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q-DIFFERENCE EQUATIONS FOR q-HYPERGEOMETRIC INTEGRALS OF TYPE G_2

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Abstract. We provide a simpler proof for an infinite product expression of Gustafson's q-beta integral of type G_2 with 4 parameters. We extend Gustafson's q-integral to a q-hypergeometric integral of type G_2 with 6 parameters. Under a constraint of the parameters called the balancing condition, we obtain two explicit forms of q-difference equations satisfied by the q-hypergeometric integral of type G_2. Taking limit for a parameter, the q-hypergeometric integral of type G_2 degenerates to Gustafson's q-integral, and one of two q-difference equations becomes that satisfied by Gustafson's q-integral. Using this we consequently have an alternative proof for the infinite product of Gustafson's q-beta integral again.

1. Introduction

The beta integral

$$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \quad (\mathrm{Re}\,\alpha>0,\ \mathrm{Re}\,\beta>0) \tag{1.1}$$

satisfies the formula

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.2}$$

where $\Gamma(\alpha)$ is the gamma function given as

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx \quad (\mathrm{Re}\,\alpha>0).$$

The formula (1.2) has a great significance in classical analysis and is applied variously not only limited to Mathematics. For instance, the orthogonal polynomials associated with the integrand of (1.1) as weight functions are called the Jacobi polynomials, and their properties are studied precisely from the view of their theories and applications. As a generalization of (1.2), the Selberg integral as a multivariable beta integral can be written as the product of gamma functions, i.e.,

$$\frac{1}{n!}\int_0^1\cdots\int_0^1\prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1}\prod_{1\le j<k\le n}|x_j-x_k|^{2\tau}\,dx_1dx_2\cdots dx_n$$
$$=\prod_{j=1}^n\frac{\Gamma(\alpha+(j-1)\tau)\Gamma(\beta+(j-1)\tau)\Gamma(j\tau)}{\Gamma(\alpha+\beta+(n+j-2)\tau)\Gamma(\tau)},$$

where $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\beta>0$ and $\mathrm{Re}\,\tau>-\min\{1/n,\mathrm{Re}\,\alpha/(n-1),\mathrm{Re}\,\beta/(n-1)\}$. This formula coincides with (1.2) if $n=1$, and is also fundamental to the theory of

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multivariable orthogonal polynomials. (See the recent reference [3] for topics relevant to the Selberg integral.)

Using Jackson integral $\int_0^1 f(x)d_qx = (1-q)\sum_{i=0}^\infty f(q^i)q^i$, q-analogues of the beta integral and the gamma function are given as

$$B_q(\alpha,\beta)=\int_0^1 x^\alpha \frac{(qx;q)_\infty}{(q^\beta x;q)_\infty}\frac{d_qx}{x}, \qquad \Gamma_q(\alpha)=\frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha},$$

where $|q|<1$. Here we used the symbol $(u;q)_\infty := \prod_{i=0}^\infty (1-q^iu)$. For simplicity we also use the symbol $(u_1,u_2,\ldots,u_n)_\infty := (u_1;q)_\infty(u_2;q)_\infty\cdots(u_n;q)_\infty$. As $q\to 1$, $B_q(\alpha,\beta)$ and $\Gamma_q(\alpha)$ become $B(\alpha,\beta)$ and $\Gamma(\alpha)$, respectively. In 1980 Askey [1] established a q-analogue of the Selberg integral given as

$$\begin{aligned}(1.3)\qquad &\int_{z_1=0}^1\int_{z_2=0}^{q^\tau z_1}\cdots\int_{z_n=0}^{q^\tau z_{n-1}}\prod_{i=1}^n z_i^\alpha\frac{(qz_i;q)_\infty}{(q^\beta z_i;q)_\infty}\\ &\times\prod_{1\le j<k\le n} z_j^{2\tau-1}\frac{(q^{1-\tau}z_k/z_j;q)_\infty}{(q^\tau z_k/z_j;q)_\infty}(z_j-z_k)\frac{d_qz_n}{z_n}\cdots\frac{d_qz_2}{z_2}\frac{d_qz_1}{z_1}\\ &=q^{\alpha\tau\binom{n}{2}+2\tau^2\binom{n}{3}}\prod_{j=1}^n\frac{\Gamma_q(\alpha+(j-1)\tau)\Gamma_q(\beta+(j-1)\tau)\Gamma_q(j\tau)}{\Gamma_q(\alpha+\beta+(n+j-2)\tau)\Gamma_q(\tau)},\end{aligned}$$

where $\mathrm{Re}\,\alpha>0$ and $\mathrm{Re}\,\alpha+(n-1)\mathrm{Re}\,\tau>0$. (In [9] an explanation for the formula (1.3) supporting this paper is provided.) After (1.3) appeared, with a great deal of researches developed in the 1990s for the Macdonald orthogonal polynomials associated with root systems, q-beta integrals possessing Weyl group symmetry associated with root systems were studied. The most typical one associated with root systems is the complex integral given as

$$(1.4)\qquad \frac{(q;q)_\infty}{2(2\pi\sqrt{-1})}\int_{|z|=1}\frac{(z^2,z^{-2};q)_\infty}{\prod_{k=1}^4(a_kz,a_kz^{-1};q)_\infty}\frac{dz}{z}=\frac{(a_1a_2a_3a_4;q)_\infty}{\prod_{1\le i<j\le 4}(a_ia_j;q)_\infty},$$

where $|a_k|<1$ $(k=1,\ldots,4)$. We call the left-hand side of (1.4) the Askey–Wilson integral (or q-beta integral of type BC_1 [8]), which gives the orthogonal norm of the Askey–Wilson orthogonal polynomials [2]. Nassrallah and Rahman [13] established an extension of (1.4) given as

$$(1.5)\qquad \frac{(q;q)_\infty}{2(2\pi\sqrt{-1})}\int_{|z|=1}\frac{(z^{\pm2},qa_6^{-1}z^{\pm1};q)_\infty}{\prod_{k=1}^5(a_kz^{\pm1};q)_\infty}\frac{dz}{z}=\frac{\prod_{i=1}^5(qa_6^{-1}a_i^{-1};q)_\infty}{\prod_{1\le i<j\le 5}(a_ia_j;q)_\infty}$$

under the balancing condition $a_1a_2\cdots a_6=q$, where $|a_k|<1$ $(k=1,\ldots,5)$. Here we used the symbols $(u^{\pm2};q)_\infty=(u^2,u^{-2};q)_\infty$ and $(cu^{\pm1};q)_\infty=(cu,cu^{-1};q)_\infty$. By taking limit $a_5\to 0$ the Askey–Wilson integral (1.4) is obtained from the Nassrallah–Rahman integral (1.5) as a special case. Although the Nassrallah–Rahman integral does not give a norm for specific orthogonal polynomials, it has higher symmetry than the Askey–Wilson integral, and consequently the proof of the formula (1.5) become simpler than that of the formula (1.4). The formulas (1.4) and (1.5) can be extended to the multidimensional case (q-Selberg contour integral of type BC_n), which gives the norm of the Macdonald–Koornwinder multivariable orthogonal polynomials, and topics around the integrals of type BC_n are still much actively researched area. We note that in the last two decades, several elliptic extensions (p,q-analogue) of the q-beta integrals have been studied, especially for those of type BC_n by van Diejen and Spiridonov [17], Spiridonov [15], Rains [14] (see also [11] for explanation supporting this paper).

On the other hand, compared with the development of the q-beta integrals associated with classical root systems, research on those associated with the exceptional root systems seems to be not fully studied yet at present. For the root system G_2 Gustafson [5, 6] showed the following.

Proposition 1.1. *Suppose that $a_k \in \mathbb{C}^*$ $(1 \le k \le 4)$ satisfy $|a_k| < 1$. Then,*

$$\begin{aligned}(1.6)\quad &\frac{(q;q)_\infty^2}{12(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\frac{\prod_{1\le i<j\le 3}(x_ix_j, x_i^{-1}x_j, x_ix_j^{-1}, x_i^{-1}x_j^{-1};q)_\infty}{\prod_{i=1}^3\prod_{k=1}^4(a_kx_i, a_kx_i^{-1};q)_\infty}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\\ &=\frac{(a_1^2a_2^2a_3^2a_4^2;q)_\infty}{(a_1a_2a_3a_4;q)_\infty}\prod_{i=1}^4\frac{(a_i;q)_\infty}{(a_i^2;q)_\infty}\prod_{1\le i<j\le 4}\frac{1}{(a_ia_j;q)_\infty}\prod_{1\le i<j<k\le 4}\frac{1}{(a_ia_ja_k;q)_\infty},\end{aligned}$$

where $x_3 = x_1^{-1}x_2^{-1}$ and $\mathbb{T}^2$ is the 2-dimensional torus given as

$$\mathbb{T}^2 = \{(x_1,x_2) \in (\mathbb{C})^2 \,|\, |x_i| = 1 \ (i = 1,2)\}$$

In spite of its simple appearance no short proofs for the formula (1.6) are known. One of our aims is to give a simpler proof for (1.6). The other aim is to investigate a generalized integral of Nassrallah–Rahman type for the case of G_2 defined as

$$(1.7)\qquad \frac{(q;q)_\infty^2}{12(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\frac{\prod_{1\le i<j\le 3}(x_i^{\pm}x_j^{\pm}, qa_6^{-1}x_i^{\pm}x_j^{\pm};q)_\infty}{\prod_{i=1}^3\prod_{k=1}^5(a_kx_i^{\pm};q)_\infty}\frac{dx_1}{x_1}\frac{dx_2}{x_2},$$

under the conditions $x_1x_2x_3 = 1$ and $a_1a_2\cdots a_6 = -q$. For simplicity here we use the symbol $(cu^{\pm}v^{\pm};q)_\infty = (cuv, cu^{-1}v, cuv^{-1}, cu^{-1}v^{-1};q)_\infty$. Gustafson's integral (the left-hand side of (1.6)) is included in (1.7) as the limiting case $a_5 \to 0$. Although the integral (1.7) no longer has product expression by gamma functions as the right-hand side of (1.6), it satisfies two independent q-difference equations of rank 2. We provide the explicit forms of these two equations (see Theorems 7.1 and 7.2). As a corollary of the theorem we can understand that, when $a_5 \to 0$, one of the q-difference equations degenerate to that of rank 1 satisfied by Gustafson's integral. This consequently gives another alternative proof for the formula (1.6).

This paper is organized as follows. After defining basic terminology in Section 2, we explain in Section 3 a way to derive the formula (1.4) for the Askey–Wilson integral (BC_1 case), before proving the formula (1.6) for Gustafson's integral of type G_2. Since the arguments for both BC_1 and G_2 cases are completely parallel, this section for BC_1 case would be instructive to understand the strategy of two steps for G_2 case. The first step is to derive the q-difference equations (two-term recurrence relations) for the integral with respect to its parameters by using q-Stokes' theorem. Once we had these recurrence relations, using them repeatedly we see that the integral can be expressed as a product form up to a multiplicative constant. The next step is to determine the indefinite constant using a special value of the integral at some specific point. We enumerate several special values for the integral of type BC_1 which are simply computed at the corresponding specific points. In Section 4 we explain the derivation of the formula (1.5) for the Nassrallah–Rahman integral (BC_1

case with a balancing condition). The argument of this section is almost parallel to that of Section 3 except the treatment of the balancing condition. In Section 5 we recall terminology of the root system G_2 and its Weyl group W. In addition to this we introduce two $\mathbb{C}$-vector subspaces $\mathcal{F}_2$ and $\mathcal{F}_4$ in the $\mathbb{C}$-vector space of W-invariant Laurent polynomials, and define the $\mathbb{C}$-bases of $\mathcal{F}_i, (i = 2, 4)$ that consist of Lagrange interpolation polynomials associated with some specific points. These bases are important and necessary to apply q-Stokes' theorem to the integrals of type G_2 when we derive q-difference equations for the integrals of type G_2 in the succeeding sections. Section 6 is devoted to a derivation for the formula (1.6) of Gustafson's integral of type G_2. This is one of our main results. Although calculation we need is more complex than the BC_1 case, the strategy of two steps is still the same as the BC_1 case. In Section 7 we define Nassrallah–Rahman type integral for G_2 case (the integral of type G_2 with a balancing condition). We compute the explicit forms of two q-difference equations which the Nassrallah–Rahman integral of type G_2 satisfies (see Theorems 7.1 and 7.2). This is the other result of ours.

Lastly we note that the contents of this paper is fundamentally based on the thesis of the second author [16]. We remark that the elliptic version of Gustafson's integral (1.6) was recently proved in [10].

