ &+ (-1)^k (1 \otimes a_k^{\mathrm{op}}) \otimes (a_1 \otimes \cdots \otimes a_{k-1}). \end{aligned}$$

$\star$ Note that

$$\begin{aligned} &b_3(1 \otimes a_1 \otimes a_2 \otimes a_3) = (a_1 \otimes 1) \otimes (a_2 \otimes a_3) \\ &- (1 \otimes 1) \otimes a_1 a_2 \otimes a_3 + (1 \otimes 1) \otimes a_1 \otimes a_2 a_3 - (1 \otimes a_3^{\mathrm{op}}) \otimes a_1 \otimes a_2. \end{aligned}$$

As well,

$$\begin{aligned} &b_2(1 \otimes a_1 \otimes a_2) = (a_1 \otimes 1) \otimes a_2 \\ &- (1 \otimes 1) \otimes a_1 a_2 + (1 \otimes a_2^{\mathrm{op}}) \otimes a_1. \end{aligned}$$

Therefore,

$$\begin{aligned}
&(b_2 \circ b_3)(1 \otimes a_1 \otimes a_2 \otimes a_3) \\
&= (a_1 a_2 \otimes 1) \otimes a_3 - (a_1 \otimes 1) \otimes a_2 a_3 + (a_1 \otimes a_3^{\mathrm{op}}) \otimes a_2 \\
&- (a_1 a_2 \otimes 1) \otimes a_3 + (1 \otimes 1) \otimes a_1 a_2 a_3 - (1 \otimes a_3^{\mathrm{op}}) \otimes a_1 a_2 \\
&+ (a_1 \otimes 1) \otimes a_2 a_3 - (1 \otimes 1) \otimes a_1 a_2 a_3 + (1 \otimes (a_2 a_3)^{\mathrm{op}}) \otimes a_1 \\
&- (a_1 \otimes a_3^{\mathrm{op}}) \otimes a_2 + (1 \otimes a_3^{\mathrm{op}}) \otimes a_1 a_2 - (1 \otimes (a_3^{\mathrm{op}} \odot a_2^{\mathrm{op}})) \otimes a_1 \\
&= 0!
\end{aligned}$$

by cancellation.

The explicit homotopy of the resolutions M' and M is given by $F : M' \to M$ defined as

$$(F\omega)(a, b, x_1, \cdots, x_k) = \langle X(x_1, b) \wedge \cdots \wedge X(x_k, b), \omega(a, b)\rangle$$

for $\omega \in M'_k = C^\infty(V^2, E_k)$ and $a, b, x_1, \cdots, x_k \in V$, so that $F\omega \in M_k = (C^\infty(A) \otimes C^\infty(V)^{\odot}) \otimes C^\infty(V^k)$.

For any given cyclic cocycle $\varphi \in cH^q(C^\infty(V))$, the following formula

$$\varphi = \omega_0 + \sum_{j \geq 1} \omega_j \in cH^q(C^\infty(V))$$

with $\omega_0 = S^j \omega_j$ (corrected), is yielded by explicit closed currents ω_j of dimension $q - 2j$, by working out the homotopy formulae explicitly.

Let W be a submanifold of V. Let $i^* : C^\infty(V) \to C^\infty(W)$ be the restriction map by the inclusion map $i : W \to V$. There is the corresponding exact sequence of algebras:

$$0 \to \ker(i^*) \to C^\infty(V) \xrightarrow{i^*} C^\infty(W) \to 0.$$

For the ordinary homology groups, we have the long exact sequence:

$$\cdots \to H_q(W) \xrightarrow{i_*} H_q(V) \to H_q(V, W) \xrightarrow[-1]{\partial} H_{q-1}(W) \to \cdots$$

with the connecting map ∂ of degree -1.

As well, from a cochain complex:

$$\begin{aligned}
0 \longrightarrow \quad & cC^n(C^\infty(W)) \xrightarrow{(i^*)^*} cC^n(C^\infty(V)) \\
\longrightarrow \; & cC^n(C^\infty(V), C^\infty(W)) = cC^n(C^\infty(V))/cC^n(C^\infty(W)) \to 0
\end{aligned}$$

there is the long exact sequence

$$\begin{aligned}
\cdots \to cH^q(C^\infty(W)) & \xrightarrow{(i^*)^*} cH^q(C^\infty(V)) \\
\to cH^q(C^\infty(V), C^\infty(W)) & \xrightarrow[+1]{\partial} cH^{q+1}(C^\infty(W)) \to
\end{aligned}$$

with the connecting map ∂ of degree $+1$.

On the other hand, the connecting map for the long exact sequence of Hochschild cohomology groups is zero, since any current on W (such as in $C^\infty(W)^*$) mapped to zero in V (as in $C^\infty(V)^*$) is zero. Thus, $\mathrm{im}(\partial) = \ker((i^*)^*)$ is zero. Then the zero $\mathrm{im}(\partial)$ may be contained in $cH^{q-1}(C^\infty(W))$.

Only trivial cyclic cocycles on $C^\infty(V)$ extend continuously to the C^*-algebra $C(V)$ of continuous functions on a compact manifold V. In fact, for any compact space X, the continuous Hochschild cohomology of $\mathfrak{A} = C(X)$ with coefficients in the bimodule $\mathfrak{A}^*$ is trivial in dimension ≥ 1. Therefore, the cyclic cohomology of A is given by

$$\begin{cases} cH^{2n}(C(X)) = cH^0(C(X)), \\ cH^{2n+1}(C(X)) = 0. \end{cases}$$

This remark is extended to arbitrary nuclear C^*-algebras.

$\star$ Note that for $\mathfrak{A} = C(X)$,

$$H^1(\mathfrak{A}, \mathfrak{A}^*) = 0 \xrightarrow{B} cH_0(\mathfrak{A}) \xrightarrow[\cong]{S} cH_2(\mathfrak{A}) \xrightarrow{I} 0 = H^2(\mathfrak{A}, \mathfrak{A}^*)$$

and for any $n \geq 0$ and in particular $n = 2j$ and $n = 2j+1$,

$$H^{n+1}(\mathfrak{A}, \mathfrak{A}^*) = 0 \xrightarrow{B} cH_n(\mathfrak{A}) \xrightarrow[\cong]{S} cH_{n+2}(\mathfrak{A}) \xrightarrow{I} 0 = H^{n+2}(\mathfrak{A}, \mathfrak{A}^*)$$

Moreover,

$$cH_{-1}(\mathfrak{A}) = 0 \xrightarrow[0]{S^\sim} cH_1(\mathfrak{A}) \xrightarrow{I} 0 = H^1(\mathfrak{A}, \mathfrak{A}^*),$$

which should hold in this sense with $S^\sim$ or another, only by I.

The next example (γ)

Let X be a topological space and G a compact group acting continuously on X.

The G-equivariant cohomology H_G^* of X is defined as the cohomology of the homotopy quotient $X_G = X \times_G EG$, where EG is the total space of the universal principal G-bundle over the classifying space BG.

In particular, $H_G^*(X)$ is a module over $H_G^*(\text{point}) = H^*(BG)$ in a natural manner.

Example 3.5. Let $G = \mathbb{T} = S^1$ the 1-dimensional torus. Then $BS^1 = P_\infty(\mathbb{C})$ and $H_{S^1}^*(\{p\}) = H^*(BS^1) = H^*(P_\infty(\mathbb{C}))$ is a polynomial ring in one generator of degree 2.

The formal analogy between cyclic cohomology and S^1-equivariant cohomology is given by the equality $B\Lambda = BS^1$, where Λ is the small category which governs cyclic cohomology by the equality $cH^*(A) = \mathrm{Ext}^*_\Lambda(A^\natural, \mathbb{C}^\natural)$ where the function $\natural$ from algebra A to Λ-modules $A^\natural$ gives the appropriate linearization of the non-abelian category of algebras (cf. [11]).

$\star$ The cyclic category Λ is the small category with one object Λ_n for each $n \in \mathbb{N}$ and with morphisms $f \in \mathrm{Hom}(\Lambda_n, \Lambda_m)$ as the homotopy classes of continuous increasing (in some sense) maps φ from S^1 to S^1, of degree one and such that $\varphi(\mathbb{Z}_{n+1}) \subset \mathbb{Z}_{m+1}$, with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

$\star$ There is a covariant functor from Λ to k-vector spaces. Let A be a unital algebra over a field k. For each $f \in \mathrm{Hom}(\Lambda_n, \Lambda_m)$, the action $f_\natural$ is given by

$$f_\natural(x_0 \otimes \cdots \otimes x_n) = y_0 \otimes \cdots \otimes y_m$$

for $x_0 \otimes \cdots \otimes x_n \in \otimes^{n+1} A = A_n^\natural$, where $y_j = \Pi_{l \in f^{-1}(j)} x_l$. This covariant functor defines a $k(\Lambda)$-module whose underlying vector space is $\oplus_{n=0}^\infty A_n^\natural$, which is denoted by $A^\natural$.

Let Γ be a countable discrete group and $A = \mathbb{C}\Gamma$ the group ring of Γ as an algebra over $\mathbb{C}$.

It follows from the following theorem of D. Burghelea [7] that there is a natural S^1-space, the S^1-equivariant cohomology with complex coefficients for which is the cyclic cohomology of A.

Recall that the free loop space Y^{S^1} of a topological space Y is defined to be the space $C(S^1, Y)$ of continuous maps from S^1 to Y with the compact-open topology.

$\star$ The compact-open topology for $C(S^1, Y)$ is generated by the subsets $V(K, U)$ of functions f of Y^{S^1} for compact subsets K of S^1 and open subsets U of Y such that $f(K) \subset U$, as sub-open-basis for the topology, as $\cap_{j=1}^n V(K_j, U_j)$ as open subsets in which.

This map space is an S^1-space by the natural action of the group S^1 on the domain S^1 by rotations.

Theorem 3.6. ([7]). *Let Γ be a discrete group and $A = \mathbb{C}\Gamma$.*

(a) *The Hochschild cohomology $H^*(A, A^*)$ is isomorphic to the cohomology $H^*((B\Gamma)^{S^1}, \mathbb{C})$ of the free loop space $(B\Gamma)^{S^1}$ of the classifying space $B\Gamma$.*

(b) *Thy cyclic cohomology $cH^*(A)$ is isomorphic to the S^1-equivariant cohomology $H^*_{S^1}((B\Gamma)^{S^1}, \mathbb{C})$.*

Moreover, the isomorphism in (b) is compatible with the module structure over $H^*(B\Lambda) = H^*(BS^1)$. As well, the long exact sequence by B-S-I under the isomorphism in (a) and (b) becomes the Gysin exact sequence relating cohomology H^* to equivariant cohomology $H^*_{S_1}$.