2. Preliminaries

Throughout this paper we fix $q \in \mathbb{C}^*$ with $|q| < 1$. We use the q-shifted factorials for $x \in \mathbb{C}$ as

$$(x; q)_\infty := \prod_{i=0}^{\infty}(1 - q^i x)$$

and

$$\begin{aligned}(x; q)_n &:= \frac{(x; q)_\infty}{(q^n x; q)_\infty} \\ &= \begin{cases} (1-x)(1-qx)(1-q^2x)\cdots(1-q^{n-1}x) & \text{if} \quad n = 1, 2, \dots, \\ 1 & \text{if} \quad n = 0, \\ \dfrac{1}{(1-q^{-1}x)(1-q^{-2}x)\cdots(1-q^{n}x)} & \text{if} \quad n = -1, -2, \dots. \end{cases}\end{aligned}$$

We also use the symbol

$$(x_1, x_2, \dots, x_m; q)_\infty := (x_1; q)_\infty (x_2; q)_\infty \cdots (x_m; q)_\infty.$$

By definition we have

$$(x^2; q)_\infty = (x, -x, q^{\frac{1}{2}}x, -q^{\frac{1}{2}}x; q)_\infty. \tag{2.1}$$

In particular, if $x = q^{\frac{1}{2}}$, then we have $(q; q)_\infty = (q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q, -q; q)_\infty$, so that

$$1 = (q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q; q)_\infty. \tag{2.2}$$

For $u, v \in \mathbb{C}^*$ we set

$$e(u, v) := u + u^{-1} - v - v^{-1} = u^{-1}(1 - uv)(1 - u/v), \tag{2.3}$$

which satisfies

$$e(u,v) = -e(v,u), \quad e(u,v) = e(u^{-1},v), \quad e(u,v) + e(v,w) = e(u,w),$$

and

$$e(u,v)e(w,x) - e(u,w)e(v,x) + e(u,x)e(v,w) = 0.$$

For $z \in \mathbb{C}^*$ we define $\vartheta(z;q)$ by the bilateral series

$$\vartheta(z;q) := \sum_{\nu=-\infty}^{\infty} (-z)^\nu q^{\binom{\nu}{2}}, \tag{2.4}$$

which converges uniformly on compact sets of $\mathbb{C}^*$ and satisfies

$$\vartheta(z;q) = (z, qz^{-1}, q; q)_\infty. \tag{2.5}$$

The identity (2.5) is called Jacobi's triple product formula [4].

3. Askey–Wilson integral

The aim of this section is to provide a way to prove the following identity.

Proposition 3.1. *Suppose that $a_k \in \mathbb{C}^*$ $(1 \le k \le 4)$ satisfy $|a_k| < 1$. Then, we have*

$$\frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \frac{dz}{z} = \frac{2}{(q;q)_\infty} \frac{(a_1a_2a_3a_4; q)_\infty}{\prod_{1\le i<j\le 4}(a_ia_j;q)_\infty}, \tag{3.1}$$

where $\mathbb{T}$ is the unit circle $\{z \in \mathbb{C} \,|\, |z| = 1\}$ traversed along the positive direction.

The left-hand side of (3.1) is called the *Askey–Wilson integral* [2, 8]. Throughout this section, we define the function $\Phi(z)$ on $\mathbb{C}^*$ by

$$\Phi(z) := \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \tag{3.2}$$

and also denote by $I(a_1,a_2,a_3,a_4)$ the left-hand side of (3.1), i.e.,

$$I(a_1,a_2,a_3,a_4) := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \Phi(z) \frac{dz}{z}. \tag{3.3}$$

On the other hand, we define $P(a_1,a_2,a_3,a_4)$ by the infinite product

$$P(a_1,a_2,a_3,a_4) := \frac{(a_1a_2a_3a_4; q)_\infty}{\prod_{1\le i<j\le 4}(a_ia_j;q)_\infty}, \tag{3.4}$$

so that the identity (3.1) is rewritten as

$$I(a_1,a_2,a_3,a_4) = \frac{2}{(q;q)_\infty} P(a_1,a_2,a_3,a_4)$$

for $a_k \in \mathbb{C}^*$ $(1 \le k \le 4)$ satisfying $|a_k| < 1$. Before proving this identity, we state the q-Stokes' theorem in the following section.

3.1. **q-Difference operator $\nabla_{q,z}$ and q-Stokes' theorem.** For an arbitrary function $f(z)$ on $\mathbb{C}^*$ we define the q-shift operator $T_{q,z}$ by $T_{q,z}f(z) = f(qz)$.

Lemma 3.2. *Let $F_+(z)$ and $F_-(z)$ be functions specified by*

$$F_+(z) := \frac{1}{z}\frac{\prod_{k=1}^4(1-a_kz)}{1-z^2} = -\frac{1}{z-z^{-1}}z^{-2}\prod_{k=1}^4(1-a_kz), \tag{3.5}$$

$$F_-(z) := F_+(z^{-1}) = z\frac{\prod_{k=1}^4(1-a_kz^{-1})}{1-z^{-2}}. \tag{3.6}$$

Then, it follows that

$$\frac{T_{q,z}\Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q,z}F_-(z)}. \tag{3.7}$$

Proof. By the definition (3.2) of $\Phi(z)$ we have

$$\begin{aligned}
\frac{\Phi(qz)}{\Phi(z)} &= \frac{(q^2z^2, q^{-2}z^{-2};q)_\infty}{(z^2,z^{-2};q)_\infty}\prod_{k=1}^4\frac{(a_kz,a_kz^{-1};q)_\infty}{(qa_kz,q^{-1}a_kz^{-1};q)_\infty}\\
&= \frac{(1-q^{-2}z^{-2})(1-q^{-1}z^{-2})}{(1-z^2)(1-qz^2)}\prod_{k=1}^4\frac{1-a_kz}{1-q^{-1}a_kz^{-1}}\\
&= -q^{-1}z^{-2}\frac{1-q^{-2}z^{-2}}{1-z^2}\prod_{k=1}^4\frac{1-a_kz}{1-q^{-1}a_kz^{-1}} = -\frac{F_+(z)}{F_-(qz)},
\end{aligned}$$

which completes the proof. □

For an arbitrary meromorphic function $\varphi(z)$ on $\mathbb{C}^*$ we define the symbol

$$\langle\varphi(z)\rangle := \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\varphi(z)\Phi(z)\frac{dz}{z} \tag{3.8}$$

In particular, from (3.3) we have $\langle 1\rangle = I(a_1,a_2,a_3,a_4)$. The following is a technical key lemma of this paper.

Proposition 3.3 (q-Stokes' theorem)**.** *For an arbitrary meromorphic function $\varphi(z)$ on $\mathbb{C}^*$, let $\nabla_{q,z}$ be operator specified by*

$$(\nabla_{q,z}\varphi)(z) := F_-(z)\varphi(z) + F_+(z)T_{q,z}\varphi(z). \tag{3.9}$$

Suppose that $|a_k|<1$ $(k=1,2,3,4)$. For an arbitrary holomorphic function $\varphi(z)$ on $\mathbb{C}^*$, it follows that

$$\langle\nabla_{q,z}\varphi(z)\rangle = 0. \tag{3.10}$$

Proof. By the definition of $\nabla_{q,z}$ we have

$$
\begin{aligned}
\langle \nabla_{q,z}\varphi(z)\rangle &= \int_{\mathbb{T}} \Phi(z)\nabla_{q,z}\varphi(z)\frac{dz}{z} = \int_{\mathbb{T}} \Phi(z)\Big(F_-(z)\varphi(z) + F_+(z)\varphi(qz)\Big)\frac{dz}{z} \\
&= \int_{\mathbb{T}} \Phi(z)\Big(F_-(z)\varphi(z) - \frac{\Phi(qz)}{\Phi(z)}F_-(qz)\varphi(qz)\Big)\frac{dz}{z} \quad \text{(from (3.7))} \\
&= \int_{\mathbb{T}} \Big(\Phi(z)F_-(z)\psi(z) - \Phi(qz)F_-(qz)\psi(qz)\Big)\frac{dz}{z} \\
&= \int_{\mathbb{T}} \Phi(z)F_-(z)\varphi(z)\frac{dz}{z} - \int_{\mathbb{T}} \Phi(qz)F_-(qz)\varphi(qz)\frac{dz}{z}.
\end{aligned} \tag{3.11}
$$

By variable change $w = qz$ for the second term of the last line of (3.11), we have $dw/w = dz/z$, and the integral over $\mathbb{T}$ changes to that over $\mathbb{T}_q := \{w \in \mathbb{C} \,|\, |w| = |q|\}$. Thus, from (3.11) we have

$$
\langle \nabla_{q,z}\varphi(z)\rangle = \int_{\mathbb{T}} \Phi(z)F_-(z)\varphi(z)\frac{dz}{z} - \int_{\mathbb{T}_q} \Phi(z)F_-(z)\varphi(z)\frac{dz}{z}.
$$

By the Cauchy theorem, in order to complete the proof, it suffices to show that the function $\Phi(z)F_-(z)\varphi(z)z^{-1}$ is holomorphic on the annulus $\{z \in \mathbb{C} \,|\, |q| \le |z| \le 1\}$. From (3.5) and (3.6), we have

$$
\begin{aligned}
\Phi(z)F_-(z)\varphi(z)z^{-1} &= \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \frac{1}{z^{-1}} \frac{\prod_{k=1}^4 (1 - a_k z^{-1})}{1 - z^{-2}} \varphi(z) z^{-1} \\
&= \frac{(z^2, qz^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, q a_k z^{-1}; q)_\infty} \varphi(z).
\end{aligned} \tag{3.12}
$$

Since $\varphi(z)$ is holomorphic on $\mathbb{C}^*$ by assumption, poles of the function $\Phi(z)F_-(z)\varphi(z)z^{-1}$ coincide with zero points of the denominator of (3.12). The set of solutions z of $(a_k z, q a_k z^{-1}; q)_\infty = 0$ are expressed as

$$
\{q^{-n}a_k^{-1} \,|\, n = 0, 1, 2, \cdots\} \cup \{q^{1+n}a_k \,|\, n = 0, 1, 2, \cdots\}.
$$

Since $|a_k| < 1$ and $|q| < 1$ by assumption, we have

$$
|q^{-n}a_k^{-1}| = \frac{1}{|q|^n |a_k|} > 1, \qquad |q^{1+n}a_k| = |q|^{1+n}|a_k| < |q|.
$$

This implies that no poles of the function $\Phi(z)F_-(z)\varphi(z)z^{-1}$ are included in the annulus $\{z \in \mathbb{C} \,|\, |q| \le |z| \le 1\}$. This completes the proof. □

3.2. **q-Difference equations for the Askey–Wilson integral.** For the proof of Proposition 3.1 we first show the q-difference equation for $I(a_1, a_2, a_3, a_4)$ as follows:

Proposition 3.4. *Suppose that $|a_k| < 1$ $(k = 1, 2, 3, 4)$. The integral $I(a_1, a_2, a_3, a_4)$ satisfies*

$$
\frac{I(qa_1, a_2, a_3, a_4)}{I(a_1, a_2, a_3, a_4)} = \frac{(1 - a_1 a_2)(1 - a_1 a_3)(1 - a_1 a_4)}{1 - a_1 a_2 a_3 a_4}. \tag{3.13}
$$

Proof. Setting $\varphi(z)=1$, $\nabla_{q,z}\varphi(z)$ is calculated as

$$
\begin{aligned}
\nabla_{q,z}\varphi(z) &= F_-(z)+F_+(z) = \frac{1}{z^{-1}}\frac{\prod_{k=1}^4(1-a_kz^{-1})}{1-z^{-2}} + \frac{1}{z}\frac{\prod_{k=1}^4(1-a_kz)}{1-z^2} \\
&= \frac{1}{z-z^{-1}}\Big[z^2(1-a_1z^{-1})(1-a_2z^{-1})(1-a_3z^{-1})(1-a_4z^{-1}) \\
&\qquad\qquad - z^{-2}(1-a_1z)(1-a_2z)(1-a_3z)(1-a_4z)\Big] \\
&= \frac{1}{z-z^{-1}}\Big[z^2(1-z^{-1}E_1+z^{-2}E_2-z^{-3}E_3+z^{-4}E_4) \\
&\qquad\qquad - z^{-2}(1-zE_1+z^2E_2-z^3E_3+z^4E_4)\Big] \\
&= \frac{(z^2-z^{-2})(1-E_4)-(z-z^{-1})(E_1-E_3)}{z-z^{-1}} \\
&= (z+z^{-1})(1-E_4)-(E_1-E_3),
\end{aligned}
\tag{3.14}
$$

where E_r $(r=0,1,\dots,4)$ is the rth elementary symmetric polynomial of $a_1,\dots,a_4$. From (3.14), using $e(a_1,z)$ defined by (2.3), $\nabla_{q,z}\varphi(z)$ is expanded as

$$
\nabla_{q,z}\varphi(z) = C_1e(a_1,z)+C_0 \tag{3.15}
$$

where the coefficients C_0 and C_1 are independent of z. We now determine C_0 and C_1. Comparing the highest degree polynomials of (3.14) and (3.15), we have $-C_1=1-E_4$, so that $C_1=-(1-a_1a_2a_3a_4)$. On the other hand, since $e(a_1,a_1)=0$, if we put $z=a_1$ in (3.15), then $C_0=\nabla_{q,z}\varphi(a_1)$. From $F_-(a_1)=0$ we also have $\nabla_{q,z}\varphi(a_1)=F_+(a_1)$. Therefore we obtain

$$
\begin{aligned}
C_0=\nabla_{q,z}\varphi(a_1)=F_+(a_1) &= \frac{1}{a_1}\frac{\prod_{k=1}^4(1-a_ka_1)}{1-a_1^2} \\
&= a_1^{-1}(1-a_1a_2)(1-a_1a_3)(1-a_1a_4).
\end{aligned}
$$

From (3.10) of q-Stokes' theorem we have

$$
0=\langle\nabla_{q,z}\varphi(z)\rangle = C_1\langle e(a_1,z)\rangle + C_0\langle 1\rangle,
$$

so that

$$
\langle e(a_1,z)\rangle = -\frac{C_0}{C_1}\langle 1\rangle = \frac{(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)}{a_1(1-a_1a_2a_3a_4)}\langle 1\rangle.
$$

Here we have $I(a_1,a_2,a_3,a_4)=\langle 1\rangle$ and $I(qa_1,a_2,a_3,a_4)=\langle a_1e(a_1,z)\rangle = a_1\langle e(a_1,z)\rangle$ because

$$
\frac{T_{q,a_1}\Phi(z)}{\Phi(z)} = \frac{(a_1z,a_1z^{-1};q)_\infty}{(qa_1z,qa_1z^{-1};q)_\infty} = (1-a_1z)(1-a_1z^{-1}) = a_1e(a_1,z).
$$

Therefore we obtain

$$
I(qa_1,a_2,a_3,a_4) = \frac{(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)}{1-a_1a_2a_3a_4}I(a_1,a_2,a_3,a_4),
$$

which completes the proof. □

By repeated use of Proposition 3.4 we have the following.

Proposition 3.5. *$I(a_1,a_2,a_3,a_4)$ coincides with $P(a_1,a_2,a_3,a_4)$ up to a multiplicative constant, i.e.,*

$$I(a_1,a_2,a_3,a_4) = c\,P(a_1,a_2,a_3,a_4). \tag{3.16}$$

where c is some constant independent of a_1,a_2,a_3,a_4.

Proof. From (3.4) $P(a_1,a_2,a_3,a_4)$ satisfies the same q-difference equation as (3.13), and (3.13) is symmetric with respect to a_1,a_2,a_3,a_4. This implies

$$\frac{I(a_1,a_2,a_3,a_4)}{P(a_1,a_2,a_3,a_4)} = \frac{I(qa_1,qa_2,qa_3,qa_4)}{P(qa_1,qa_2,qa_3,qa_4)} = \lim_{N\to\infty}\frac{I(q^Na_1,q^Na_2,q^Na_3,q^Na_4)}{P(q^Na_1,q^Na_2,q^Na_3,q^Na_4)}. \tag{3.17}$$

From (3.4) we have

$$\lim_{N\to\infty} P(q^Na_1,q^Na_2,q^Na_3,q^Na_4) = 1. \tag{3.18}$$

On the other hand, if we set c as

$$c = \lim_{N\to\infty} I(q^Na_1,q^Na_2,q^Na_3,q^Na_4) = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}} (z^2,z^{-2};q)_\infty \frac{dz}{z}, \tag{3.19}$$

then, using (3.17) and (3.18) we obtain (3.16). □

In order to compute the constant c in (3.16) as $c = 2/(q;q)_\infty$ we want to know a special value of the integral $I(a_1,a_2,a_3,a_4)$. In the following section, we show several special values of $I(a_1,a_2,a_3,a_4)$ at specific points.

3.3. **Special values of the Askey–Wilson integral.** In this subsection, we show special values of $I(a_1,a_2,a_3,a_4)$ at four specific points

$$(a_1,a_2,a_3,a_4) = (0,0,0,0),\ (0,-1,q^{\frac12},-q^{\frac12}),\ (1,-1,q^{\frac12},-q^{\frac12}) \text{ and } (0,0,q^{\frac12},-q^{\frac12}).$$

The evaluation at $(a_1,a_2,a_3,a_4) = (1,-1,q^{\frac12},-q^{\frac12})$ is the most simplest case.

Lemma 3.6.

$$I(1,-1,q^{\frac12},-q^{\frac12}) = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{dz}{z} = 1, \quad P(1,-1,q^{\frac12},-q^{\frac12}) = \frac{(q;q)_\infty}{2}.$$

Proof. Using (2.1) we have

$$\begin{aligned} &I(1,-1,q^{\frac12},-q^{\frac12}) \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{(z^2,z^{-2};q)_\infty}{(z,z^{-1},-z,-z^{-1},q^{\frac12}z,q^{\frac12}z^{-1},-q^{\frac12}z,-q^{\frac12}z^{-1};q)_\infty}\frac{dz}{z} = \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{dz}{z}, \end{aligned}$$

which is equal to 1 due to Cauchy's residue theorem. On the other hand, by definition (3.4) we have

$$\begin{aligned} P(1,-1,q^{\frac12},-q^{\frac12}) &= \frac{(q;q)_\infty}{(-1,q^{\frac12},-q^{\frac12},-q^{\frac12},q^{\frac12},-q;q)_\infty} = \frac{2(q;q)_\infty}{(-1,q^{\frac12},-q^{\frac12};q)_\infty^2} \\ &= \frac{(q;q)_\infty}{2(-q,q^{\frac12},-q^{\frac12};q)_\infty^2} = \frac{(q;q)_\infty}{2} \quad \text{(from (2.2))}, \end{aligned}$$

which completes the proof. □

Using (3.19) we can actually compute the constant c in Proposition 3.5 directly as

Lemma 3.7.

$$I(0,0,0,0)=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^2,z^{-2};q)_\infty\frac{dz}{z}=\frac{2}{(q;q)_\infty},\qquad P(0,0,0,0)=1.$$

Proof. Using Jacobi's theta function $\vartheta(x;q)$ defined as (2.4), we have

$$\begin{aligned}\frac{(q;q)_\infty}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^2,z^{-2};q)_\infty\frac{dz}{z}&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(1-z^{-2})z^{-1}(z^2,qz^{-2},q;q)_\infty dz\\&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^{-1}-z^{-3})\vartheta(z^2;q)dz=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^{-1}-z^{-3})\sum_{n=-\infty}^{\infty}(-z^2)^nq^{\binom{n}{2}}dz.\end{aligned}$$

For the integrand of the above expression, the point $z=0$ is the unique essential singularity, and its residue is calculated from the Laurent expansion

$$\begin{aligned}(z^{-1}-z^{-3})\sum_{n=-\infty}^{\infty}(-z^2)^nq^{\binom{n}{2}}&=(z^{-1}-z^{-3})(\cdots+q^3z^{-4}-qz^{-2}+1-z^2+qz^4-\cdots)\\&=(\cdots+q^3z^{-5}-qz^{-3}+z^{-1}-z+qz^3-\cdots)\\&\quad-(\cdots+q^3z^{-8}-qz^{-6}+z^{-3}-z^{-1}+qz-\cdots),\end{aligned}$$

whose residue (coefficient of z^{-1}) is 2. Therefore Cauchy's residue theorem implies

$$\frac{(q;q)_\infty}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^2,z^{-2};q)_\infty\frac{dz}{z}=2,$$

which shows $I(0,0,0,0)$ in Lemma 3.7. From (3.18) we have $P(0,0,0,0)=1$. □

The evaluations at $(a_1,a_2,a_3,a_4)=(0,-1,q^{\frac12},-q^{\frac12})$ and $(0,0,q^{\frac12},-q^{\frac12})$ are almost the same as that at $(0,0,0,0)$.

Lemma 3.8.

$$I(0,-1,q^{\frac12},-q^{\frac12})=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z,z^{-1};q)_\infty\frac{dz}{z}=\frac{2}{(q;q)_\infty},\tag{3.20}$$

$$P(0,-1,q^{\frac12},-q^{\frac12})=1.$$

Proof. From (2.1), $I(0,-1,q^{\frac12},-q^{\frac12})$ is expressed as

$$\begin{aligned}I(0,-1,q^{\frac12},-q^{\frac12})&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{(z^2,z^{-2};q)_\infty}{(-z,-z^{-1},q^{\frac12}z,q^{\frac12}z^{-1},-q^{\frac12}z,-q^{\frac12}z^{-1};q)_\infty}\frac{dz}{z}\\&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{(z,-z,q^{\frac12}z,-q^{\frac12}z,z^{-1},-z^{-1},q^{\frac12}z^{-1},-q^{\frac12}z^{-1};q)_\infty}{(-z,-z^{-1},q^{\frac12}z,q^{\frac12}z^{-1},-q^{\frac12}z,-q^{\frac12}z^{-1};q)_\infty}\frac{dz}{z}\\&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z,z^{-1};q)_\infty\frac{dz}{z}.\end{aligned}$$

From (2.5) we have

$$\begin{aligned}\frac{(q;q)_\infty}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z,z^{-1};q)_\infty\frac{dz}{z}&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(1-z^{-1})z^{-1}(z,qz^{-1},q;q)_\infty dz\\&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^{-1}-z^{-2})\vartheta(z;q)dz=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^{-1}-z^{-2})\sum_{n=-\infty}^{\infty}(-z)^nq^{\binom{n}{2}}dz,\end{aligned}$$

whose integrand has the unique essential singularity at $z=0$, and is expanded as

$$\begin{aligned}(z^{-1}-z^{-2})\sum_{n=-\infty}^{\infty}(-z)^n q^{\binom{n}{2}} &= (z^{-1}-z^{-2})(\cdots+q^3z^{-2}-qz^{-1}+1-z+qz^2-\cdots)\\ &=(\cdots+q^3z^{-3}-qz^{-2}+z^{-1}-1+qz-\cdots)\\ &\quad-(\cdots+q^3z^{-4}-qz^{-3}+z^2-z^{-1}+q-\cdots).\end{aligned}$$

The residue of the above expansion is 2 and the Cauchy's residue theorem implies

$$\frac{(q;q)_\infty}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z,z^{-1};q)_\infty\frac{dz}{z}=2,$$

which is equivalent to (3.20). Besides, $P(0,-1,q^{\frac12},-q^{\frac12})=(0;q)_\infty/(-q^{\frac12},q^{\frac12},-q;q)_\infty$ $=1$. □

Lemma 3.9.

$$I(0,0,q^{\frac12},-q^{\frac12})=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^2,z^{-2};q^2)_\infty\frac{dz}{z}=\frac{2}{(q^2;q^2)_\infty}, \tag{3.21}$$

$$P(0,0,q^{\frac12},-q^{\frac12})=\frac{1}{(-q;q)_\infty}, \tag{3.22}$$

and $I(0,0,q^{\frac12},-q^{\frac12})/P(0,0,q^{\frac12},-q^{\frac12})=2/(q;q)_\infty$.

Proof. From (2.1), $I(0,0,q^{\frac12},-q^{\frac12})$ is expressed as

$$\begin{aligned}I(0,0,q^{\frac12},-q^{\frac12})&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{(z^2,z^{-2};q)_\infty}{(q^{\frac12}z,q^{\frac12}z^{-1},-q^{\frac12}z,-q^{\frac12}z^{-1};q)_\infty}\frac{dz}{z}\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{(z,-z,q^{\frac12}z,-q^{\frac12}z,z^{-1},-z^{-1},q^{\frac12}z^{-1},-q^{\frac12}z^{-1};q)_\infty}{(q^{\frac12}z,q^{\frac12}z^{-1},-q^{\frac12}z,-q^{\frac12}z^{-1};q)_\infty}\frac{dz}{z}\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z,-z,z^{-1},-z^{-1};q)_\infty\frac{dz}{z}=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^2,z^{-2};q^2)_\infty\frac{dz}{z}.\end{aligned}$$

Using (2.4) for $\vartheta(x;q^2)$ we have

$$\begin{aligned}\frac{(q^2;q^2)_\infty}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^2,z^{-2};q^2)_\infty\frac{dz}{z}&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(1-z^{-2})z^{-1}(z^2,q^2z^{-2},q^2;q^2)_\infty dz\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^{-1}-z^{-3})\vartheta(z^2;q^2)dz=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^{-1}-z^{-3})\sum_{n=-\infty}^{\infty}(-z^2)^nq^{2\binom{n}{2}}dz\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^{-1}-z^{-3})\sum_{n=-\infty}^{\infty}(-z^2)^nq^{n(n-1)}dz,\end{aligned}$$

whose integrand has the unique essential singularity at $z=0$, and is expanded as

$$\begin{aligned}&(z^{-1}-z^{-3})\sum_{n=-\infty}^{\infty}(-z^2)^nq^{n(n-1)}\\ &=(z^{-1}-z^{-3})(\cdots+q^6z^{-4}-q^2z^{-2}+1-z^2+q^2z^4-\cdots)\\ &=(\cdots+q^6z^{-5}-q^2z^{-3}+z^{-1}-z+q^2z^3-\cdots)\\ &\quad-(\cdots+q^6z^{-8}-q^2z^{-6}+z^{-3}-z^{-1}+q^2z-\cdots).\end{aligned}$$

The residue of the above expansion is 2 and the Cauchy's residue theorem implies

$$\frac{(q^2;q^2)_\infty}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z^2,z^{-2};q^2)_\infty\frac{dz}{z}=2,$$

which is equivalent to (3.21). On the other hand, by definition we have (3.22). Thus,

$$\frac{I(0,0,q^{\frac12},-q^{\frac12})}{P(0,0,q^{\frac12},-q^{\frac12})}=\frac{2/(q^2;q^2)_\infty}{1/(-q;q)_\infty}=\frac{2(-q;q)_\infty(q;q)_\infty}{(q^2;q^2)_\infty(q;q)_\infty}=\frac{2(q^2;q^2)_\infty}{(q^2;q^2)_\infty(q;q)_\infty}=\frac{2}{(q;q)_\infty},$$

which completes the proof. □

We conclude this section with the following.