As the following corollary, obtained is that the cyclic cohomology of $\mathbb{C}\Gamma$ is computed in terms of the cohomology of the subgroups of Γ as follows.

For any $g \in \Gamma$, let $C_g = \{h \in \Gamma \mid gh = hg\}$ be the centralizer of g in Γ.

For any $g \in \Gamma$, let $N_g = C_g / g^{\mathbb{Z}}$ be the quotient of C_g by the central subgroup $g^{\mathbb{Z}} = \langle g \rangle$ generated by g.

Let $\langle \Gamma \rangle$ be the set of conjugacy classes of Γ.

$\star$ Namely,

$$\langle \Gamma \rangle = \Gamma / \mathrm{Inn}(\Gamma)$$

where $\mathrm{Inn}(\Gamma)$ means the group of inner automorphisms of Γ defined as $\mathrm{ad}(h)(g) = [h, g] = hgh^{-1}$ for $g, h \in \Gamma$.

Let $\langle\Gamma\rangle_f$ be the set of conjugacy classes of elements of Γ of finite order and $\langle\Gamma\rangle_f^c$ its complement in $\langle\Gamma\rangle$.

The groups C_g and N_g only depend upon the conjugacy class $[g] \in \langle\Gamma\rangle$ of $g \in \Gamma$.

$\star$ Note that if $k \in [g]$, then there is $x \in \Gamma$ such that $k = xgx^{-1}$. If $h \in C_g$, then $k(xhx^{-1}) = xghx^{-1}$ and $(xhx^{-1})k = xhgx^{-1}$, and thus $xhx^{-1} \in C_k$. Hence, $h \in \mathrm{ad}(x^{-1})C_k$.

Corollary 3.7. ([7]). *It is obtained that*

$$H^*(\mathbb{C}\Gamma, (\mathbb{C}\Gamma)^*) \cong \Pi_{[g]\in\langle\Gamma\rangle} H^*(C_g, \mathbb{C}).$$

Moreover,

$$cH^*(\mathbb{C}\Gamma) \cong \Pi_{[g]\in\langle\Gamma\rangle_f}(H^*(N_g, \mathbb{C}) \otimes cH^*(\mathbb{C})) \times \Pi_{[g]\in\langle\Gamma\rangle_f^c} H^*(N_g, \mathbb{C}).$$

The structure of $cH^*(\mathbb{C})$ as an $cH^*(\mathbb{C})$-module is the decomposition over the finite conjugacy classes, with the operator S on $cH^*(\mathbb{C})$, and another structure is the decomposition over the ininite conjugacy classes, with the operator S the product given by the 2-cocycle $\omega_g \in H^2(N_g, \mathbb{C})$ of the central extension:

$$0 \to \mathbb{Z} = \langle g\rangle \to C_g \to N_g \to 1$$

where there are the inclusion $\mathbb{Z} \subset \mathbb{C}$ and the map $H^2(N_g, \mathbb{Z}) \to H^2(N_g, \mathbb{C})$ used.

In particular, the infinite conjugacy classes may contribute non-trivially to the periodic cyclic cohomology $H^*(\mathbb{C}\Gamma)$.

For the proof of the theorem above, with $A = \mathbb{C}\Gamma$, the associated cyclic vector space $c(A)$ is defined to be the linear space of the following cyclic set (Y_n, d_n^j, s_n^j, t_n), whose faces, degeneracies, and permutations are given respectively by the maps of which,

$$\begin{aligned}
&Y_n = \Gamma^{n+1} = \{(g_0, g_1, \cdots, g_n) \,|\, g_0, \cdots, g_n \in \Gamma\},\\
&d_n^j(g_0, \cdots, g_n) = (g_0, \cdots, g_j g_{j+1}, \cdots, g_n) \in Y_{n-1}, \quad 0 \le j \le n-1,\\
&d_n^n(g_0, \cdots, g_n) = (g_n g_0, g_1, \cdots, g_{n-1}) \in Y_{n-1},\\
&s_n^j(g_0, \cdots, g_n) = (g_0, \cdots, g_j, 1, g_{j+1}, \cdots, g_n) \in Y_{n+1},\\
&t_n(g_0, \cdots, g_n) = (g_n, g_0, \cdots, g_{n-1}) \in Y_n.
\end{aligned}$$

It then follows that the Hochschild and cyclic H cohomology of $A = \mathbb{C}\Gamma$ is the cohomology and S^1-equivariant cohomology of the geometric realization $|Y|$ of Y. Namely,

$$H^*(A, A^*) = H^*(|Y|, \mathbb{C}), \quad cH^*(A) = H^*_{S^1}(|Y|)$$

where the cyclic structure of Y allows $|Y|$ endowed with a canonical action of S^1.

$\star$ The geometric realization $|X|$ of a simplicial set $X = \cup_{n\ge0} X_n$ is the quotient of the topological space $X \times \Delta$:

$$X \times_\Delta \Delta = \cup_{n\ge0}(X_n \times_{\Delta_n} \Delta_n)$$

by the equivalence relation which identifies $(x, \alpha_*(y))$ with $(\alpha^*(x), y)$ for any morphism α of Δ, where Δ is the small category whose objects are the totally ordered finite sets $\{0 < 1 < \cdots < n\} = \Delta_n$, $n \in \mathbb{N}$ and whose morphisms are the increasing maps.

The S^1-space $|Y|$ is S^1-equivariantly homeomorphic to the space $C_{S^1}(\Gamma) = cf(S^1, \Gamma)$ of Γ-valued configurations of the oriented circle.

A Γ-valued configuration on S^1 is a map $\alpha : S^1 \to \Gamma$ such that $\alpha(\theta) = 1_\Gamma$ except on, a finite subset of S^1, called the support denoted by $\mathrm{supp}(\alpha)$.

The topology of the configuration space $cf(S^1, \Gamma)$ is generated by the open sets

$$\begin{aligned}&U(I_1, \cdots, I_k, g_1, \cdots, g_k)\\&\quad = \{\alpha \in cf(S^1, \Gamma) \,|\, \mathrm{supp}(\alpha) \subset \cup_{j=1}^k I_j, \Pi_{\theta \in I_j} \alpha(\theta) = g_j \in \Gamma (1 \le j \le k)\},\end{aligned}$$

where I_j are open intervals of S^1, and the product is the time-ordered product of the values of α at the times $\theta_1 < \cdots < \theta_k$, where $\mathrm{supp}(\alpha) \cap I_j = \{\theta_1, \cdots, \theta_k\}$.

There is the natural homeomorphism $h : |Y| \to cf(S^1, \Gamma)$ defined as follows.

At the set theoretic level, we have the decomposition

$$|Y| = \cup_{n=0}^\infty (Y_n \setminus \deg(Y_n)) \times \mathrm{int}(\Delta_n),$$

where the degeneracy $\deg(Y_n)$ is the union of the images of the maps $s_{n-1}^i : Y_{n-1} \to Y_n$, and $\mathrm{int}(\Delta_n)$ is the interior of the n-simplex

$$\Delta_n = (\lambda_0, \cdots, \lambda_n) \,|\, \lambda_i \ge 0 (0 \le i \le n), \sum_{i=0}^n \lambda_i = 1\}.$$

$\star$ Note that $\Delta_0 = \{1\}$, $\mathrm{int}(\Delta_0) = \emptyset$, and

$$\Delta_1 = \{(\lambda_0, \lambda_1) \,|\, \lambda_0, \lambda_1 \ge 0, \lambda_0 + \lambda_1 = 1\},$$

which is homeomorphic to the closed interval $[0, 1]$, so that $\mathrm{int}(\Delta_1) \approx (0, 1)$ the open interval. As well, Δ_2 is homeomorphic to 2-dimensional closed ball, and hence $\mathrm{int}(\Delta_2) \approx (0, 1)^2$.

For $((g_i)_{i=0}^n, (\lambda_i)_{i=0}^n) \in (Y_n \setminus \deg(Y_n)) \times \mathrm{int}(\Delta_n)$, the image under h is the configuration α with support contained in the set

$$\{0, \lambda_0, \lambda_0 + \lambda_1, \cdots, \lambda_0 + \cdots + \lambda_{n-1}\}$$

and such that

$$\alpha(0) = g_0, \alpha(\lambda_0) = g_1, \alpha(\sum_{k=0}^{i-1} \lambda_k) = g_i (2 \le i \le n).$$

Since $(g_i) \in Y_n \setminus \deg(Y_n)$, we have $g_i \ne 1_\Gamma$ for $i \ne 0$, but g_0 may be 1_Γ. This is the case where the cardinality of the support of the configuration α is equal to n.

It is then checked that the map $h : |Y| \to cf(S^1, \Gamma)$ is an S^1-equivariant homeomorphism.

Proposition 3.8. (due to J. Milnor and G. Segal). *The configuration space $cf(S^1,\Gamma)$ is S^1-equivariantly weakly homotopy equivalent to the free loop space $(B\Gamma)^{S^1} = C(S^1, B\Gamma)$.*

That theorem above follows from this proposition.

Let $I = S^1 \setminus \{0\} \subset S^1 = \mathbb{R}/\mathbb{Z}$ be the open interval. Then $cf(I,\Gamma)$ is canonically homeomorphic to $B\Gamma$.

The above weak homotopy equivalence J associated to a configuration $\alpha \in cf(S^1,\Gamma)$ the loop $\beta = J(\alpha)$, where $\beta(\theta)$ for θ is equal to the restriction to I of $\alpha \circ R_\theta$, that is in $cf(I,\Gamma) \approx B\Gamma$, where R_θ is the rotation defined by $R_\theta(t) = t+\theta$ for $t \in S^1 = \mathbb{R}/\mathbb{Z}$.