Proof of Proposition 3.1. Since the constant c of (3.16) in Proposition 3.5 is independent of a_1,a_2,a_3,a_4, for any case of Lemmas 3.6–3.9 the constant c equals

$$\frac{I(0,0,0,0)}{P(0,0,0,0)}=\frac{I(0,-1,q^{\frac12},-q^{\frac12})}{P(0,-1,q^{\frac12},-q^{\frac12})}=\frac{I(1,-1,q^{\frac12},-q^{\frac12})}{P(1,-1,q^{\frac12},-q^{\frac12})}=\frac{I(0,0,q^{\frac12},-q^{\frac12})}{P(0,0,q^{\frac12},-q^{\frac12})}=\frac{2}{(q;q)_\infty}.$$

This was the claim of Proposition 3.1. □

4. Nassrallah–Rahman integral

The aim of this section is to provide a way to prove the following identity of the Nassrallah–Rahman integral.

Proposition 4.1. *Suppose that $a_k\in\mathbb{C}^*$ $(1\le k\le 5)$ satisfy $|a_k|<1$. Under the condition $a_1a_2a_3a_4a_5a_6=q$, we have*

$$\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\frac{(z^2,z^{-2},qa_6^{-1}z,qa_6^{-1}z^{-1};q)_\infty}{\prod_{k=1}^{5}(a_kz,a_kz^{-1};q)_\infty}\frac{dz}{z}=\frac{2}{(q;q)_\infty}\frac{\prod_{i=1}^{5}(qa_6^{-1}a_i^{-1};q)_\infty}{\prod_{1\le i<j\le 5}(a_ia_j;q)_\infty}. \tag{4.1}$$

Throughout this section, we define the function $\Phi(z)$ on $\mathbb{C}^*$ by

$$\Phi(z):=\frac{(z^2,z^{-2},qa_6^{-1}z,qa_6^{-1}z^{-1};q)_\infty}{\prod_{k=1}^{5}(a_kz,a_kz^{-1};q)_\infty}, \tag{4.2}$$

and also denote by $J(a_1,a_2,a_3,a_4,a_5,a_6)$ the left-hand side of (4.1). i.e.,

$$J(a_1,a_2,a_3,a_4,a_5,a_6):=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}\Phi(z)\frac{dz}{z}. \tag{4.3}$$

On the other hand, we define $Q(a_1,a_2,a_3,a_4,a_5,a_6)$ by the infinite product

$$Q(a_1,a_2,a_3,a_4,a_5,a_6):=\frac{\prod_{i=1}^{5}(qa_6^{-1}a_i^{-1};q)_\infty}{\prod_{1\le i<j\le 5}(a_ia_j;q)_\infty}. \tag{4.4}$$

4.1. Definition of q-difference operator $\nabla_{q,z}$.

Lemma 4.2. *Let $F_+(z)$ and $F_-(z)$ be functions specified by*

$$F_+(z):=\frac{1}{z^2}\frac{\prod_{k=1}^{6}(1-a_kz)}{1-z^2}=-\frac{1}{z-z^{-1}}z^{-3}\prod_{k=1}^{6}(1-a_kz), \tag{4.5}$$

$$F_-(z):=F_+(z^{-1})=z^2\frac{\prod_{k=1}^{6}(1-a_kz^{-1})}{1-z^{-2}}. \tag{4.6}$$

Then, it follows that

$$\frac{T_{q,z}\Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q,z}F_-(z)}.$$

Proof. By the definition (4.2) of $\Phi(z)$ we have

$$\begin{aligned}
\frac{\Phi(qz)}{\Phi(z)} &= \frac{(q^2z^2, q^{-2}z^{-2};q)_\infty (q^2a_6^{-1}z, a_6^{-1}z^{-1};q)_\infty}{(z^2, z^{-2};q)_\infty (qa_6^{-1}z, qa_6^{-1}z^{-1};q)_\infty} \frac{\prod_{k=1}^5 (a_kz, a_kz^{-1};q)_\infty}{\prod_{k=1}^5 (qa_kz, q^{-1}a_kz^{-1};q)_\infty} \\
&= \frac{(1-q^{-2}z^{-2})(1-q^{-1}z^{-2})}{(1-z^2)(1-qz^2)} \frac{1-a_6^{-1}z^{-1}}{1-qa_6^{-1}z} \prod_{k=1}^5 \frac{1-a_kz}{1-q^{-1}a_kz^{-1}} \\
&= -(q^{-1}z^{-2})^2 \frac{1-q^{-2}z^{-2}}{1-z^2} \prod_{k=1}^6 \frac{1-a_kz}{1-q^{-1}a_kz^{-1}} = -\frac{F_+(z)}{F_-(qz)},
\end{aligned}$$

which completes the proof. □

For an arbitrary meromorphic function $\varphi(z)$ on $\mathbb{C}^*$ we define the symbol

$$\langle \varphi(z) \rangle := \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \varphi(z)\Phi(z)\frac{dz}{z}$$

Proposition 4.3. *For an arbitrary meromorphic function $\varphi(z)$ on $\mathbb{C}^*$, let $\nabla_{q,z}$ be operator specified by*

$$(\nabla_{q,z}\varphi)(z) := F_-(z)\varphi(z) + F_+(z)T_{q,z}\varphi(z). \tag{4.7}$$

Suppose that $|a_k| < 1$ $(k = 1, \ldots, 5)$. For an arbitrary holomorphic function $\varphi(z)$ on $\mathbb{C}^*$, it follows that

$$\langle \nabla_{q,z}\varphi(z) \rangle = 0. \tag{4.8}$$

Proof. The argument of the proof is paralleled with that of Proposition 3.3 and we omit the details. □

4.2. **q-Difference equations for the Nassrallah–Rahman integral.** In this subsection we explain a derivation of the q-difference equations for the Nassrallah–Rahman integral given as follows:

Proposition 4.4. *Suppose that $|a_k| < 1$ $(k = 1, \ldots, 5)$. Under the condition*

$$a_1a_2a_3a_4a_5a_6 = q,$$

the integral $J(a_1, \ldots, a_6)$ satisfies

$$J(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6) = J(a_1, a_2, a_3, a_4, a_5, a_6) \prod_{k=2}^5 \frac{1-a_1a_k}{1-qa_k^{-1}a_6^{-1}}. \tag{4.9}$$

This proposition is equivalent to the following.

Proposition 4.5. *Suppose that $|a_k| < 1$ $(k = 1, \ldots, 5)$. Under the condition*

$$a_1a_2a_3a_4a_5a_6 = 1,$$

the integral $J(a_1, \ldots, a_6)$ satisfies

$$J(qa_1, a_2, a_3, a_4, a_5, a_6) = J(a_1, a_2, a_3, a_4, a_5, qa_6) \prod_{k=2}^5 \frac{1-a_1a_k}{1-a_k^{-1}a_6^{-1}}, \tag{4.10}$$

which is equivalent to

$$\langle e(a_1,z)\rangle = \langle e(a_6,z)\rangle \frac{a_6^2}{a_1^2}\prod_{k=2}^{5}\frac{1-a_1a_k}{1-a_ka_6}. \tag{4.11}$$

Remark. If we consider the substitution $a_6 \to q^{-1}a_6$ for Proposition 4.5, then the balancing condition changes as $a_1a_2a_3a_4a_5a_6 = 1 \to a_1a_2a_3a_4a_5(q^{-1}a_6) = 1$, i.e., $a_1a_2a_3a_4a_5a_6 = q$ and (4.10) becomes (4.9). We prove Proposition 4.5 instead of Proposition 4.4.

Proof. We will prove (4.11) in Lemma 4.6. Here we just confirm the equivalence between (4.10) and (4.11). Since $T_{q,a_1}\Phi(z)/\Phi(z) = a_1 e(a_1,z)$ and $T_{q,a_6}\Phi(z)/\Phi(z) = a_6^{-1}e(a_6,z)$ by definition, we have

$$J(qa_1,a_2,a_3,a_4,a_5,a_6) = a_1\langle e(a_1,z)\rangle, \quad J(a_1,a_2,a_3,a_4,a_5,qa_6) = a_6^{-1}\langle e(a_6,z)\rangle,$$

so that

$$\frac{J(qa_1,a_2,a_3,a_4,a_5,a_6)}{J(a_1,a_2,a_3,a_4,a_5,qa_6)} = a_1a_6\frac{\langle e(a_1,z)\rangle}{\langle e(a_6,z)\rangle}. \tag{4.12}$$

On the other hand, under the condition $a_1a_2a_3a_4a_5a_6 = 1$, we have

$$\prod_{k=2}^{5}\frac{1-a_1a_k}{1-a_k^{-1}a_6^{-1}} = \frac{1}{a_1a_2a_3a_4a_5a_6}\prod_{k=2}^{5}\frac{1-a_1a_k}{1-a_k^{-1}a_6^{-1}} = \frac{a_6^3}{a_1}\prod_{k=2}^{5}\frac{1-a_1a_k}{1-a_ka_6}. \tag{4.13}$$

(4.13) and (4.13) implies the equivalence between (4.10) and (4.11) □

Lemma 4.6. *Suppose that $|a_k| < 1$ $(k = 1,\dots,5)$. Under the condition*

$$a_1a_2a_3a_4a_5a_6 = 1,$$

we have

$$C_1\langle e(a_1,z)\rangle + C_6\langle e(a_6,z)\rangle = 0,$$

where the coefficients C_1 and C_6 are given as

$$C_1 = \frac{a_1}{a_6(a_6-a_1)}\prod_{k=2}^{5}(1-a_ka_6), \tag{4.14}$$

$$C_6 = \frac{a_6}{a_1(a_1-a_6)}\prod_{k=2}^{5}(1-a_1a_k). \tag{4.15}$$

Proof. Taking $\varphi(z)$ in (4.7) as $\varphi(z) = 1$, $\nabla_{q,z}\varphi(z)$ is written as

$$\begin{aligned}
\nabla_{q,z}\varphi(z) &= F_-(z) + F_+(z)\\
&= \frac{1}{z^{-2}}\frac{\prod_{k=1}^{6}(1-a_kz^{-1})}{1-z^{-2}} + \frac{1}{z^2}\frac{\prod_{k=1}^{6}(1-a_kz)}{1-z^2}\\
&= \frac{1}{z-z^{-1}}\Big(\frac{1}{z^{-3}}\prod_{k=1}^{6}(1-a_kz^{-1}) - \frac{1}{z^3}\prod_{k=1}^{6}(1-a_kz)\Big)\\
&= \frac{1}{z-z^{-1}}\Big(S_0z^3 - S_1z^2 + S_2z - S_3 + S_4z^{-1} - S_5z^{-2} + S_6z^{-3}\\
&\qquad - S_0z^{-3} + S_1z^{-2} - S_2z^{-1} + S_3 - S_4z + S_5z^2 - S_6z^3\Big)
\end{aligned}$$

$$(4.16) \qquad = (S_0 - S_6)(z^2 + z^{-2}) + (S_5 - S_1)(z + z^{-1}) + (S_0 + S_2 - S_4 - S_6),$$

where S_r $(r = 0, 1, \ldots, 6)$ is the rth elementary symmetric polynomial of $a_1, \ldots, a_6$. The condition $a_1a_2a_3a_4a_5a_6 = 1$ implies $S_0 - S_6 = 0$. From (4.16), we have

$$\nabla_{q,z}\varphi(z) = (S_5 - S_1)(z + z^{-1}) + (S_0 + S_2 - S_4 - S_6) \in \mathbb{C}(z + z^{-1}) \oplus \mathbb{C}\,1.$$

Since the set $\{e(a_1, z), e(a_6, z)\}$ forms a $\mathbb{C}$-basis of the space $\mathbb{C}(z+z^{-1}) \oplus \mathbb{C}\,1$, we can expand $\nabla_{q,z}\varphi(z)$ as

$$(4.17) \qquad \nabla_{q,z}\varphi(z) = C_1 e(a_1, z) + C_6 e(a_6, z),$$

where C_1 and C_6 are some constants independent of z. Thus we can write

$$(4.18) \qquad F_-(z) + F_+(z) = C_1 e(a_1, z) + C_6 e(a_6, z).$$

Since $e(a_1, a_1) = 0$ and $F_-(a_1) = 0$, (4.18) with $z = a_1$ implies

$$F_+(a_1) = C_6 e(a_6, a_1).$$

In the same way, (4.18) with $z = a_6$ implies $F_+(a_6) = C_1 e(a_1, a_6)$. Thus we obtain

$$C_1 = \frac{F_+(a_6)}{e(a_1, a_6)} = \frac{a_1}{a_6(a_6 - a_1)} \prod_{k=2}^{5} (1 - a_k a_6),$$

$$C_6 = \frac{F_+(a_1)}{e(a_6, a_1)} = \frac{a_6}{a_1(a_1 - a_6)} \prod_{k=2}^{5} (1 - a_1 a_k).$$

Applying Proposition 4.3 to (4.17), we obtain

$$C_1 \langle e(a_1, z) \rangle + C_6 \langle e(a_6, z) \rangle = \langle \nabla_{q,z}\varphi(z) \rangle = 0,$$

which completes the proof. □

Proposition 4.7. *Suppose that* $|a_k| < 1$ $(k = 1, \ldots, 5)$. *Under the condition*

$$a_1a_2a_3a_4a_5a_6 = q,$$

$J(a_1, \ldots, a_6)$ *coincides with* $Q(a_1, \ldots, a_6)$ *up to a multiplicative constant, i.e.,*

$$(4.19) \qquad J(a_1, a_2, a_3, a_4, a_5, a_6) = c\, Q(a_1, a_2, a_3, a_4, a_5, a_6),$$

where c *is some constant independent of* $a_1, \ldots, a_6$.

Proof. By the definition (4.4), $Q(a_1, \ldots, a_6)$ satisfies the same q-difference equation as (4.9), i.e.,

$$(4.20) \qquad Q(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6) = Q(a_1, a_2, a_3, a_4, a_5, a_6) \prod_{k=2}^{5} \frac{1 - a_1 a_k}{1 - q a_k^{-1} a_6^{-1}}.$$

Thus, under the condition $a_1a_2a_3a_4a_5a_6 = q$, considering the ratio of (4.9) and (4.20),

$$\frac{J(a_1, a_2, a_3, a_4, a_5, a_6)}{Q(a_1, a_2, a_3, a_4, a_5, a_6)} = \frac{J(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6)}{Q(qa_1, a_2, a_3, a_4, a_5, q^{-1}a_6)},$$

which is symmetric with respect to $a_1, \cdots, a_5$. Therefore we obtain

$$
\begin{aligned}
\frac{J(a_1,a_2,a_3,a_4,a_5,a_6)}{Q(a_1,a_2,a_3,a_4,a_5,a_6)} &= \frac{J(qa_1,qa_2,qa_3,qa_4,qa_5,q^{-5}a_6)}{Q(qa_1,qa_2,qa_3,qa_4,qa_5,q^{-5}a_6)} \\
&= \frac{J(q^N a_1,q^N a_2,q^N a_3,q^N a_4,q^N a_5,q^{-5N}a_6)}{Q(q^N a_1,q^N a_2,q^N a_3,q^N a_4,q^N a_5,q^{-5N}a_6)} \\
&= \lim_{N\to\infty} \frac{J(q^N a_1,q^N a_2,q^N a_3,q^N a_4,q^N a_5,q^{-5N}a_6)}{Q(q^N a_1,q^N a_2,q^N a_3,q^N a_4,q^N a_5,q^{-5N}a_6)}.
\end{aligned} \tag{4.21}
$$

From (4.4), we have

$$
\begin{aligned}
&\lim_{N\to\infty} Q(q^N a_1,q^N a_2,q^N a_3,q^N a_4,q^N a_5,q^{-5N}a_6) \\
&= \lim_{N\to\infty} \frac{\prod_{i=1}^5 (q^{1+4N}a_6^{-1}a_i^{-1};q)_\infty}{\prod_{1\le i<j\le 5}(q^{2N}a_ia_j;q)_\infty} = 1,
\end{aligned} \tag{4.22}
$$

and putting c as $c = \lim\limits_{N\to\infty} J(q^N a_1,q^N a_2,q^N a_3,q^N a_4,q^N a_5,q^{-5N}a_6)$ we obtain

$$
\begin{aligned}
c &= \lim_{N\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2,z^{-2},q^{1+5N}a_6^{-1}z,q^{1+5N}a_6^{-1}z^{-1};q)_\infty}{\prod_{i=1}^5 (q^N a_i z, q^N a_i z^{-1};q)_\infty} \frac{dz}{z} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2,z^{-2};q)_\infty \frac{dz}{z}.
\end{aligned} \tag{4.23}
$$

Therefore, (4.21), (4.22) and (4.23) imply (4.19). □

Proof of proposition 4.1. Since we already knew that

$$\frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} (z^2,z^{-2};q)_\infty \frac{dz}{z} = \frac{2}{(q;q)_\infty}$$

from Lemma 3.7, the constant c as (4.23) is equal to $2/(q;q)_\infty$. □

4.3. **Special values of the Nassrallah–Rahman integral.** We can find the special values that are simply computed.

Lemma 4.8.

$$
\begin{aligned}
J(1,-1,q^{\frac12},-q^{\frac12},a_5,a_5^{-1}) &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{dz}{(1-a_5z)(z-a_5)} = \frac{1}{1-a_5^2}, \\
Q(1,-1,q^{\frac12},-q^{\frac12},a_5,a_5^{-1}) &= \frac{(q;q)_\infty}{2(1-a_5^2)}.
\end{aligned}
$$

Proof. When $a_1=1$, $a_2=-1$, $a_3=q^{\frac12}$, $a_4=-q^{\frac12}$, the balancing condition $a_1a_2a_3a_4a_5a_6 = q$ implies $a_5a_6 = 1$Using $a_6 = a_5^{-1}$ the integral $J(a_1,\dots,a_6)$ is computed as

$$
\begin{aligned}
&J(1,-1,q^{\frac12},-q^{\frac12},a_5,a_5^{-1}) \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2,z^{-2},qa_5z,qa_5z^{-1};q)_\infty}{(z,z^{-1},-z,-z^{-1},q^{\frac12}z,q^{\frac12}z^{-1},-q^{\frac12}z,-q^{\frac12}z^{-1},a_5z,a_5z^{-1};q)_\infty} \frac{dz}{z} \\
&= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{1}{(1-a_5z)(1-a_5z^{-1})} \frac{dz}{z} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{dz}{(1-a_5z)(z-a_5)}
\end{aligned}
$$

$$= \operatorname*{Res}_{z=a_5} \frac{dz}{(1-a_5 z)(z-a_5)} \quad \text{(by Cauchy's residue theorem with } |a_5|<1)$$
$$= \lim_{z\to a_5} \frac{1}{1-a_5 z} = \frac{1}{1-a_5^2}.$$

On the other hand, from (4.4), $Q(1,-1,q^{\frac12},-q^{\frac12},a_5,a_5^{-1})$ is computed as

$$Q(1,-1,q^{\frac12},-q^{\frac12},a_5,a_5^{-1}) = \frac{(qa_5,-qa_5,q^{\frac12}a_5,-q^{\frac12}a_5,q;q)_\infty}{(-1,q^{\frac12},-q^{\frac12},a_5,-q^{\frac12},q^{\frac12},-a_5,-q,q^{\frac12}a_5,-q^{\frac12}a_5;q)_\infty}$$
$$= \frac{(qa_5,-qa_5,q;q)_\infty}{(-1,q^{\frac12},-q^{\frac12},a_5,-q^{\frac12},q^{\frac12},-a_5,-q;q)_\infty} = \frac{(qa_5,-qa_5,q;q)_\infty}{2(a_5,-a_5;q)_\infty(-q,q^{\frac12},-q^{\frac12};q)_\infty^2}$$
$$= \frac{(q;q)_\infty}{2(1-a_5)(1+a_5)}, \quad \text{(from (2.2))}$$

which completes the proof. □

The constant c in Proposition 4.7 is also obtained from

$$c = \frac{J(1,-1,q^{\frac12},-q^{\frac12},a_5,a_5^{-1})}{Q(1,-1,q^{\frac12},-q^{\frac12},a_5,a_5^{-1})} = \frac{1/(1-a_5^2)}{(q;q)_\infty/2(1-a_5^2)} = \frac{2}{(q;q)_\infty},$$

which also completes the proof of Proposition 4.1.

4.4. **The relation between the Askey–Wilson and Nassrallah–Rahman integrals.** Since the Askey–Wilson integral $I(a_1,\dots,a_4)$ is obtained from the Nassrallah–Rahman integral $J(a_1,\dots,,a_6)$ as a limiting case of $a_5\to 0$, we can understand $I(a_1,a_2,a_3,a_4)$, is a special value of $J(a_1,\dots,,a_6)$. Conversely $J(a_1,\dots,a_6)$ is obtained from $I(a_1,\dots,a_4)$ using the following relation. (See [7] for the BC_n case.)

Proposition 4.9. *Suppose that $|a_k|<1$ $(k=1,\dots,5)$ and the condition*

$$a_1a_2a_3a_4a_5a_6 = q.$$

Then,

$$J(a_1,a_2,a_3,a_4,a_5,a_6) = I(a_1,a_2,a_3,a_4)\prod_{k=1}^{4}\frac{(qa_k^{-1}a_6^{-1};q)_\infty}{(a_ka_5;q)_\infty}. \tag{4.24}$$

Proof. Using the recurrence relation (4.9) for $J(a_1,\dots,a_6)$ repeatedly we obtain

$$J(a_1,a_2,a_3,a_4,a_5,a_6) = J(a_1,a_2,a_3,a_4,qa_5,q^{-1}a_6)\prod_{k=1}^{4}\frac{1-qa_k^{-1}a_6^{-1}}{1-a_ka_5}$$
$$= J(a_1,a_2,a_3,a_4,q^Na_5,q^{-N}a_6)\prod_{k=1}^{4}\frac{(qa_k^{-1}a_6^{-1};q)_N}{(a_ka_5;q)_N}$$
$$= \lim_{N\to\infty} J(a_1,a_2,a_3,a_4,q^Na_5,q^{-N}a_6)\prod_{k=1}^{4}\frac{(qa_k^{-1}a_6^{-1};q)_\infty}{(a_ka_5;q)_\infty},$$

where we also have

$$\lim_{N\to\infty} J(a_1,a_2,a_3,a_4,q^Na_5,q^{-N}a_6)$$

$$= \lim_{N\to\infty} \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}, q^{1+N}a_6^{-1}z, q^{1+N}a_6^{-1}z^{-1}; q)_\infty}{(q^N a_5 z, q^N a_5 z^{-1}; q)_\infty \prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \frac{dz}{z}$$
$$= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{T}} \frac{(z^2, z^{-2}; q)_\infty}{\prod_{k=1}^4 (a_k z, a_k z^{-1}; q)_\infty} \frac{dz}{z} = I(a_1, a_2, a_3, a_4),$$

which implies (4.24). □

5. Root system of type G_2

5.1. Root system of type G_2 and its Weyl group W. In this subsection, we follow [10] for the basic terminology of the root system of type G_2 and its Weyl group W.

Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be the standard basis of $\mathbb{R}^3$ with the inner product $(\cdot,\cdot)$ satisfying $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, and let V be the hyperplane in $\mathbb{R}^3$ with equation $\xi_1 + \xi_2 + \xi_3 = 0$, i.e., $V = \{\xi \in \mathbb{R}^3 \mid (\xi, \varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 0\}$.

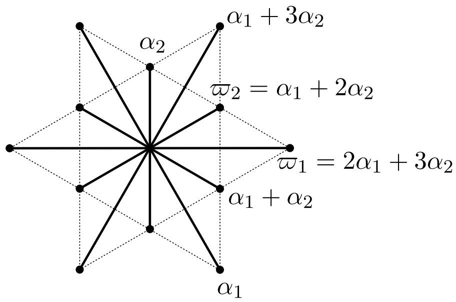

Figure 1. Root system R

Let $R \subset V$ be the root system of type G_2 given by

$$R = \{\pm\bar\varepsilon_1, \pm\bar\varepsilon_2, \pm\bar\varepsilon_3\} \cup \{\pm(\bar\varepsilon_1 - \bar\varepsilon_2), \pm(\bar\varepsilon_1 - \bar\varepsilon_3), \pm(\bar\varepsilon_2 - \bar\varepsilon_3)\},$$

where $\bar\varepsilon_i = \varepsilon_i - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3$. We refer the setting of the root system of type G_2 to Macdonald's book [12]. We fix the set of simple roots $\{\alpha_1, \alpha_2\} \subset R$ given by

$$\alpha_1 = \bar\varepsilon_1 - \bar\varepsilon_2 = \varepsilon_1 - \varepsilon_2, \qquad \alpha_2 = \bar\varepsilon_2 = (-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3)/3.$$

The set of positive roots is given by

$$\begin{aligned} R^+ &= \{\bar\varepsilon_1, \bar\varepsilon_2, -\bar\varepsilon_3\} \cup \{\bar\varepsilon_1 - \bar\varepsilon_2, \bar\varepsilon_1 - \bar\varepsilon_3, \bar\varepsilon_2 - \bar\varepsilon_3\} \\ &= \{\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\} \cup \{\alpha_1, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}. \end{aligned}$$

We also fix the set of fundamental weights $\{\varpi_1, \varpi_2\}$ by $(\alpha_i^\vee, \varpi_j) = \delta_{ij}$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. This implies that

$$\varpi_1 = 2\alpha_1 + 3\alpha_2, \qquad \varpi_2 = \alpha_1 + 2\alpha_2.$$

Let P and Q be the weight lattice and root lattice defined by $P = \mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2$ and $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$, respectively. For the root system G_2, the root lattice Q coincides with the weight lattice P.

Let W be the Weyl group of type G_2 generated by orthogonal reflections s_α ($\alpha \in R$) with respect to the hyperplane perpendicular to $\alpha \in R$, which are given by $s_\alpha(\xi) = \xi - (\alpha^\vee, \xi)\alpha$. The group W is generated by the reflections $s_i = s_{\alpha_i} : V \to V$ ($i = 1, 2$), and is isomorphic to the dihedral group of order 12. Moreover, W is explicitly written as

$$W = \{(s_1 s_2)^k, (s_1 s_2)^k s_2 \,|\, k = 0, 1, \ldots, 5\}, \tag{5.1}$$

The element $w_0 = (s_1 s_2)^3$ is the *longest element* of W. Note also that the inner product and the reflections are uniquely extended linearly to $V_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} V$.