The classifying space $B\Gamma$ is defined to be the geometric realization $|X|$ of the simplicial set $X = \cup_{n\geq 1} X_n$, given by

$$\begin{aligned}
&X_n = \Gamma^n = \{(g_1,\cdots,g_n)\,|\,g_1,\cdots,g_n \in \Gamma\},\\
&d_n^0(g_1,\cdots,g_n) = (g_2,\cdots,g_n), \quad d_n^n(g_1,\cdots,g_n) = (g_1,\cdots,g_{n-1}) \in X_{n-1}\\
&d_n^i(g_1,\cdots,g_n) = (g_1,\cdots,g_ig_{i+1},\cdots,g_n) \in X_{n-1}, \quad 1 \leq i \leq n-1,\\
&s_n^i(g_1,\cdots,g_n) = (g_1,\cdots,g_i,1_\Gamma,g_{i+1},\cdots,g_n) \in X_{n+1}, \quad 0 \leq i \leq n.
\end{aligned}$$

The natural homeomorphism $cf(I,\Gamma) \approx |X|$ associates to a configuration α on $I = (0,1)$ with $\mathrm{supp}(\alpha) = \{t_1,\cdots,t_n\}$ with $t_i < t_{i+1}$ and $\alpha(t_i) = g_i$, the following element of $(X_n \setminus \deg(X_n)) \times \mathrm{int}(\Delta_n)$:

$$(g_1,\cdots,g_n) \times (t_1, t_2-t_1, t_3-t_2,\cdots,t_n-t_{n-1}, 1-t_n).$$

$\star$ Note that $t_1 > 0$, $t_{i+1} - t_i > 0$, $1 - t_n > 0$, and

$$t_1 + (t_2-t_1) + (t_3-t_2) + \cdots + (t_n - t_{n-1}) + (1-t_n) = 1.$$

For the proof of the proposition above, we compares the two row fibrations in the following diagram:

$$\begin{array}{ccccc}
\Gamma & \longrightarrow & cf(S^1,\Gamma) \approx |Y| & \xrightarrow{\text{res}} & cf(I = S^1 \setminus \{0\},\Gamma)\\
\big\downarrow & & {\scriptstyle J}\big\downarrow & & \big\downarrow\\
\Omega(B\Gamma) = (B\Gamma)^{[0,1],0,1,\star} & \longrightarrow & (B\Gamma)^{S^1} = C(S^1,B\Gamma) & \xrightarrow[\text{at 0 and 1}]{\text{ev}} & B\Gamma \approx |X|
\end{array}$$

where for $\alpha \in cf(S^1,\Gamma)$, its restriction $\mathrm{res}(\alpha)$ to I determines α up to the value $\alpha(0) \in \Gamma$, as the first row fibration. In the second row fibration, $\Omega B\Gamma$ is the loop space of $B\Gamma$, that is, $C([0,1],B\Gamma)_{0,1,\star}$ of all continuous closed paths on $B\Gamma$ with the same point $\star$ at 0 and 1 fixed. The vertical side arrows $cf(I,\Gamma) \to B\Gamma$ and $\Gamma \to \Omega B\Gamma$ are weak homotopy equivalences, and thus so is the middle arrow J.

The last example (δ)

Let V be a smooth manifold and Γ a discrete group acting on V by diffeomorphisms. Let $A = C_c^\infty(V) \rtimes \Gamma$ the crossed product algebra. The cyclic cohomology of A is described in the following.

The analogues of the above results of Burghelea are obtained by Feigin and Tsygan [15], Nistor [26], and Brylinski [5], [6]. It is obtained by Connes [12] the periodic cyclic cohomology $H^*(A)$ contains the twisted cohomology groups $H^*_\tau(V_\Gamma, \mathbb{C})$ as a direct factor.

Let V_Γ be the homotopy quotient $V \times_\Gamma E\Gamma$, where $E\Gamma \to B\Gamma$ is the universal principal Γ-bundle over the classifying space $B\Gamma$, with Γ as fibers. Let τ be the real vector bundle on V_Γ associated to the Γ-equivariant tangent bundle TV of V.

The crossed product algebra $A = C_c^\infty(V) \rtimes \Gamma$ is defined to be the convolution algebra $C_c^\infty(V \times \Gamma)*$ of smooth functions with compact support on $V \times \Gamma$. The convolution product is defined by

$$(f_1 f_2)(x, q) = (f_1 * f_2)(x, g) = \sum_{g_1 g_2 = g} f_1(x, g_1) f_2(x g_1, g_2),$$

for $f_1, f_2 \in A$, $(x, g) \in V \times \Gamma$.

$\star$ Note that since $g_2 = g_1^{-1} g$ and $g_1 = h \in \Gamma$ is arbitrary,

$$(f_1 * f_2)(x, g) = \sum_{h \in \Gamma} f_1(x, h) f_2(xh, h^{-1} g).$$

Associativity is checked as

$$\begin{aligned}
&((f_1 * f_2) * f_3)(x, g) = \sum_{h \in \Gamma} (f_1 * f_2)(x, h) f_3(xh, h^{-1} g) \\
&= \sum_{h \in \Gamma} \sum_{k \in \Gamma} f_1(x, k) f_2(xk, k^{-1} h) f_3(xh, h^{-1} g), \\
&(f_1 * (f_2 * f_3))(x, g) = \sum_{k \in \Gamma} f_1(x, k)(f_2 * f_3)(xk, k^{-1} g) \\
&= \sum_{k \in \Gamma} f_1(x, k) \sum_{h \in \Gamma} f_2(xk, h) f_3(xkh, h^{-1} k^{-1} g), \quad kh = l \in \Gamma, \\
&= \sum_{k \in \Gamma} f_1(x, k) \sum_{l \in \Gamma} f_2(xk, k^{-1} l) f_3(xl, l^{-1} g),
\end{aligned}$$

so that $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$ holds.

The group Γ acts on V from the right so that the action $V \times \Gamma \ni (x, g) \mapsto xg \in V$ satisfies that $x(g_1 g_2) = (x g_1) g_2$. The action is free if each action by $g \in \Gamma$, $g \neq 1$ has no fixed point. The action is proper if the map from $V \times \Gamma$ to $V \times V$ defined by $(x, g) \mapsto (x, xg)$ is proper.

If Γ acts on V, to be free and proper, then the quotient space $X = V/\Gamma$ is Hausforff and is a manifold of dimension equal to $\dim V$. In this case, the crossed product C^*-algebra $C_0(V) \rtimes \Gamma$ is strongly Morita equivalent to $C_0(X)$, due to Rieffel [28].

When V is compact, for $g \in G$, define the element $u_g \in C_c^\infty(V \times \Gamma)*$ by

$$u_g(x, k) = \begin{cases} 0, & k \neq g, \\ 1, & k = g \end{cases} \qquad x \in V, k \in \Gamma.$$

When V is not compact, this element is not defined in A, but in a multiplier of A.

In both cases, any element f of A can be uniquely written as a finite sum

$$f = \sum f_g u_g, \quad f_g \in C_c^\infty(V).$$

$\star$ Since Γ is a discrete group, then a compact subset of Γ is a finite set. There is the following algebraic rule

$$(u_g h u_{g^{-1}})(x) = h(xg), \quad h \in C_c^\infty(V), x \in V, g \in \Gamma.$$

$\star$ Note that for $h \in C_c^\infty(V \times \Gamma)$,

$$\begin{aligned}
u_g(hu_{g^{-1}})(x,s) &= \sum_{h\in\Gamma} u_g(x,h)(hu_{g^{-1}})(xh, h^{-1}s) \\
&= u_g(x,g)(hu_{g^{-1}})(xg, g^{-1}s) \\
&= 1 \sum_{k\in\Gamma} h(xg,k) u_{g^{-1}}(xgk, k^{-1}g^{-1}s), \quad k^{-1}g^{-1}s = g^{-1}, \\
&= h(xg, g^{-1}sg) u_{g^{-1}}(xsg^{-1}, g^{-1}) \\
&= h(xg, g^{-1}sg).
\end{aligned}$$

Since the homotopy quotient V_Γ is the geometric realization of a simplicial manifold, the twisted cohomology $H_\tau^*(V_\Gamma)$ can be described as the cohomology of a double complex [2, Theorem 4.5].

More explicitly, the space $E\Gamma$ can be viewed as the geometric realization of the simplicial set Γ^∞, where $\Gamma_n^\infty = \Gamma^{n+1}$ with

$$\begin{aligned}
d_i(g_0, \cdots, g_n) &= (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n) \in \Gamma_{n-1}^\infty, \\
s_j(g_0, \cdots, g_n) &= (g_0, \cdots, g_j, g_j, \cdots, g_n) \in \Gamma_{n+1}^\infty.
\end{aligned}$$

The group Γ acts on both V and Γ^∞ from the right. It then follows from [2, Theorem 4.5] that the τ-twisted cohomology $H_\tau^*(V_\Gamma, \mathbb{C})$ is the cohomology of the bicomplex of Γ-invariant simplicial τ-twisted forms on $V \times \Gamma^\infty$.

A twisted form on V is now viewed as the same thing as a smooth de Rham current in the following sense. To the twisted form ω, associated is the smooth current C with values

$$C(\alpha) = \int_V \alpha \wedge \omega, \quad \deg(\alpha) + \deg(\omega) = \dim V$$

for any differential form α with compact support and degree condition above.

The double complex (C^*, d_1, d_2) is used in computing $H_\tau^*(V_\Gamma, \mathbb{C})$ in terms of currents on V rather than twisted forms, to be more convenient to describe.

Let $C^{n,m} = \{0\}$ for $n < 0$ or $m > 0$ or $m < -\dim V$. Otherwise, let $C^{n,m}$ be the space of totally anti-symmetric maps γ from Γ^{n+1} to $\Omega'_{-m}(V)$ the (dual) space of de Rham currents of dimension $-m$ on V, $0 \le -m \le \dim V$, which satisfy

$$\gamma(g_0 g, g_1 g, \cdots, g_n g) = \gamma(g_0, \cdots, g_n) g, \quad g, g_1, \cdots, g_n \in \Gamma.$$

The coboundary $d_1 : C^{n,m} \to C^{n+1,m}$ is given by

$$(d_1\gamma)(g_0, \cdots, g_{n+1}) = (-1)^m \sum_{j=0}^{n+1} (-1)^j \gamma(g_0, \cdots, g_{j-1}, g_{j+1}, \cdots, g_{n+1}).$$

$\star$ Note that for $\gamma \in C^{1,m}$,

$$\begin{aligned} d_1(d_1\gamma)(g_0, g_1, g_2, g_3) &= (-1)^m (d_1\gamma)(g_1, g_2, g_3) - (-1)^m (d_1\gamma)(g_0, g_2, g_3) \\ &\quad + (-1)^m (d_1\gamma)(g_0, g_1, g_3) - (-1)^m (d_1\gamma)(g_0, g_1, g_2) \\ &= (-1)^m \gamma(g_2, g_3) - (-1)^m \gamma(g_1, g_3) + (-1)^m \gamma(g_1, g_2) \\ &\quad - (-1)^m \gamma(g_2, g_3) + (-1)^m \gamma(g_0, g_3) - (-1)^m \gamma(g_0, g_2) \\ &\quad + (-1)^m \gamma(g_1, g_3) - (-1)^m \gamma(g_0, g_3) + (-1)^m \gamma(g_0, g_1) \\ &\quad - (-1)^m \gamma(g_1, g_2) + (-1)^m \gamma(g_0, g_2) - (-1)^m \gamma(g_0, g_1) \\ &= 0! \end{aligned}$$

The coboundary $d_2 : C^{n,m} \to C^{n,m+1}$ is the de Rham (dual) boundary

$$(d_2\gamma)(g_0, \cdots, g_n) = d^t(\gamma(g_0, \cdots, g_n)) = \gamma(g_0, \cdots, g_n) \circ d,$$

where d is the usual differential boundary as

$$d : \Omega_{-(m+1)}(V) \to \Omega_{-(m+1)+1}(V) = \Omega_{-m}(V).$$

$\star$ Note that for $\gamma : \Gamma^{n+1} \to \Omega'_{-m}(V)$,

$$\gamma(g_0, \cdots, g_n) \circ d \in \Omega'_{-m-1}(V) = \Omega'_{-(m+1)}(V).$$

It then follows that

Proposition 3.9. *The twisted cohomology $H^*_\tau(V_\Gamma, \mathbb{C})$ is isomorphic to the cohomology of the bicomplex (C^*, d_1, d_2) above, with d_2 as a shift in dimension of $\dim V$, and with d_1 as an alternative degeneration on Γ^*, where we may let*

$$C^* = C^{*,*} = \mathrm{Map}_{asy}(\Gamma^*, \Omega'_{-*}(V)).$$

This holds without regard to the regularity imposed on the currents.