We fix the set of fundamental coweights $\{\omega_1, \omega_2\}$ by $(\omega_i, \alpha_j) = \delta_{ij}$, so that $\omega_1 = \varpi_1$, $\omega_2 = 3\varpi_2$. Let $P^\vee$ be the coweight lattice defined by $P^\vee = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. For $c \in \mathbb{C}$ and $\omega \in P^\vee$ we denote by $S_{c,\omega}$ the c-shift operator with respect to ω for functions $f(\zeta)$ on $V_\mathbb{C}$ by

$$S_{c,\omega} f(\zeta) = f(\zeta + c\,\omega). \tag{5.2}$$

We also define action of the Weyl group W on $f(\zeta)$ by

$$w.f(\zeta) = f(w^{-1}\zeta) \quad (w \in W). \tag{5.3}$$

We consider the mapping from $V_\mathbb{C}$ to $(\mathbb{C}^*)^2$ by

$$\zeta \mapsto z = (e^{2\pi\sqrt{-1}(\zeta,\alpha_1)}, e^{2\pi\sqrt{-1}(\zeta,\alpha_2)}). \tag{5.4}$$

If we write $\zeta \in V_\mathbb{C}$ with the fundamental coweights by $\zeta = \zeta_1\omega_1 + \zeta_2\omega_2$, then the above mapping is written as $\zeta \mapsto z = (e^{2\pi\sqrt{-1}\zeta_1}, e^{2\pi\sqrt{-1}\zeta_2})$. For $\lambda \in P$, we write $z^\lambda = e^{2\pi\sqrt{-1}(\zeta,\lambda)}$. In particular, for $\lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2 \in Q = P$ we have the expression $z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2}$, where $z_i = z^{\alpha_i}$. Through (5.3) and (5.4), for $w \in W$ we can define $w.z^\lambda = z^{w\lambda}$, i.e.,

$$w.z^\lambda = w.e^{2\pi\sqrt{-1}(\zeta,\lambda)} = e^{2\pi\sqrt{-1}(w^{-1}\zeta,\lambda)} = e^{2\pi\sqrt{-1}(\zeta,w\lambda)} = z^{w\lambda},$$

and we can also define $w.f(z)$ for functions $f(z) = f(z_1, z_2)$ on $(\mathbb{C}^*)^2$ as

$$w.f(z) = f(w.z_1, w.z_2) = f(z^{w\alpha_1}, z^{w\alpha_2}),$$

so that, for instance, we have

$$s_1.f(z_1, z_2) = f(z_1^{-1}, z_1 z_2), \quad s_2.f(z_1, z_2) = f(z_1 z_2^3, z_2^{-1}), \tag{5.5}$$

and

$$w_0.f(z_1, z_2) = (s_1 s_2)^3.f(z_1, z_2) = f(z_1^{-1}, z_2^{-1}). \tag{5.6}$$

We say that a function $f(z)$ is W-symmetric if $w.f(z) = f(z)$ for all $w \in W$, and that $f(z)$ is W-skew symmetric if $w.f(z) = (\operatorname{sgn} w) f(z)$ for all $w \in W$. By chain rule for differential forms, we have

$$\begin{aligned} \frac{d(s_1.z_1)}{s_1.z_1} &= -\frac{dz_1}{z_1}, \quad \frac{d(s_1.z_2)}{s_1.z_2} = \frac{dz_1}{z_1} + \frac{dz_2}{z_2} \\ \frac{d(s_2.z_1)}{s_2.z_1} &= \frac{dz_1}{z_1} + 3\frac{dz_2}{z_2}, \quad \frac{d(s_2.z_2)}{s_2.z_2} = -\frac{dz_2}{z_2}. \end{aligned} \tag{5.7}$$

We fix $q = e^{2\pi\sqrt{-1}\tau}$, where $\operatorname{Im}\tau > 0$. If we consider a function $f(z) = f(z_1, z_2)$ on $(\mathbb{C}^*)^2$ as the function on $V_{\mathbb{C}}$ through (5.4), then the q-shift operators for $f(z)$ with respect to z_i $(i = 1, 2)$

$$T_{q,z_1} f(z) = f(qz_1, z_2), \qquad T_{q,z_2} f(z) = f(z_1, qz_2)$$

are induced by the τ-shift operators S_{τ,ω_i} with respect to $\omega_i \in P^\vee$ $(i = 1, 2)$, respectively.

Using the notation $x_i = e^{2\pi\sqrt{-1}(\zeta,\bar{\varepsilon}_i)}$ $(i = 1, 2, 3)$, we have

$$x_1 x_2 x_3 = e^{2\pi\sqrt{-1}(\zeta,\bar{\varepsilon}_1+\bar{\varepsilon}_2+\bar{\varepsilon}_3)} = 1$$

and the variable change $(z_1, z_2) \mapsto (x_1, x_2)$ of $(\mathbb{C}^*)^2$, where

$$x_1 = z_1 z_2, \quad x_2 = z_2 \quad \text{and} \quad \frac{dx_1}{x_1} = \frac{dz_1}{z_1} + \frac{dz_2}{z_2}, \quad \frac{dx_2}{x_2} = \frac{dz_2}{z_2}. \tag{5.8}$$

Though using the coordinates (x_1, x_2, x_3) with $x_1 x_2 x_3 = 1$ instead of (z_1, z_2) we sometimes have simple expressions for functions on $(\mathbb{C}^*)^2$ in appearance, like the integrand shown in the left-hand side of (1.6) for instance, we use the coordinates (z_1, z_2) of $(\mathbb{C}^*)^2$ associated with simple roots in the succeeding sections.

5.2. W **invariant Laurent polynomials.** For a function $f(z)$ on $(\mathbb{C}^*)^2$ we define

$$\mathcal{A}f(z) := \sum_{w \in W} (\operatorname{sgn} w) w.f(z)$$

which we call the W-skew symmetrization of $f(z)$. By definition $\mathcal{A}f(z)$ is W-skew symmetric. The dominance ordering $<$ on P is defined by

$$\mu \le \lambda \quad \Longleftrightarrow \quad \lambda - \mu \in Q_+ = \mathbb{N}\alpha_1 + \mathbb{N}\alpha_2,$$

for $\mu, \lambda \in P$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We set

$$P_+ := \{\lambda \in P \,|\, (\alpha_1, \lambda) \ge 0, (\alpha_2, \lambda) \ge 0\} = \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2,$$

whose elements are called the *dominant weights*.

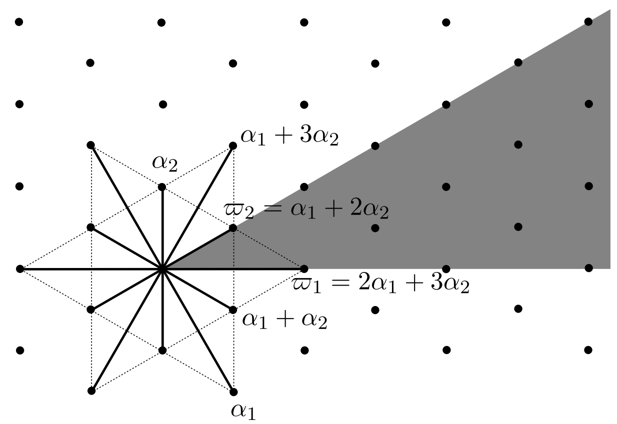

FIGURE 2. Dominant weights in P_+

For $\rho := \frac{1}{2}\sum_{\alpha \in R_+} \alpha = 3\alpha_1 + 5\alpha_2 = \varpi_1 + \varpi_2$, $\mathcal{A}(z^\rho)$ satisfies

$$\mathcal{A}(z^\rho) = z_1^{-3} z_2^{-5} (1 - z_2)(1 - z_1 z_2)(1 - z_1 z_2^2)(1 - z_1)(1 - z_1 z_2^3)(1 - z_1^2 z_2^3), \tag{5.9}$$

which is called the *Weyl denominator*. For $\lambda \in P_+$ we set

$$s_\lambda(z) := \frac{\mathcal{A}(z^{\rho+\lambda})}{\mathcal{A}(z^\rho)}, \qquad m_\lambda(z) := \sum_{\mu \in W\lambda} z^\mu,$$

where $W\lambda = \{w\lambda \,|\, w \in W\}$. The functions $s_\lambda(z)$ and $m_\lambda(z)$ are W-invariant Laurent polynomials, and $s_\lambda(z)$ are expanded as

$$s_\lambda(z) = m_\lambda(z) + \sum_{\substack{\mu \in P_+ \\ \mu < \lambda}} c_\mu m_\mu(z).$$

For instance, we see

$$\begin{aligned}
s_0(z) &= 1, \\
s_{\varpi_2}(z) &= m_{\varpi_2}(z) + 1, \\
s_{\varpi_1}(z) &= m_{\varpi_1}(z) + m_{\varpi_2}(z) + 2, \\
s_{2\varpi_2}(z) &= m_{2\varpi_2}(z) + m_{\varpi_1}(z) + 2m_{\varpi_2}(z) + 3, \\
&\ \vdots
\end{aligned}$$

and

$$\begin{aligned}
m_0(z) &= 1, \\
m_{\varpi_2}(z) &= z_2 + z_2^{-1} + z_1 z_2 + z_1^{-1} z_2^{-1} + z_1 z_2^2 + z_1^{-1} z_2^{-2}, \\
m_{\varpi_1}(z) &= z_1 + z_1^{-1} + z_1 z_2^3 + z_1^{-1} z_2^{-3} + z_1^2 z_2^3 + z_1^{-2} z_2^{-3}, \\
m_{2\varpi_2}(z) &= z_2^2 + z_2^{-2} + z_1^2 z_2^2 + z_1^{-2} z_2^{-2} + z_1^2 z_2^4 + z_1^{-2} z_2^{-4}, \\
&\ \vdots
\end{aligned}$$

We denote by $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]^W$ the set of W-invariant Laurent polynomials, which satisfies

$$\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]^W = \bigoplus_{\lambda \in P_+} \mathbb{C} s_\lambda(z) = \bigoplus_{\lambda \in P_+} \mathbb{C} m_\lambda(z).$$

Lemma 5.1. *For $n \in \mathbb{N}$, let $\mathcal{F}_n$ be $\mathbb{C}$ vector space defined by*

$$\mathcal{F}_n := \bigoplus_{\substack{\lambda \in P_+ \\ \lambda \le n\varpi_2}} \mathbb{C} m_\lambda(z). \tag{5.10}$$

Then the dimension of $\mathcal{F}_n$ as a $\mathbb{C}$-vector space is given as

$$\dim_{\mathbb{C}} \mathcal{F}_{2m} = (m+1)^2, \quad \dim_{\mathbb{C}} \mathcal{F}_{2m+1} = (m+1)(m+2) \qquad (m = 0, 1, 2, \ldots).$$

For any $a \in \mathbb{C}^*$ and any $z = (z_1, z_2) \in (\mathbb{C}^*)^2$ we define $g(a; z)$ by

$$\begin{aligned}
g(a; z) &:= e(a, z_2) e(a, z_1 z_2) e(a, z_1 z_2^2) \\
&= a^{-3}(1 - a z_2)(1 - a z_2^{-1})(1 - a z_1 z_2) \\
&\qquad \times (1 - a z_1^{-1} z_2^{-1})(1 - a z_1 z_2^2)(1 - a z_1^{-1} z_2^{-2}),
\end{aligned} \tag{5.11}$$

where the symbol $e(u,v)$ is by (2.3). The expansion of $g(a;z)$ by $m_\lambda(z)$ ($\lambda \in P_+$) is given as

$$
\begin{aligned}
g(a;z) &= -m_{2\varpi_2}(z) + (a+a^{-1})m_{\varpi_1}(z) \\
&\qquad - \Big((a+a^{-1})^2 - (a+a^{-1})\Big)m_{\varpi_2}(z) + \Big((a+a^{-1})^3 - 2\Big)m_0(z),
\end{aligned} \tag{5.12}
$$

and the expansion of $g(a;z)$ by $s_\lambda(z)$ ($\lambda \in P_+$) is given as

$$
\begin{aligned}
g(a;z) &= -s_{2\varpi_2}(z) + (a+a^{-1}+1)s_{\varpi_1}(z) - \Big((a+a^{-1})^2 - 1\Big)s_{\varpi_2}(z) \\
&\qquad + \Big((a+a^{-1})^3 + (a+a^{-1})^2 - 2(a+a^{-1}) - 2\Big)s_0(z) \\
&= -s_{2\varpi_2}(z) + \frac{(1-a^3)}{a(1-a)}s_{\varpi_1}(z) - \frac{(1-a^6)}{a^2(1-a^2)}s_{\varpi_2}(z) \\
&\qquad + \frac{(1-a^3)(1-a^8)}{a^3(1-a)(1-a^4)}s_0(z).
\end{aligned} \tag{5.13}
$$

From (5.12), we see

$$
g(a;z) \in \mathcal{F}_2, \quad g(a;z)g(b;z) \in \mathcal{F}_4
$$

5.3. **$\mathbb{C}$-vector space $\mathcal{F}_2$.** By the definition (5.10) of $\mathcal{F}_n$, we have

$$
\mathcal{F}_2 = \mathbb{C}m_0(z) \oplus \mathbb{C}m_{\varpi_2}(z) \oplus \mathbb{C}m_{\varpi_1}(z) \oplus \mathbb{C}m_{2\varpi_2}(z),
$$

so that $\dim_{\mathbb{C}} \mathcal{F}_2 = 4$. Here we introduce two bases of the $\mathbb{C}$-vector space $\mathcal{F}_2$. For this purpose, we define two types of specific points as follows: Let p_{ij}, p^*_{ij} ($1 \le i < j \le 6$) be points in $(\mathbb{C}^*)^2$ specified by

$$
\mathrm{p}_{ij} := (a_i/a_j, a_j) \in (\mathbb{C}^*)^2, \tag{5.14}
$$

(so that we have $z_2 = a_j,\ z_1z_2 = a_i,\ z_1z_2^2 = a_ia_j$ if $z = \mathrm{p}_{ij}$) and

$$
\mathrm{p}^*_{ij} := (a_i^2/a_j, a_j/a_i) \in (\mathbb{C}^*)^2, \tag{5.15}
$$

(so that we have $z_2 = a_j/a_i,\ z_1z_2 = a_i,\ z_1z_2^2 = a_j$ if $z = \mathrm{p}^*_{ij}$).

5.3.1. **The basis $\{e_0(z), e_{\varpi_2}(z), e_{\varpi_1}(z), e_{2\varpi_2}(z)\}$.** We construct a basis of $\mathbb{C}$-vector space $\mathcal{F}_2$. We first define

$$
\begin{aligned}
e_0(z) &:= 1, \\
e_{\varpi_2}(z) &:= m_{\varpi_2}(z) - m_{\varpi_2}(\mathrm{p}_{12}), \\
e_{2\varpi_2}(z) &:= -g(a_1;z) = m_{2\varpi_2}(z) - (a_1 + a_1^{-1})m_{\varpi_1}(z) + \cdots .
\end{aligned}
$$

Moreover we define $e_{\varpi_1}(z)$ as

$$
e_{\varpi_1}(z) := m_{\varpi_1}(z) + c_{\varpi_2}m_{\varpi_2}(z) + c_0, \tag{5.16}
$$

where the coefficients c_{ϖ_2} and c_0 are uniquely determined to be satisfied as

$$
e_{\varpi_1}(\mathrm{p}_{12}) = 0 \quad \text{and} \quad e_{\varpi_1}(\mathrm{p}^*_{12}) = 0.
$$

We remark that the coefficients c_{ϖ_2} and c_0 are actually given as

$$
c_{\varpi_2} := -\frac{m_{\varpi_1}(\mathrm{p}_{12}) - m_{\varpi_1}(\mathrm{p}^*_{12})}{m_{\varpi_2}(\mathrm{p}_{12}) - m_{\varpi_2}(\mathrm{p}^*_{12})}, \quad c_0 := \frac{m_{\varpi_1}(\mathrm{p}_{12})m_{\varpi_2}(\mathrm{p}^*_{12}) - m_{\varpi_1}(\mathrm{p}^*_{12})m_{\varpi_2}(\mathrm{p}_{12})}{m_{\varpi_2}(\mathrm{p}_{12}) - m_{\varpi_2}(\mathrm{p}^*_{12})}.
$$

for $e_{\varpi_1}(z)$.

Remark. The values of $e_{\varpi_1}(\mathrm{p}_{1j})$ and $e_{\varpi_1}(\mathrm{p}_{1j}^*)$ are actually given

$$e_{\varpi_1}(\mathrm{p}_{1j}) = e(a_2, a_j)e(a_2, a_1a_j), \quad e_{\varpi_1}(\mathrm{p}_{1j}^*) = e(a_2, a_j)e(a_2, a_1/a_j),$$

respectively. (We do not use these explicit forms in this paper.)

By definition the set $\{e_0(z),\ e_{\varpi_2}(z),\ e_{\varpi_1}(z),\ e_{2\varpi_2}(z)\}$ is linearly independent.

	p_{12}	p_{12}^*	p_{13}	p_{23}
$e_0(z)$	1	1	1	1
$e_{\varpi_2}(z)$	0	$*$	$*$	$*$
$e_{\varpi_1}(z)$	0	0	$*$	$*$
$e_{2\varpi_2}(z)$	0	0	0	$*$

In particular, since we have

$$e_{\varpi_2}(z) = m_{\varpi_2}(z) - m_{\varpi_2}(\mathrm{p}_{12}) = s_{\varpi_2}(z) - s_{\varpi_2}(\mathrm{p}_{12}) \tag{5.17}$$

we can actually compute $e_{\varpi_2}(\mathrm{p}_{1j})$ and $e_{\varpi_2}(\mathrm{p}_{1j}^*)$ as

$$\begin{aligned} e_{\varpi_2}(\mathrm{p}_{1j}) &= -(1+a_1)(1-a_2/a_j)(1-a_1a_2a_j), \\ e_{\varpi_2}(\mathrm{p}_{1j}^*) &= -a_1^{-1}a_2^{-1}(1+a_1)(1-a_1a_2/a_j)(1-a_2a_j), \end{aligned} \tag{5.18}$$

which will be used in §6.3.

Proposition 5.2. *The set* $\{e_0(z),\ e_{\varpi_2}(z),\ e_{\varpi_1}(z),\ e_{2\varpi_2}(z)\}$ *forms a basis of* $\mathbb{C}$*-vector space* $\mathcal{F}_2$*, i.e.,*

$$\mathcal{F}_2 = \mathbb{C}e_0(z) \oplus \mathbb{C}e_{\varpi_2}(z) \oplus \mathbb{C}e_{\varpi_1}(z) \oplus \mathbb{C}e_{2\varpi_2}(z).$$

In particular, $\dim_{\mathbb{C}} \mathcal{F}_2 = 4$.

5.3.2. **The basis $\{g(a_1;z), g(a_2;z), g(a_3;z), G(z)\}$ of $\mathcal{F}_2$.**

Proposition 5.3. *suppose that* $a_1,\dots,a_4 \in \mathbb{C}^*$ *satisfy* $a_i \neq a_j$ *and* $a_ia_j \neq 1$ *for* $1 \le i < j \le 4$*. Then, the set* $\{g(a_1;z),\dots,g(a_4;z)\}$ *forms a basis of* $\mathbb{C}$*-vector space* $\mathcal{F}_2$*, i.e.,*

$$\mathcal{F}_2 = \mathbb{C}g(a_1;z) \oplus \mathbb{C}g(a_2;z) \oplus \mathbb{C}g(a_3;z) \oplus \mathbb{C}g(a_4;z).$$

In particular, $\dim_{\mathbb{C}} \mathcal{F}_2 = 4$.

Proof. Since we already know that $g(a;z) \in \mathcal{F}_2$, it suffices to prove that the set $\{g(a_1;z),\dots,g(a_4;z)\}$ is linearly independent. From the expansion (5.12) of $g(a;z)$ in terms of $m_\lambda(z)$, we have

$$\begin{bmatrix} g(a_1;z) \\ g(a_2;z) \\ g(a_3;z) \\ g(a_4;z) \end{bmatrix} = \begin{bmatrix} (a_1+a_1^{-1})^3-2 & -(a_1+a_1^{-1})^2+(a_1+a_1^{-1}) & a_1+a_1^{-1} & -1 \\ (a_2+a_2^{-1})^3-2 & -(a_2+a_2^{-1})^2+(a_2+a_2^{-1}) & a_2+a_2^{-1} & -1 \\ (a_3+a_3^{-1})^3-2 & -(a_3+a_3^{-1})^2+(a_3+a_3^{-1}) & a_3+a_3^{-1} & -1 \\ (a_4+a_4^{-1})^3-2 & -(a_4+a_4^{-1})^2+(a_4+a_4^{-1}) & a_4+a_4^{-1} & -1 \end{bmatrix} \begin{bmatrix} m_0(z) \\ m_{\varpi_2}(z) \\ m_{\varpi_1}(z) \\ m_{2\varpi_2}(z) \end{bmatrix}.$$

The determinant of the above transition matrix is computed as

$$\begin{vmatrix} (a_1+a_1^{-1})^3-2 & -(a_1+a_1^{-1})^2+(a_1+a_1^{-1}) & a_1+a_1^{-1} & -1 \\ (a_2+a_2^{-1})^3-2 & -(a_2+a_2^{-1})^2+(a_2+a_2^{-1}) & a_2+a_2^{-1} & -1 \\ (a_3+a_3^{-1})^3-2 & -(a_3+a_3^{-1})^2+(a_3+a_3^{-1}) & a_3+a_3^{-1} & -1 \\ (a_4+a_4^{-1})^3-2 & -(a_4+a_4^{-1})^2+(a_4+a_4^{-1}) & a_4+a_4^{-1} & -1 \end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix} (a_1+a_1^{-1})^3 & (a_1+a_1^{-1})^2 & a_1+a_1^{-1} & 1 \\ (a_2+a_2^{-1})^3 & (a_2+a_2^{-1})^2 & a_2+a_2^{-1} & 1 \\ (a_3+a_3^{-1})^3 & (a_3+a_3^{-1})^2 & a_3+a_3^{-1} & 1 \\ (a_4+a_4^{-1})^3 & (a_4+a_4^{-1})^2 & a_4+a_4^{-1} & 1 \end{vmatrix} \text{ (by elementary row operations)} \\
&= \prod_{1\le i<j\le 4} (a_j+a_j^{-1}-a_i+a_i^{-1}) \qquad \text{(by Vandermonde determinant)} \\
&= \prod_{1\le i<j\le 4} e(a_i,a_j),
\end{aligned}
$$

which is divisible by $e(a_i,a_j)$ $(1\le i<j\le 4)$. This implies that the above determinant does not vanish if and only if $a_i\ne a_j$ and $a_ia_j\ne 1$ for $1\le i<j\le 4$. Therefore we obtain the fact that the set $\{g(a_1;z),\dots,g(a_4;z)\}$ is linearly independent. □

Let $\mathrm{p}_{ij}\in(\mathbb{C}^*)^2$ be the point defined in (5.14). From the definition (5.11), $g(a_k;z)$ is evaluated as

$$
\begin{aligned}
g(a_k;\mathrm{p}_{ij}) &= e(a_k,a_i)e(a_k,a_j)e(a_k,a_ia_j) \\
&= a_k^{-3}(1-a_ka_i)(1-a_ka_i^{-1})(1-a_ka_j)(1-a_ka_j^{-1})(1-a_ka_ia_j)(1-a_ka_i^{-1}a_j^{-1}),
\end{aligned}
$$

so that we have

$$
g(a_i;\mathrm{p}_{ij}) = g(a_j;\mathrm{p}_{ij}) = 0. \tag{5.19}
$$

We now define the function $G(z)\in\mathcal{F}_2$ as

$$
\begin{aligned}
G(z) := \frac{1}{\prod_{i=1}^3 e(a_4;a_i)} \Bigg[g(a_4;z) - \frac{g(a_4;\mathrm{p}_{23})}{g(a_1;\mathrm{p}_{23})} g(a_1;z) \\
- \frac{g(a_4;\mathrm{p}_{13})}{g(a_2;\mathrm{p}_{13})} g(a_2;z) - \frac{g(a_4;\mathrm{p}_{12})}{g(a_3;\mathrm{p}_{12})} g(a_3;z) \Bigg],
\end{aligned} \tag{5.20}
$$

which will be used later.

Remark. The function $G(z)$ seems to be very important for the integral of type G_2. In fact the elliptic version of the function $G(z)$ plays an essential role in research of the elliptic Gustafson integral of type G_2. See [10].

Lemma 5.4. *The function $G(z)$ satisfies*

$$
G(\mathrm{p}_{12}) = G(\mathrm{p}_{13}) = G(\mathrm{p}_{23}) = 0. \tag{5.21}
$$

and

$$
G(\mathrm{p}_{14}) = \frac{(1-a_1^2)(1-a_2a_4^{-1})(1-a_3a_4^{-1})(1-a_1a_2a_4)(1-a_1a_3a_4)}{(1-a_1)(1-a_2a_3)(1-a_1a_2a_3^{-1})(1-a_1a_3a_2^{-1})(1-a_1a_2a_3)}. \tag{5.22}
$$

Proof. By the definition (5.20) of $G(z)$ we immediately have (5.21). (5.22) is due to a direct calculation. □

Remark. For an arbitrary $x\in\mathbb{C}^*$, we can confirm

$$
G(a_1/x,x) = \frac{(1-a_1^2)(1-a_2x^{-1})(1-a_3x^{-1})(1-a_1a_2x)(1-a_1a_3x)}{(1-a_1)(1-a_2a_3)(1-a_1a_2a_3^{-1})(1-a_1a_3a_2^{-1})(1-a_1a_2a_3)}. \tag{5.23}
$$

We see that (5.22) is only a special case corresponding to $x=a_4$ of (5.23). This is something interesting since the definition of $G(z)$ depends on a_4.