Note that the filtration of the cohomology of the bicomplex, given by the maximal value of $n - m$ on the support of cocycles $\gamma_{n,m} \in C^{n,m}$ does depend on the regularity.

Example 3.10. Let $V = S^1$, $\Gamma = \mathbb{Z}$, and that Γ acts on V by a diffeomorphism $\varphi \in \mathrm{Dif}^+(S^1)$ (preserving the orientation), with a Liouville rotation number, but not C^∞-conjugate to a rotation (cf. [18]). The homotopy quotient V_Γ is the mapping torus, that is, the quotient of $V \times \mathbb{R}$ by the diffeomorphism $\varphi^\sim$ defined by

$$\varphi^\sim(x, s) = (\varphi(x), s + 1), \quad x \in V = S^1, s \in \mathbb{R}.$$

By construction, it is a 2-torus with S^1 as fibers over $B\Gamma = \mathbb{R}/\mathbb{Z}$. Its cohomotopy group $\pi^1(V_\Gamma)$ has another generator given by a continuous map $V_\Gamma \to V = S^1$. There is the corresponding cocycle in the bicomplex (C^*, d_1, d_2) to be computed. Since the group Γ preserves the orientation, then the twisting by τ is ignored. Thus, there is a canonical map from $\pi^1(V_\Gamma)$ to $H^1_\tau(V_\Gamma, \mathbb{C})$. By using arbitrary currents, the corresponding cocycle is described as follows. It is given by the following element of $C^{0,0} = \mathrm{Map}_{asy}(\Gamma, \Omega'_0(V))$, that is, $\gamma_0 \in \Omega'_0$ the unique Γ-invariant probability measure on $V = S^1$. This measure is zero dimensional, with $d_1\gamma = d_2\gamma = 0$.

$\star$ Note that

$$(d_1\gamma)(g_0, g_1) = \gamma(g_1) - \gamma(g_0).$$

Moreover, it holds by the invariance that

$$\gamma(g_1) = \gamma(g_1)g_1^{-1}g_0 = \gamma(g_1 g_1^{-1} g_0) = \gamma(g_0).$$

Also, $(d_2\gamma)(g_0) = \gamma(g_0) \circ d = 0$, because of the zero dimensionality killing forms with dimension more than zero.

Since φ is not C^∞-conjugate to a rotation, the current γ_0 is not smooth. We have to describe a smooth cocycle γ' in the above bicomplex belonging to the same cohomology class.

Let $\gamma'_0 \in \Omega'_0$ be any smooth 0-dimensional current such that $\langle 1, \gamma'_0 \rangle = 1$. It is not φ-invariant, but the equation

$$\varphi\gamma'_0 - \gamma'_0 = d^t\gamma'_1 = \gamma'_1 \circ d$$

is solved by γ'_1 a smooth 1-dimensional current on $V = S^1$. This yields the desired smooth cocycle γ'.

The current γ_0 gives rise to a trace on the crossed product $A = C^\infty_c(V) \rtimes \Gamma$, while the cocycle γ' gives rise to a cyclic 2-cocycle on A, with the same class in periodic cyclic cohomology.

Described in full generality is a morphism Φ of bicomplexes from (C^*, d_1, d_2) to the (b, B) bicomplex of the algebra A. This implies the desired map from $H^*_\tau(V_\Gamma, \mathbb{C})$ to the periodic cyclic cohomology of A.

The construction of the morphism Φ is applied also to smooth groupoids for which the maps r and s are étale, but in the case of $G = V \rtimes \Gamma$, there are special features due to the total anti-symmetry of the cochains $\gamma(g_0, \cdots, g_n)$.

Let us introduce an auxiliary graded differential algebra exploiting. As an algebra, that is the crossed product $B \rtimes_\alpha \Gamma$, where B is the graded tensor product

$$B = A^*(V) \otimes \wedge^*\mathbb{C}\Gamma' = B_1 \otimes B_2$$

of the graded algebra $A^*(V)$ of smooth compactly supported differential forms on V by the exterior algebra $\wedge^*\mathbb{C}\Gamma'$ of the linear space $\mathbb{C}\Gamma'$ with the basis of δ_g for $g \in \Gamma$, with $\delta_e = 0$ for e the unit of Γ, as $\mathbb{C}\Gamma' = \mathbb{C}_0\Gamma$. The action α of Γ on B by automorphisms is defined by the tensor product such that $\alpha_g = \alpha_{1,g} \otimes \alpha_{2,g}$ for $g \in \Gamma$.

The action α_1 of Γ on $B_1 = A^*(V)$ is the natural action commuting with the differential, defined as

$$\alpha_{1,g}(f)(x) = f(xg), \quad f \in C_c^\infty(V), x \in V, g \in \Gamma$$

and extended.

⋆ For $g_1, g_2 \in \Gamma$, $x \in V$,

$$\begin{aligned}\alpha_{1,g_1}(\alpha_{1,g_2}(f))(x) &= \alpha_{1,g_2}(f)(xg_1) \\ &= f((xg_1)g_2) = f(x(g_1g_2)) = \alpha_{1,g_1g_2}(f)(x).\end{aligned}$$

The action α_2 of Γ on $B_2 = \wedge^*\mathbb{C}\Gamma'$ is given by the equality

$$\alpha_{2,g}\delta_k = \delta_{kg^{-1}} - \delta_{g^{-1}}, \quad g, k \in \Gamma,$$

and it preserves the subspace $\mathbb{C}\Gamma' = \wedge^1\mathbb{C}\Gamma'$.

⋆ For $g_1, g_2, k \in \Gamma$,

$$\begin{aligned}\alpha_{2,g_1}(\alpha_{2,g_2}(\delta_k)) &= \alpha_{2,g_1}(\delta_{kg_2^{-1}} - \delta_{g_2^{-1}}) \\ &= \delta_{kg_2^{-1}g_1^{-1}} - \delta_{g_2^{-1}g_1^{-1}} = \delta_{k(g_1g_2)^{-1}} - \delta_{(g_1g_2)^{-1}} = \alpha_{2,g_1g_2}(\delta_k).\end{aligned}$$

Also, if $g = e$, then $\alpha_{2,e}(\delta_k) = \delta_k - \delta_e = \delta_k$ for any $k \in \Gamma$. If $g = k$, then $\alpha_{2,k}(\delta_k) = \delta_e - \delta_{k^{-1}} = -\delta_{k^{-1}}$. If $\alpha_{2,g}(\delta_k) = 0$, then $kg^{-1} = g^{-1}$. Thus, $k = e$.

Since the action $\alpha = \alpha_1 \otimes \alpha_2$ of Γ on $B = B_1 \otimes B_2$ preserves the bi-grading of B, the crossed product $B \rtimes_\alpha \Gamma$ has a canonical bi-grading.

Any generic element of $B \rtimes_\alpha \Gamma$ is written as a finite sum $\sum_{g\in\Gamma} b_g u_g$ for $b_g \in B$.

The algebra B is endowed with the differential $d = d_1 \otimes d_2$ defined by

$$d(\omega \otimes \varepsilon) = d\omega \otimes \varepsilon, \quad \omega \in B_1 = A^*(V), \varepsilon \in B_2,$$

where $d\omega = d_1\omega$ is the usual differential of forms, and B_2 is endowed with the zero differential $d_2 = 0$.

The differential d of the crossed product algebra $B \rtimes_\alpha \Gamma$ is defined by

$$d(bu_g) = (db)u_g + (-1)^{\deg b} bu_g\delta_g, \quad b \in B, g \in \Gamma,$$

where this is defined for b homogeneous in B and extended by linearity with degrees.

Lemma 3.11. *The crossed product $B \rtimes_\alpha \Gamma$ is a graded differential algebra by the differential d above.*

Proof. By the construction above, $B \rtimes_\alpha \Gamma$ is a bigraded algebra.

It is shown that the two components d_2 and d_1 of d given by

$$d_2(bu_g) = (-1)^{\deg b} bu_g\delta_g, \quad d_1(bu_g) = (db)u_g$$

for $b \in B$ (homogeneous), $g \in \Gamma$ are derivations of $B \rtimes_\alpha \Gamma$ such that

$$d_2^2 = d_1^2 = d_2 d_1 + d_1 d_2 = 0$$

so that $d^2 = 0$.

It is clear that d_1 is a derivation with $d_1^2 = 0$, since $d_1^2(bu_g) = (d^2 b)u_g = 0$.

$\star$ For $b, b' \in B$ and $g, h \in \Gamma$,

$$\begin{aligned} d_1((bu_g)(b'u_h)) &= d_1(b(u_g b' u_{g^{-1}})u_{gh}) \\ &= d(b(u_g b' u_{g^{-1}}))u_{gh} \\ &= d(b)(u_g b' u_{g^{-1}})u_{gh} + bd(u_g b' u_{g^{-1}})u_{gh} \\ &= d_1(bu_g)b'u_h + bu_g u_{g^{-1}} d(u_g b' u_{g^{-1}})u_g u_h. \end{aligned}$$

Moreover, as a possible equation,

$$u_{g^{-1}} d(u_g b' u_{g^{-1}}) u_g u_h = d(b')u_h = d_1(b'u_h).$$

Namely, $\mathrm{Ad}_{g^{-1}} \circ d \circ \mathrm{Ad}_g = d$ is required. That is, $\mathrm{Ad}_g = \alpha_g$ commutes with d, as a question? But it seems that $\alpha_{1,g}(d(f))(x) = d(f)(xg)$ and $d(\alpha_{1,g}(f))(x) = d(f(\cdot g))(x)$ are slightly different?

To check that d_2 is a derivation, we need to show that for $g_1, g_2 \in \Gamma$,

$$d_2(u_{g_1 g_2}) = (d_2 u_{g_1})u_{g_2} + u_{g_1} d_2 u_{g_2}.$$

$\star$ This means that

$$u_{g_1 g_2}\delta_{g_1 g_2} = u_{g_1}\delta_{g_1} u_{g_2} + u_{g_1} u_{g_2} \delta_{g_2}.$$

On the other hand,

$$\begin{aligned} u_{g_1}\delta_{g_1} u_{g_2} &= u_{g_1 g_2} u_{g_2^{-1}} \delta_{g_1} u_{g_2} \\ &= u_{g_1 g_2} \alpha_{g_2^{-1}} \delta_{g_1} = u_{g_1 g_2}(\delta_{g_1 g_2} - \delta_{g_2}), \end{aligned}$$

which shows the derivation rule of d_2 above.

Since $\delta_g^2 = 0$, we have $d_2^2 = 0$.

$\star$ Note that

$$\begin{aligned} d_2^2(bu_g) &= d_2((-1)^{\deg b} bu_g \delta_g) \\ &= (-1)^{\deg b} d_2(b(u_g \delta_g u_{g^{-1}})u_g) \\ &= (-1)^{\deg b} d_2(b(\delta_{gg^{-1}} - \delta_{g^{-1}})u_g) \\ &= (-1)^{1+\deg b}(-1)^{\deg(b\delta_{g^{-1}})} b\delta_{g^{-1}} u_g \delta_g. \end{aligned}$$

Moreover,

$$\delta_{g^{-1}} u_g \delta_g = \delta_{g^{-1}} \alpha_g(\delta_g) u_g = \delta_{g^{-1}}(\delta_{gg^{-1}} - \delta_{g^{-1}}) u_g = 0!$$

Finally,

$$\begin{aligned}(d_2 \circ d_1)(bu_g) &= d_2((db)u_g) = (-1)^{\deg db}(db)u_g\delta_g \\ &= (-1)^{\deg b+1}(db)\alpha_g(\delta_g)u_g = (-1)^{\deg b+2}(db)\delta_{g^{-1}}u_g, \\ (d_1 \circ d_2)(bu_g) &= d_1((-1)^{\deg b} bu_g\delta_g) = (-1)^{\deg b}\alpha_1(b\alpha_g(\delta_g)u_g) \\ &= (-1)^{\deg b}\alpha_1 b(\delta_{gg^{-g}} - \delta_{g^{-1}})u_g) = (-1)^{\deg b+1}\alpha_1(b\delta_{g^{-1}}u_g) \\ &= (-1)^{\deg b+1}(db)\delta_{g^{-1}}u_g,\end{aligned}$$

which implies $d_2 d_1 + d_1 d_2 = 0$. □

Let $\gamma \in C^{n,m} = \mathrm{Map}_{asy}(\Gamma^{n+1}, \Omega_{-m}(V)')$ be a cochain in the bicomplex (C^*, d_1, d_2). We associate to γ a linear form $\gamma^\sim$ on $B \rtimes_\alpha \Gamma$ defined by

$$\gamma^\sim(\omega \otimes (\delta_{g_1} \cdots \delta_{g_n}) = \langle \omega, \gamma(1, g_1, \cdots, g_n)\rangle, \quad g_j \in \Gamma, \omega \in A^{-m}(V) = \Omega_{-m}(V),$$

and $\gamma^\sim(bu_g) = 0$ for $g \neq 1_\Gamma$ or for $b \notin A^{-m}(V) \otimes \wedge^n$.

The relation between the coboundaries d_1 and d_2 of C^* and the derivations d_2 and d_1 of $B \rtimes_\alpha \Gamma$ that are replaced by c_2 and c_1 is given by,

Lemma 3.12. *For $\gamma \in C^{n,m}$, the following holds that for $a_1, a_2, a \in B \rtimes_\alpha \Gamma$,*
(a) $\gamma^\sim(a_1 a_2 - (-1)^{\deg a_1 \deg a_2} a_2 a_1) = (-1)^{\deg a_1}(d_1\gamma)^\sim(a_1 c_2(a_2))$,
(b) $\gamma^\sim(da) = \gamma^\sim(c_1(a)) = (d_2\gamma)^\sim(a)$, where d can be replaced with c_1.

Proof. (a) We may assume that $a_j = b_j u_{g_j}$ with $b_j \in B$ for $j = 1, 2$, and that $g_1 g_2 = 1$, so that $g_2 g_1 = 1$.

⋆ Note that

$$(b_1 u_{g_1})(b_2 u_{g_2}) = b_1(\alpha_{g_1}(b_2)) u_{g_1 g_2}.$$

If $g_1 g_2 \neq 1$, then its value under $\gamma^\sim$ is zero.

Then $a_1 a_2 = b_1 \alpha_{g_1}(b_2)$ and

$$\begin{aligned}(-1)^{\deg a_1 \deg a_2} a_2 a_1 &= (-1)^{\deg a_1 \deg a_2} b_2 \alpha_{g_2}(b_1) \\ &= \alpha_{g_2}(b_1 \alpha_{g_1}(b_2)) = \alpha_{g_2}(b_1)\alpha_{g_1 g_2}(b_2) = \alpha_{g_2}(b_1) b_2\end{aligned}$$

by using graded commutativity of B, with $\deg a_j = \deg b_j$ in this case.

On the other hand,

$$a_1 c_2(a_2) = b_1 u_{g_1}(-1)^{\deg a_2} b_2 u_{g_2}\delta_{g_2} = (-1)^{\deg a_2} b_1 \alpha_{g_1}(b_2)\delta_{g_2}$$

Thus, with $b = b_1 \alpha_{g_1}(b_2))$, it is enough to show that for $g \in \Gamma$, $b \in B$,

$$\gamma^\sim(b - \alpha_g(b)) = (-1)^{\deg b}(d_1\gamma)^\sim(b\delta_g).$$

We may assume that $b = \omega \otimes \delta_{g_1} \cdots \delta_{g_n}$ with $g_j \in \Gamma, \omega \in A^{-m}(V) = \Omega_{-m}(V)$, so that $\deg b = n - m$ (not $n + m$).

We have $\alpha_g(b) = \alpha_{1,g}(\omega) \otimes \alpha_{2,g}(\delta_{g_1} \cdots \delta_{g_n})$ and

$$\alpha_{2,g}(\delta_{g_1} \cdots \delta_{g_n}) = (\delta_{g_1 g^{-1}} - \delta_{g^{-1}}) \cdots (\delta_{g_n g^{-1}} - \delta_{g^{-1}}) = \cdots$$

which can be expanded term by term. The equation claimed above follows from the invariance of γ by Γ-action and the definition of d_1.

$\star$ Consider the case of $b = \omega \otimes \delta_{g_1}$. Then, with $-m = 1 = \deg b$,

$$\begin{aligned}
&\gamma^{\sim}(b - \alpha_g(b)) = \gamma^{\sim}(\omega \otimes \delta_{g_1} - \alpha_g(\omega) \otimes (\delta_{g_1 g^{-1}} - \delta_{g^{-1}})) \\
&= \langle \omega, \gamma(1, g_1) \rangle - \langle \alpha_g(\omega), \gamma(1, g_1 g^{-1}) - \gamma(1, g^{-1}) \rangle, \\
&(d_1\gamma)^{\sim}(b\delta_g) = (d_1\gamma)^{\sim}(\omega \otimes \delta_{g_1}\delta_g) = \langle \omega, (d_1\gamma)(1, g_1, g) \rangle \\
&= \langle \omega, -\gamma(g_1, g) + \gamma(1, g) - \gamma(1, g_1) \rangle.
\end{aligned}$$

Moreover, it should hold that

$$\begin{aligned}
&\langle \alpha_g(\omega), \gamma(1, g_1 g^{-1}) - \gamma(1, g^{-1}) \rangle \\
&= \langle \omega, \gamma(1, g_1 g^{-1})g - \gamma(1, g^{-1})g \rangle \\
&= \langle \omega, \gamma(g, g_1) - \gamma(g, 1) \rangle = \langle \omega, \gamma(g_1, g) - \gamma(1, g) \rangle
\end{aligned}$$

by the symmetry. Hence the case is proved.

(b) We may assume $a = bu_g$ with $g \neq 1$. Then

$$\begin{aligned}
&\gamma^{\sim}(da) = \gamma^{\sim}((db)u_g + (-1)^{\deg b} bu_g\delta_g) = \gamma^{\sim}(c_1(a)) + \gamma^{\sim}(c_2(a)) \\
&= \gamma^{\sim}(\{(db) + (-1)^{\deg b} b\alpha_g(\delta_g)\}u_g) = 0 = 0 + 0.
\end{aligned}$$

On the other hand,

$$(d_2\gamma)^{\sim}(a) = (\gamma \circ d)^{\sim}(a) = \gamma^{\sim}(da).$$

Note that $d = c_1$ on B, while $c_2 = 0$ on B. $\square$

Theorem 3.13. (A general construction of cyclic cocycles of $A = C_c^\infty(V) \rtimes \Gamma$ [12]).

(a) There is the following morphism Φ of the bicomplex (C^*, d_1, d_2) to the (b, B) bicomplex of A. For $\gamma \in C^{n,m} = \mathrm{Map}_{asy}(\Gamma^{n+1}, \Omega'_{-m}(V))$, with $l = n - m + 1$, the $\Phi(\gamma)$ is the l-linear form on A given by

$$\Phi(\gamma)(x_0, \cdots, x_l) = \lambda_{n,m} \sum_{j=0}^{l} (-1)^{j(l-j)} \gamma^{\sim}(dx_{j+1} \cdots dx_l x_0 dx_1 \cdots dx_j)$$

for $x_j \in A$, with $\lambda_{n,m} = \frac{n!}{(l+1)!} = \frac{n!}{(n-m)!}$.

(b) The corresponding map in cohomology groups gives a canonical inclusion map $\Phi_* : H_\tau^*(V_\Gamma, \mathbb{C}) \to H^*(A)$ of $H_\tau^*(V_\Gamma, \mathbb{C})$ as a direct factor of the periodic cyclic cohomology of A.

Proof. (a) We compute $b(\Phi(\gamma))(x_0, \cdots, x_{l+1})$.

Consider the functional ρ_j

$$(x_0, \cdots, x_l) \mapsto \gamma^{\sim}(dx_{j+1} \cdots (dx_l) x_0 dx_1 \cdots dx_j).$$

The Hochschild coboundary of this functional gives

$$(-1)^j \gamma^{\sim}(a_j x_{j+1} - x_{j+1} a_j), \quad a_j = dx_{j+2} \cdots (dx_{l+1}) x_0 dx_1 \cdots dx_j.$$

$\star$ Note that for $j = 1$, $l = 1$ for ρ_1,

$$\begin{aligned}
&(b\rho_1)(x_0, x_1, x_2) = \rho(x_0 x_1, x_2) - \rho(x_0, x_1 x_2) + \rho(x_2 x_0, x_1) \\
&= \gamma^{\sim}(x_0 x_1 dx_2) - \gamma^{\sim}(x_0 d(x_1 x_2)) + \gamma^{\sim}(x_2 x_0 dx_1) \\
&= -\gamma^{\sim}(x_0 (dx_1) x_2) + \gamma^{\sim}(x_2 x_0 dx_1) \\
&= (-1)^j \gamma^{\sim}((x_0 dx_1) x_2 - x_2 (x_0 dx_1)), \quad \text{with } x_0 dx_1 = a_1.
\end{aligned}$$

Therefore, it follows from the lemma (a) above that

$$b(\Phi(\gamma))(x_0, \cdots, x_{l+1}) = (-1)^l \lambda_{n,m} \sum_{j=0}^{l} (-1)^{j(l-j+1)} (d_1\gamma)^{\sim}(a_j c_2(x_{j+1})).$$

Since $d_1^2 = 0$, $(d_1\gamma)^{\sim}$ is a graded trace on $B \rtimes_\alpha \Gamma$ by the lemma above. We can rewrite the equation above as

$$b\Phi(\gamma)(x_0, \cdots, x_{l+1}) = \lambda_{n,m} \sum_{j=0}^{l} (d_1\gamma)^{\sim}(x_0 dx_1 \cdots dx_j c_2(x_{j+1}) dx_{j+2} \cdots dx_{l+1}).$$

By using $dx_k = c_2(x_k) + c_1(x_k)$, consider the product

$$x_0 dx_1 \cdots dx_{l+1} = x_0 (c_2(x_1) + c_1(x_1)) \cdots (c_2(x_{l+1}) + c_1(x_{l+1}))$$

and the terms with c_2 appearing $n + 1$ times. Then

$$b\Phi(\gamma)(x_0, \cdots, x_{l+1}) = (n + 1)\lambda_{n,m} (d_1\gamma)^{\sim}(x_0 dx_1 \cdots dx_{l+1}).$$

$\star$ Note that

$$\begin{aligned}
&(d_1\gamma)^{\sim}(x_0 dx_1) = (d_1\gamma)^{\sim}(x_0 (c_2(x_1) + c_1(x_1)) \\
&= (d_1\gamma)^{\sim}(x_0 (c_2(x_1)) + (d_1\gamma)^{\sim}(x_0 c_1(x_1)).
\end{aligned}$$

Since $(d_1\gamma)^{\sim}$ is a graded trace on $B \rtimes_\alpha \Gamma$, we have

$$\Phi(d_1\gamma)(x_0, \cdots, x_{l+1}) = \lambda_{n+1,m} (l + 2)(d_1\gamma)^{\sim}(x_0 dx_1 \cdots dx_{l+1}).$$

$\star$ With $l = n - m + 1$,

$$\lambda_{n+1,m}(l+2) = \frac{(n+1)!}{((n+1) - m + 2)!}(n - m + 3) = (n+1)\frac{n!}{(l+1)!} = (n+1)\lambda_{n,m}.$$

It then follows that $\Phi(d_1\gamma) = b\Phi(\gamma)$.

$\star$ Note that it is shown above that the following diagram commutes:

$$\begin{tikzcd} C^{n,m} \arrow[r, "\Phi"] \arrow[d, "d_1"'] & C^{l=n-m+1} \arrow[d, "b"] \\ C^{n+1,m} \arrow[r, "\Phi"] & C^{l+1=(n+1)-m+1} \end{tikzcd}$$

We next compute $B\Phi(\gamma)$. We have

$$B_0\Phi(\gamma)(x_0, \cdots, x_{l-1}) = \lambda_{n,m} \sum_{j=0}^{l} (-1)^{j(l-j)} \gamma^{\sim}(dx_j \cdots dx_{l-1} dx_0 \cdots dx_{j-1}).$$

$\star$ Recall that

$$B_0\Phi(\gamma)(x_0, \cdots, x_{l-1}) = \Phi(\gamma)(1, x_0, \cdots, x_{l-1}) - (-1)^l \Phi(\gamma)(x_0, \cdots, x_{l-1}, 1).$$

Hence, the second term is zero since $d1 = 0$, and x_0 is replaced by 1 in the first term for $\Phi(\gamma)$.

Thus we have

$$B\Phi(\gamma)(x_0, \cdots, x_{l-1}) = (l+1)\lambda_{n,m} \sum_{j=0}^{l-1} (-1)^{j(l-1)} \gamma^{\sim}(dx_j \cdots dx_{l-1} dx_0 \cdots dx_{j-1})$$

since $B = P_c B_0$.

By the lemma (b) above, we have the above equation equal to

$$(l+1)\lambda_{n,m} \sum_{j=0}^{l-1} (-1)^{(j-1)(l-j)} (d_2\gamma)^{\sim}(dx_j \cdots (dx_{l-1}) x_0 dx_1 \cdots dx_{j-1}).$$

Therefore, $B\Phi(\gamma) = \Phi(d_2\gamma)$ is obtained.

$\star$ Note that it is shown above that the following diagram commutes:

$$\begin{tikzcd} C^{n,m} \arrow[r, "\Phi"] \arrow[d, "d_2"'] & C^{l=n-m+1} \arrow[d, "B"] \\ C^{n,m+1} \arrow[r, "\Phi"] & C^{l-1=n-(m+1)+1} \end{tikzcd}$$

(b) A natural retraction $\lambda : H^*(A) \to H^*_\tau(V_\Gamma, \mathbb{C})$ using localization may be described in the context of foliations.

The conclusion then follows from the equality $\lambda \circ \Phi_* = \mathrm{id}$.

$\star$ Note that the following is commutative:

$$\begin{tikzcd} H^*_\tau(V_\Gamma, \mathbb{C}) \arrow[r, "\Phi_*"] \arrow[d, "\mathrm{id}"'] & H^*(A) \arrow[d, equal] \\ H^*_\tau(V_\Gamma, \mathbb{C}) & H^*(A) \arrow[l, "\lambda"'] \end{tikzcd}$$

$\square$

Remark 3.14. (a) In the case of V as a point, the above construction gives a cycle $((\wedge^*\mathbb{C}\Gamma') \rtimes \Gamma, d, \gamma^\sim)$ on the algebra $A = \mathbb{C}\Gamma$, for any group cocycle $\gamma \in Z^n(\Gamma, \mathbb{C})$ represented by a totally anti-symmetric right invariant cochain $\gamma : \Gamma^{n+1} \to \mathbb{C}$ with $d_1\gamma = 0$. The algebra $(\wedge^*\mathbb{C}\Gamma') \rtimes \Gamma$ is a non-trivial quotient of the universal differential algebra $\Omega\mathbb{C}\Gamma$.

(b) There is the analogue of the above construction to be explained below, in the general case of smooth groupoids G such that r and s are étale maps.

The bicomplex (C^*, d_1, d_2) is now the bicomplex of twisted differential forms on the simplicial manifold $\mathrm{Mr}(G)$ which is the nerve of the small category G.

$\star$ The nerve $\mathrm{Mr}(C)$ of a small category C is a simplicial set, elements of which are the composable n-tuples of morphisms belonging to C, where the faces d_i and the degeneracies s_j are obtained by using composition of adjacent morphisms and the identity morphism. Namely,

$$\begin{aligned} &d_0(f_1, \cdots, f_n) = (f_2, \cdots, f_n), \quad d_n(f_1, \cdots, f_n) = (f_1, \cdots, f_{n-1}), \\ &d_i(f_1, \cdots, f_n) = (f_1, \cdots, f_i f_{i+1}, \cdots, f_n), \quad (1 \leq i \leq n-1), \quad \text{and} \\ &s_i(f_1, \cdots, f_n) = (f_1, \cdots, f_i, 1, f_{i+1}, \cdots, f_n). \end{aligned}$$

The classifying space BC of a small category C is defined to be the geometric realization of the simplicial set $\mathrm{Mr}(C)$.

This is applied to discrete groups viewed as small categories with a single object.

Described is the bicomplex in terms of currents, so that $C^{n,m}$ is the space of de Rham currents of dimension $-m$ on the manifold

$$G^{(n)} = \{(\gamma_1, \cdots, \gamma_n) \in G^n \,|\, s(\gamma_i) = r(\gamma_{i+1}), \quad 1 \leq i \leq n-1\}.$$

The first coboundary d_1 is given by the simplicial $d_1 = (-1)^m \sum_{j=0}^n (-1)^j d_j^*$, where

$$\begin{aligned} &d_0(\gamma_1, \cdots, \gamma_n) = (\gamma_2, \cdots, \gamma_n), \quad d_n(\gamma_1, \cdots, \gamma_n) = (\gamma_1, \cdots, \gamma_{n-1}), \\ &d_j(\gamma_1, \cdots, \gamma_n) = (\gamma_1, \cdots, \gamma_j\gamma_{j+1}, \cdots, \gamma_n), \quad (1 \leq j \leq n-1). \end{aligned}$$

Note that in the formula above, currents are pulled back. This is possible because of only using étale maps.

The second coboundary d_2 is the de Rham $d^t = \circ d$, transposed as in the formula before.

We may restrict to the normalized subcomplex of currents which vanish if any γ_j is a unit of G the groupoid, of if $\gamma_1 \cdots \gamma_n$ is a unit.

There is the analogue of the bigraded differential algebra $(B \rtimes_\alpha \Gamma, c_2, c_1)$.

As a linear space, the space $C^{n,m}$ of elements of $(\wedge^*\mathbb{C}\Gamma') \rtimes \Gamma$ of bidegree (n, m) corresponds to the quotient space of the space of compactly supported smooth differential forms of degree m on $G^{(n+1)}$ by the subspace of such forms with support in the set

$$\{(\gamma_0, \cdots, \gamma_n) \in G^{n+1} \,|\, \gamma_j \text{ is a unit of } G \text{ for some } j \neq 0\}.$$

The differential c_1 in this case is given by the ordinary differential of forms.

The product is given by

$$(\omega_1\omega_2)(\gamma_0,\cdots,\gamma_{n_1},\cdots,\gamma_{n_1+n_2})$$
$$= \sum_{\gamma\gamma'=\gamma_{n_1}} \omega_1(\gamma_0,\cdots,\gamma_{n_1-1},\gamma)\wedge\omega_2(\gamma',\gamma_{n_1+1},\cdots,\gamma_{n_1+n_2})$$
$$+ \sum_{j=0}^{n_1-2}(-1)^{n_1-j-1}\sum_{\gamma\gamma'=\gamma_j}\omega_1(\gamma_0,\cdots,\gamma_{j-1},\gamma,\gamma',\cdots,\gamma_{n_1-1})\wedge\omega_2(\gamma_{n_1},\cdots,\gamma_{n_1+n_2}),$$

where the étale maps $r, s : G \to G^{(0)}$ are used to identify the corresponding cotangent spaces and perform the wedge product.

The differential c_2 is given by

$$(c_2\omega)(\gamma_0,\cdots,\gamma_{n+1}) = \begin{cases} 0, & \text{unless } \gamma_0 \text{ is a unit of } G \text{ groupoid,} \\ \omega(\gamma_1,\cdots,\gamma_{n+1}), & \text{if } \gamma_0 \text{ is a unit.} \end{cases}$$

For γ a cochain of $C^{n,m}$ in the bicomplex (C^*, d_1, d_2) as a current, associated is the linear map $\gamma^\sim$ on $C^{n,m}$, obtained from the push-forward of the current by the map

$$G^{(n)} \ni (\gamma_1,\cdots,\gamma_n) \mapsto ((\gamma_1\cdots\gamma_n)^{-1},\gamma_1,\cdots,\gamma_n) \in G^{(n+1)}.$$

Then the lemma above for the relation under $\gamma^\sim$ holds, but the part (a) only holds for $a_2 \in C^{0,0} = C_c^\infty(G)$ as an important difference. This is because of the loss of the total anti-symmetry of cochains.

The map Φ is defined as in the theorem above, and this theorem still holds because its proof only uses the weaker form of the lemma above.

Many concrete examples of cyclic cocycles on $C_c^\infty(G)$ may be constructed by using the map Φ.

For a complete description of the cyclic H cohomology for crossed product algebras, we may refer to [15], [26], [6], and [16].

4 Groupoids and the more somepoids

A few or some added below to remember the dream.

A groupoid is a set G with a distinguished subset $G^{(0)}$ as the set of units of G, two range and source maps $r, s : G \to G^{(0)}$, and a law of composition $\circ$:

$$\circ : G^{(2)} = \{(\gamma_1,\gamma_2) \in G\times G \,|\, s(\gamma_1) = r(\gamma_2)\} \to G$$

such that

$$\begin{aligned}
&(1)\quad s(\gamma_1\circ\gamma_2)=s(\gamma_2), r(\gamma_1\circ\gamma_2)=r(\gamma_1),\quad (\gamma_1,\gamma_2)\in G^{(2)},\\
&(2)\quad s(x)=r(x)=x,\quad x\in G^{(0)},\\
&(3)\quad \gamma\circ s(\gamma)=\gamma, r(\gamma)\circ\gamma=\gamma,\quad \gamma\in G,\\
&(4)\quad (\gamma_1\circ\gamma_2)\circ\gamma_3=\gamma_1\circ(\gamma_2\circ\gamma_3),\\
&(5)\quad \gamma\circ\gamma^{-1}=r(\gamma), \gamma^{-1}\circ\gamma=s(\gamma)
\end{aligned}$$

where each $\gamma\in G$ has a two-sided inverse $\gamma^{-1}\in G$.

$$\gamma=\gamma_1\circ\gamma_2 \;=\!=\!=\; s(\gamma)=s(\gamma_2)\xrightarrow{\gamma_2} r(\gamma_2)=s(\gamma_1)\xrightarrow{\gamma_1} r(\gamma)=r(\gamma_1).$$

Let X be a set and $R\subset X\times X$ an equivalence relation so that it holds that $(x,x)\in R$ for any $(x,x)\in X^2$, if $(x,y)\in R$, then $(y,x)\in R$, and if $(x,y),(y,z)\in R$, then $(x,z)\in R$. A groupoid by an equivalence relation is obtained by letting $G=R$, $G^{(0)}$ the diagonal of $X\times X$ contained in R, $r(x,y)=x$ (identified with (x,x)), $s(x,y)=y$ for any $\gamma=(x,y)\in R\subset X^2$, and

$$(x,y)\circ(y,z)=(x,z),\quad (x,y)^{-1}=(y,x).$$

$\star$ Check that

$$\begin{aligned}
&(1)\quad s((x,y)\circ(y,z))=z=s(y,z),\quad r((x,y)\circ(y,z))=x=r(x,y),\\
&(2)\quad s(x,x)=x=r(x,x)=(x,x),\\
&(3)\quad (x,y)\circ y=(x,y)\circ(y,y)=(x,y),\quad x\circ(x,y)=(x,y),\\
&(4)\quad ((x,y)\circ(y,z))\circ(z,w)=(x,w)=(x,y)\circ((y,z)\circ(z,w)),\\
&(5)\quad (x,y)\circ(y,x)=(x,x)=r(x,y),\quad (y,x)\circ(x,y)=(y,y)=s(x,y).
\end{aligned}$$

A groupoid by a group Γ is defined by taking $G=\Gamma$, $G^{(0)}=\{e\}$ the unit of Γ, and the law of composition by the group law.

$\star$ Check that for $g,g_1,g_2,g_3\in\Gamma$,

(1) $s(g_1g_2)=e=s(g_2)$, $r(g_1g_2)=e=r(g_1)$, (2) $s(e)=r(e)=e$, (3) $ge=g=eg$, (4) $(g_1g_2)g_3=g_1(g_2g_3)$, (5) $gg^{-1}=e=g^{-1}g$.

Suppose now that a group Γ acts on a set X by an action α such that $\alpha: X\times\Gamma\to X$, $\alpha(x,g)=xg$, and $(xg_1)g_1=x(g_1g_2)$ for $x\in X$, $g,g_1,g_2\in\Gamma$.

A groupoid by a group action on a space is defined by taking $G=X\times\Gamma$, $G^{(0)}=X\times\{e\}$, and $r(x,g)=x$, $s(x,g)=xg$ for $(x,g)\in X\times\Gamma$, and

$$(x,g_1)(xg_1,g_2)=(x,g_1g_2),\quad\text{and}\quad (x,g)^{-1}=(xg,g^{-1}).$$

This groupoid is said to be the semi-direct product of X by Γ, and is denoted by $X\rtimes\Gamma$.

$\star$ Check that

$$\begin{aligned}
&(1) \quad s((x,g_1)(xg_1,g_2)) = s(x,g_1g_2) = xg_1g_2 = s(xg_1,x_2),\\
&\qquad r((x,g_1)(xg_1,g_2)) = r(x,g_1g_2) = x = r(x,g_1),\\
&(2) \quad s(x,e) = x = r(x,e),\\
&(3) \quad (x,g)xg = (x,g)(xg,e) = (x,g),\\
&\qquad x(x,g) = (x,e)(x,g) = (x,g),\\
&(4) \quad ((x,g_1)(xg_1,g_2))(xg_1g_2,g_3) = (x,g_1g_2)(xg_1g_2,g_3) = (x,(g_1g_2)g_3),\\
&\qquad (x,g_1)((xg_1,g_2))(xg_1g_2,g_3)) = (x,g_1)(xg_1,g_2g_3) = (x,g_1g_2g_3),\\
&(5) \quad (x,g)(x,g)^{-1} = (x,g)(xg,g^{-1}) = (x,e) = x = r(x,g),\\
&\qquad (x,g)^{-1}(x,g) = (xg,g^{-1})(x,g) = (xg,e) = xg = s(x,g).
\end{aligned}$$

There is a natural locally compact topology for all such groupoids G given above such that the fibers $G^x = r^{-1}(x)$ for $x \in G^{(0)}$ as the inverse images by the map r are discrete.

This what allows us to define the convolution algebra of certain functions f, h on G with convolution defined as

$$(f * h)(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)h(\gamma_2).$$

We may refer to [27], [3], [4] for the general case of locally compact groupoids.

A smooth groupoid is defined to be a groupoid G with a differential structure on G and $G^{(0)}$ such that the maps $r, s : G \to G^{(0)}$ are submersions, and the inclusion map $G^{(0)} \to G$ is smooth, as is the composition map $G^{(2)} \to G$.

The general notion is due to C. Ehresmann.

The specific definition is due to J. Prodines. It is proved that in a smooth groupoid G, all the maps

$$s : G^x = \{\gamma \in G \,|\, r(\gamma) = x\} \to G^{(0)}$$

are subimmersions.

$\star$ Recall from [25] some basic for manifolds in the following.

A C^r-class (Kyu) map $f : M \to N$ of manifolds is said to be immersion if for any $p \in M$, the differential linear map $(df)_p : T_p(M) \to T_{f(p)}(N)$ is one-to-one (or injective).

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(\theta) = (\cos\theta, \sin\theta)$. Then the Jacobi matrix for f is

$$J_f = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \quad df_\theta(v\frac{d}{dx}) = -v\sin\theta\frac{\partial}{\partial x} + v\cos\theta\frac{\partial}{\partial y}.$$

Since $\sin^2\theta + \cos^2\theta = 1$, then $J_f \neq (0,0)^t$. Thus, df_θ is one-to-one and onto.

A C^r-class map $f : M \to N$ of manifolds is said to be submersion if for any $p \in M$, the differential linear map $(df)_p : T_p(M) \to T_{f(p)}(N)$ is onto (or surjective). In other words, by definition, any point of M is regular, so that there are no critical points of M.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(p) = f(x,y) = x^2 + y^2$. Then the Jacobi matrix for f is $J_f = (f_x, f_y) = (2x, 2y)$, and the differential df is given by

$$T_p(\mathbb{R}^2) \ni v_1 \tfrac{\partial}{\partial x} + v_2 \tfrac{\partial}{\partial y} \xrightarrow[J_f]{df_p} J_f \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tfrac{d}{dx} = 2(xv_1 + yv_2) \tfrac{d}{dx} \in T_{f(p)}(\mathbb{R}).$$

It then follows that any nonzero point $(x,y) \neq (0,0)$ of $\mathbb{R}^2$ is regular, but the origin $(0,0)$ only is a critical point.

A subimmersion may be defined to be a map which is immersion and submersion.

A continuous map $f : M \to N$ of manifolds is said to be proper if compact is the inverse image $f^{-1}(K)$ for any compact subset K of N. $\square$

The convolution algebra of a smooth groupoid G is defined by the notion of a $\frac{1}{2}$-density on a smooth manifold.

Let $\Omega^{\frac{1}{2}}$ be the line bundle over G, with the fiber $\Omega_\gamma^{\frac{1}{2}}$ at $\gamma \in G$ with $r(\gamma) = x$ and $s(\gamma) = y$, given by the linear space of maps

$$\rho : (\wedge^k T_\gamma(G^x)) \otimes (\wedge^k T_\gamma(G_y)) \to \mathbb{C}$$

such that $\rho(\lambda v) = |\lambda|^{\frac{1}{2}} \rho(v)$ for $\lambda \in \mathbb{R}$, where

$$G_y = \{\gamma \in G \,|\, s(\gamma) = y\},$$

and $k = \dim T_\gamma(G^x) = \dim T_\gamma(G_y)$ is the dimension of the fibers of the submersions $r : G \to G^{(0)}$ and $s : G \to G^{(0)}$.

The linear space $C_c^\infty(G, \Omega^{\frac{1}{2}})$ of compactly supported smooth sections of $\Omega^{\frac{1}{2}}$ over G is then endowed with the convolution product

$$(a * b)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} a(\gamma_1) b(\gamma_2), \quad a, b \in C_c^\infty(G, \Omega^{\frac{1}{2}}),$$

where the integral makes sense since it is the integral of a 1-density, for $a(\gamma_1)b(\gamma_1^{-1}\gamma)$ on the manifold G^x with $x = r(\gamma) = r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$.

Example (α). Let M be a compact manifold and $G = M \times M$ the groupoid where r and s are the two coordinate projections

$$G \to M = G^{(0)} = \{(x,x) \in G \,|\, x \in M\},$$

and the composition is given by $(x,y) \circ (y,z) = (x,z)$ for $x, y, z \in M$. The convolution algebra is then the algebra of smoothing kernels on the manifold M.

Example (β). Let G be a Lie group, as a groupoid with $G^{(0)}$ trivial, in a trivial way. Then the convolution algebra $C_c^\infty(G, \Omega^{\frac{1}{2}})$ is of smooth 1-densities on G.

$\star$ In this case, $G^x = G^e = G_y = G_e = G$ since $r, s : G \to G^{(0)} = \{e\}$.

Proposition 4.1. *Let G be a smooth groupoid and let $A = C_c^\infty(G, \Omega^{\frac{1}{2}})$ be the convolution algebra of compactly supported smooth $\frac{1}{2}$-densities, with involution defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$ for $f \in A$.*

Then for each $x \in G^{(0)}$, an involutive representation π_x of A in the Hilbert space $L^2(G_x)$ is defined by

$$(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1)\xi(\gamma_1^{-1}\gamma), \quad \gamma \in G_x, \xi \in L^2(G_x).$$

The completion of A by the supreme operator norm $\|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|$ over $G^{(0)}$ defines a C^*-algebra, denoted by $C_r^*(G)$, named as the reduced groupoid C^*-algebra of G.

We may refer to [9], [27] for the proof.

As well, as in the case of discrete groups, the full groupoid C^*-algebra $C^*(G)$ of a smooth groupoid G is defined to be the completion of the involutive algebra $C_c^\infty(G, \Omega^{\frac{1}{2}})$ by the maximal norm

$$\|f\|_{\max} = \sup_\pi \|\pi(f)\|, \quad \pi : C_c^\infty(G, \Omega^{\frac{1}{2}}) \to H_\pi$$

involutive representations on Hilbert spaces H_π.

There is the canonical surjective homomorphism from $C^*(G)$ to $C_r^*(G)$.

The coincidence between the full and reduced groupoid C^*-algebras is related to the notion of amenability for G. We may refer to [27] on this topic.

The tangent groupoid G of a manifold M is defined as follows.

Let $G = (M \times M \times (0,1]) \cup TM$, where TM is the total space of the tangent bundle over M.

Let $G^{(0)} = M \times [0,1]$ with inclusion of G defined by

$$\begin{aligned} &M \times (0,1] \ni (x,\varepsilon) \mapsto (x,x,\varepsilon) \in M \times M \times (0,1], \\ &M \times \{0\} \ni (x,0) \mapsto x \in M \subset TM \end{aligned}$$

where the last inclusion is given as the zero section $(x,0)$ on M. The range and source maps are given by respectively

$$\begin{aligned} &r(x,y,\varepsilon) = (x,\varepsilon), \quad r(x,X_x) = (x,0), \quad X_x \in T_x(M), \\ &s(x,y,\varepsilon) = (y,\varepsilon), \quad s(x,X_x) = (x,0). \end{aligned}$$

The composition is defined by

$$\begin{aligned} &(x,y,\varepsilon) \circ (y,z,\varepsilon) = (x,z,\varepsilon), \\ &(x,X_x) \circ (x,Y_x) = (x, X_x + Y_x), \quad X_x, Y_x \in T_x(M). \end{aligned}$$

The groupoid G is the union

$$G_1 \cup G_2 = (M \times M \times (0,1])) \cup TM$$

as a union groupoid of groupoids, with G_1 the product of the groupoid $M \times M$ of Example (α) by the set $(0, 1]$ as a set groupoid such as $H = H^{(0)}$ and with $G_2 = TM = \cup_{x\in M} T_x M$ the groupoid as a union groupoid of the tangent spaces $T_x(M)$ as groups.

The decomposition $G = G_1 \cup G_2$ is a disjoint union as true set theoretically, but not at the manifold level.

Let G be endowed with the manifold structure by its identification with the space obtained by blowing up the diagonal $\Delta = M$ in the cartesian square $M \times M$.

The topology of G is defines as that G_1 is an open subset of G and a sequence $(x_n, y_n, \varepsilon_n)$ of elements of $G_1 = M \times M \times (0, 1]$ with $\varepsilon \to 0 + 0$ as $n \to \infty$ converges to a tangent vector (x, X_x), $X_x \in T_x(M)$ if and only if both x_n and y_n converge to x, and $\frac{x_n - x_n}{\varepsilon_n} \to X_x$.

The last limit makes sense in any local chart around x independently of any choice. In this way, a manifold with boundary is obtained, and a local chart around a boundary point $(x, X_x) \in TM$ is provided by a choice of Rieman metric on M and the following map of an open subset of $TM \times [0, 1]$ to G, defined as

$$\begin{aligned} \psi(x, X_x, \varepsilon) &= (x, \exp(-\varepsilon X), \varepsilon) \in M \times M \times (0, 1], \quad \varepsilon > 0, \\ \psi(x, X, 0) &= (x, X_x) \in TM. \end{aligned}$$

Proposition 4.2. *The groupoid G with the the above manifold structure is a smooth one on.*

$\star$ Note that

$$J_r(x, y, \varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_s(x, y, \varepsilon) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$J_r(x, y, \varepsilon) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_3 \end{pmatrix}, \quad J_s(x, y, \varepsilon) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_3 \end{pmatrix},$$

which implies that $r, s : G \to G^{(0)}$ are submersions (locally).

The tangent groupoid of the manifold M is denoted by $G_{M,TM}$.

The algebraic structure of the C^*-algebra of this groupoid $G_{M,TM}$ is obtained from the inclusion of $G_2 = TM$ as a closed subgroupoid of $G_{M,TM}$ with complement $G_1 = M \times M \times (0, 1]$.

Proposition 4.3. (1) *To the decomposition $G_{M,TM} = G_1 \cup G_2$ as a union of an open subgroupoid and a closed subgroupoid corresponds the short exact sequence of C^*-algebras*

$$0 \to C^*(G_1) \to C^*(G_{M,TM}) \xrightarrow{\sigma} C^*(G_2) \to 0.$$

(2) *The groupoid C^*-algebra $C^*(G_1)$ is isomorphic to $C_0((0, 1]) \otimes \mathbb{K}$, where $\mathbb{K}$ is the elementary C^*-algebra of all compact operators on a Hilbert space.*

(3) *The groupoid C^*-algebra $C^*(G_2)$ is isomorphic to $C_0(T^*M)$ on the dual tangent bundle T^*M over M, by the group Fourier transform from $C^*(T_xM)$ to $C_0(T_x^*M)$ for each $x \in M$ by the duality.*

Corollary 4.4. *The grouppoid C^*-algebra $C^*(G_1)$ is contractible to zero.*

Proof. The C^*-algebra $C_0((0,1])$ is contractible in the sense that there is a pointwise norm continuous family θ_λ of endomorphisms for $\lambda \in [0,1]$ such that θ_0 is the identity map and θ_1 is the zero map. It then follows that the tensor product of $C_0((0,1])$ with any C^*-algebra is also contractible. □

Corollary 4.5. *There are the K-theory isomorphisms*

$$K_j(C^*(G_{M,TM})) \cong K_j(C^*(G_2)) \cong K^j(T^*M), \quad j = 0, 1.$$

Proof. There is the six-term exact sequence of K-theory groups as in the following:

$$\begin{array}{ccccc} K_0(C^*(G_1)) & \longrightarrow & K_0(C^*(G_{M,TM})) & \xrightarrow{\sigma_*} & K_0(C^*(G_2)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(G_2)) & \xleftarrow{\sigma_*} & K_1(C^*(G_{M,TM})) & \longleftarrow & K_1(C^*(G_1)) \end{array}$$

with $K_j(C^*(G_2)) \cong 0$.

As well, the K-theory K_j (of stable equivalent classes of projections and unitaries of matrix algebras) of the C^*-algebra $C_0(T^*M)$ of continuous complex-valued functions on T^*M vanishing at infinity is identified with the topological K-theory K^j (of stable isomorphism classes of complex vector bundles) of the dual tangent (or cotangent) space T^*M. □

Let $C^*(G_{M,TM}) \to \mathbb{K} \cong C^*(M \times M)$ be the transpose of the inclusion map from $M \times M$ to $G_{M,TM}$ defined by sending $(x,y) \in M^2$ to $(x,y,1) \in M^2 \times (0,1]$.

Lemma 4.6. *The Atiyah-Singer analytic index map is given by*

$$\mathrm{index}_a = \rho_* \circ (\sigma_*)^{-1} : K^0(T^*M) \xrightarrow{\sigma_*^{-1}} K_0(C^*(G_{M,TM})) \xrightarrow{\rho_*} K_0(\mathbb{K}) = \mathbb{Z}.$$

The quotient map $\sigma : C^*(G_{M,TM}) \to C^*(G_2) \cong C_0(T^*M)$ is the same as the symbol map of the pseudodifferential calculus for asymptotic pseudodifferential operators (cf. [17]).

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