Corollary 5.5. *Let* $\mathrm{p}_{ij}^* \in (\mathbb{C}^*)^2$ *be the point defined by* (5.15). *Then we have*

$$g(a_1;\mathrm{p}_{12}^*) = g(a_2;\mathrm{p}_{12}^*) = 0 \tag{5.24}$$

and

$$G(\mathrm{p}_{12}^*) = G(\mathrm{p}_{21}^*) = \frac{(1-a_1^2)(1-a_2^2)}{(1-a_1a_2a_3^{-1})(1-a_1a_2a_3)} = \frac{(1-a_1^2)(1-a_2^2)}{a_1a_2\, e(a_1a_2,a_3)}. \tag{5.25}$$

Proof. By the definition (5.11) we can immediately confirm (5.24). (5.25) is a special case of (5.23) corresponding to $x = a_2/a_1$. □

From the arguments above the set $\{g(a_1;z), g(a_2;z), g(a_3;z), G(z)\}$ also forms a basis of $\mathbb{C}$-vector space $\mathcal{F}_2$, i.e.,

$$\mathcal{F}_2 = \mathbb{C}g(a_1;z) \oplus \mathbb{C}g(a_2;z) \oplus \mathbb{C}g(a_3;z) \oplus \mathbb{C}G(z).$$

	p_{23}	p_{13}	p_{12}	p_{12}^*
$g(a_1;z)$	$*$	0	0	0
$g(a_2;z)$	0	$*$	0	0
$g(a_3;z)$	0	0	$*$	$*$
$G(z)$	0	0	0	$*$

5.4. **$\mathbb{C}$-vector space $\mathcal{F}_4$.** By the definition of (5.10) of $\mathcal{F}_n$, we have

$$\mathcal{F}_4 = \mathbb{C}m_0(z) \oplus \mathbb{C}m_{\varpi_2}(z) \oplus \mathbb{C}m_{\varpi_1}(z) \oplus \mathbb{C}m_{2\varpi_2}(z) \oplus \mathbb{C}m_{\varpi_1+\varpi_2}(z) \oplus \mathbb{C}m_{3\varpi_2}(z) \oplus \mathbb{C}m_{2\varpi_1}(z) \oplus \mathbb{C}m_{\varpi_1+2\varpi_2}(z) \oplus \mathbb{C}m_{4\varpi_2}(z),$$

so that $\dim_{\mathbb{C}} \mathcal{F}_4 = 9$. Here we construct a basis of the $\mathbb{C}$-vector space $\mathcal{F}_4$.

We use the abbreviation

$$g_{ij}(z) = g(a_i;z)g(a_j;z). \tag{5.26}$$

We denote by $G_l^{ijk}(z)$ the function $G(z) \in \mathcal{F}_2$ substituting $a_1 \to a_i$, $a_2 \to a_j$, $a_3 \to a_k$, $a_4 \to a_l$ in its definition (5.20), i.e.,

$$G_l^{ijk}(z) := \frac{1}{\prod_{m\in\{i,j,k\}} e(a_l;a_m)}\Bigg[g(a_l;z) - \frac{g(a_l;\mathrm{p}_{jk})}{g(a_i;\mathrm{p}_{jk})}g(a_i;z) - \frac{g(a_l;\mathrm{p}_{ik})}{g(a_j;\mathrm{p}_{ik})}g(a_j;z) - \frac{g(a_l;\mathrm{p}_{ij})}{g(a_k;\mathrm{p}_{ij})}g(a_k;z)\Bigg]. \tag{5.27}$$

As we saw in (5.21), we have

$$G_l^{ijk}(\mathrm{p}_{ij}) = G_l^{ijk}(\mathrm{p}_{ik}) = G_l^{ijk}(\mathrm{p}_{jk}) = 0.$$

Moreover, if we set

$$\begin{aligned} G_1(z) &:= g(a_1;z)G_5^{234}(z),\\ G_2(z) &:= g(a_2;z)G_5^{134}(z),\\ G_3(z) &:= g(a_3;z)G_5^{124}(z) \end{aligned} \tag{5.28}$$

then $G_1(z), G_2(z), G_3(z) \in \mathcal{F}_4$, and by definition we have

$$G_1(\mathrm{p}_{ij}) = G_2(\mathrm{p}_{ij}) = G_3(\mathrm{p}_{ij}) = 0 \quad (1 \le i < j \le 4).$$

From (5.24) and (5.25), we have

$$G_1(\mathrm{p}_{23}^*) = g(a_1;\mathrm{p}_{23}^*)G_5^{234}(\mathrm{p}_{23}^*) = e(a_1,a_3/a_2)e(a_1,a_2)e(a_1,a_3)\frac{(1-a_2^2)(1-a_3^2)}{a_2a_3\,e(a_2a_3,a_4)},$$
$$G_1(\mathrm{p}_{13}^*) = 0,\quad G_1(\mathrm{p}_{12}^*) = 0.$$

(5.29)

	p_{34}	p_{24}	p_{14}	p_{23}	p_{13}	p_{12}	p_{23}^*	p_{13}^*	p_{12}^*
g_{12}	$*$	0	0	0	0	0	0	0	0
g_{13}	0	$*$	0	0	0	0	0	0	0
g_{23}	0	0	$*$	0	0	0	0	0	0
g_{14}	0	0	0	$*$	0	0	$*$	0	0
g_{24}	0	0	0	0	$*$	0	0	$*$	0
g_{34}	0	0	0	0	0	$*$	0	0	$*$
G_1	0	0	0	0	0	0	$*$	0	0
G_2	0	0	0	0	0	0	0	$*$	0
G_3	0	0	0	0	0	0	0	0	$*$

From the argument above we have the following:

Lemma 5.6. *The set* $\{g_{ij}(z)\,|\,1\le i<j\le 4\}\cup\{G_1(z),G_2(z),G_3(z)\}$ *forms a basis of* $\mathbb{C}$*-vector space* $\mathcal{F}_4$*, i.e.,*

$$\mathcal{F}_4 = \bigoplus_{1\le i<j\le 4}\mathbb{C}g_{ij}(z)\oplus\bigoplus_{i=1}^{3}\mathbb{C}G_i(z).$$

In particular, $\dim_{\mathbb{C}}\mathcal{F}_4 = 9$.

6. Gustafson integral (Askey–Wilson integral of type G_2)

In this section we give an alternative proof of Proposition 1.1 for Gustafson's q-integral of type G_2.

6.1. **Gustafson integral of type G_2.** Let $\Phi(z)$ be function of $z=(z_1,z_2)\in(\mathbb{C}^*)^2$ defined by

(6.1)
$$\Phi(z) := \Phi_+(z)\Phi_+(z^{-1}),$$

where $z^{-1}=(z_1^{-1},z_2^{-1})$ and

$$\Phi_+(z) := \frac{(z_2,z_1z_2,z_1z_2^2,z_1,z_1z_2^3,z_1^2z_2^3;q)_\infty}{\prod_{k=1}^4(a_kz_2,a_kz_1z_2,a_kz_1z_2^2;q)_\infty}.$$

We define $I(a_1,a_2,a_3,a_4)$ as

$$I(a_1,a_2,a_3,a_4) := \frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\Phi(z)\frac{dz_1}{z_1}\frac{dz_2}{z_2},$$

and also define $P(a_1,a_2,a_3,a_4)$ as

(6.2)
$$P(a_1,a_2,a_3,a_4) := \frac{(a_1^2a_2^2a_3^2a_4^2;q)_\infty}{(a_1a_2a_3a_4;q)_\infty}\prod_{i=1}^4\frac{(a_i;q)_\infty}{(a_i^2;q)_\infty}\prod_{1\le i<j\le 4}\frac{1}{(a_ia_j;q)_\infty}\prod_{1\le i<j<k\le 4}\frac{1}{(a_ia_ja_k;q)_\infty}.$$

Using $I(a_1,a_2,a_3,a_4)$ and $P(a_1,a_2,a_3,a_4)$ Proposition 1.1 is stated as the following

Theorem 6.1 (Gustafson [5, 6])**.** *Suppose that* $a_k \in \mathbb{C}^*$ $(1 \le k \le 4)$ *satisfy* $|a_k| < 1$. *Then,*

$$I(a_1,a_2,a_3,a_4) = \frac{12}{(q;q)_\infty^2} P(a_1,a_2,a_3,a_4).$$

The aim of this section is to give an alternative proof for the above theorem. One of key steps for our proof is the following.

Proposition 6.2. $I(a_1,a_2,a_3,a_4)$ *satisfy the following* q*-difference equation.*

$$\frac{I(qa_1,a_2,a_3,a_4)}{I(a_1,a_2,a_3,a_4)} = \frac{(1+a_1)(1-qa_1^2)\prod_{i=2}^4(1-a_1a_i)\prod_{2\le j<k\le 4}(1-a_1a_ja_k)}{(1+a_1a_2a_3a_4)(1-qa_1^2a_2^2a_3^2a_4^2)}. \tag{6.3}$$

Proof. We state the detail later in §6.3 □

Remark. Proposition 6.2 is also obtained as a corollary of Theorem 7.2 for the Nassrallah–Rahman integral of type G_2 defined in §7.1. See Remark of Theorem 7.2.

Once we had the recurrence relation (6.3) of Proposition 6.2, by repeated use of (6.3) we have the following.

Proposition 6.3. $I(a_1,a_2,a_3,a_4)$ *coincides with* $P(a_1,a_2,a_3,a_4)$ *up to a multiplicative constant, i.e.,*

$$I(a_1,a_2,a_3,a_4) = c\,P(a_1,a_2,a_3,a_4). \tag{6.4}$$

where c *is some constant independent of* a_1,a_2,a_3,a_4.

Proof. From (6.2), $P(a_1,a_2,a_3,a_4)$ satisfies the same q-difference equation as (6.3), and (6.3) is symmetric with respect to a_1,a_2,a_3,a_4. This implies

$$\frac{I(a_1,a_2,a_3,a_4)}{P(a_1,a_2,a_3,a_4)} = \frac{I(qa_1,qa_2,qa_3,qa_4)}{P(qa_1,qa_2,qa_3,qa_4)} = \lim_{N\to\infty}\frac{I(q^Na_1,q^Na_2,q^Na_3,q^Na_4)}{P(q^Na_1,q^Na_2,q^Na_3,q^Na_4)} \tag{6.5}$$

From (6.2) we have

$$\lim_{N\to\infty} P(q^Na_1,q^Na_2,q^Na_3,q^Na_4) = 1 \tag{6.6}$$

On the other hand, if we set c as

$$\begin{aligned} c &= \lim_{N\to\infty} I(q^Na_1,q^Na_2,q^Na_3,q^Na_4) \\ &= \frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}(z_2,z_1z_2,z_1z_2^2,z_1,z_1z_2^3,z_1^2z_2^3;q)_\infty \\ &\qquad \times (z_2^{-1},z_1^{-1}z_2^{-1},z_1^{-1}z_2^{-2},z_1^{-1},z_1^{-1}z_2^{-3},z_1^{-2}z_2^{-3};q)_\infty \frac{dz_1}{z_1}\frac{dz_2}{z_2}, \end{aligned}$$

then, from (6.5) and (6.6), we obtain (6.4). □

It is necessary to compute a special value of $I(a_1,a_2,a_3,a_4)$ for proving that c in (6.4) is equal to $12/(q;q)_\infty^2$. Here we compute $I(1,0,0,0)$.

Lemma 6.4.

$$\begin{aligned} I(1,0,0,0) &= \frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}(z_1, z_1z_2^3, z_1^2z_2^3; q)_\infty(z_1^{-1}, z_1^{-1}z_2^{-3}, z_1^{-2}z_2^{-3}; q)_\infty \frac{dz_1}{z_1}\frac{dz_2}{z_2} \\ &= \frac{6}{(q;q)_\infty^2}. \end{aligned}$$

Proof. We state the detail in §6.4. For its proof we use the triple product identity (2.5) of Jacobi's theta function. □

Proof of Theorem 6.1. From the definition (6.2) of $P(a_1, a_2, a_3, a_4)$ we have

$$P(1,0,0,0) = \frac{1}{(-1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_\infty} = \frac{1}{2(-q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_\infty} = \frac{(q;q)_\infty}{2(q, -q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}; q)_\infty} = \frac{1}{2}.$$

Combining this with Lemma 6.4, we can compute c in Proposition 6.3 as

$$c = \frac{I(1,0,0,0)}{P(1,0,0,0)} = \frac{6/(q;q)_\infty^2}{1/2} = \frac{12}{(q;q)_\infty^2}.$$

From (6.4) this concludes our proof of Theorem 6.1. □

6.2. q-Stokes' theorem.

Lemma 6.5. *Let $F_+(z)$ and $F_-(z)$ be functions specified as*

$$F_+(z) := \frac{(z_1^2z_2^3)^{-\frac{1}{2}}\prod_{k=1}^4(1-a_kz_1z_2)(1-a_kz_1z_2^2)}{(1-z_1z_2)(1-z_1z_2^2)(1-z_1)(1-z_1z_2^3)(1-z_1^2z_2^3)}$$

and

$$\begin{aligned} F_-(z) &:= F_+(z^{-1}) \\ &= \frac{(z_1^2z_2^3)^{\frac{1}{2}}\prod_{k=1}^4(1-a_kz_1^{-1}z_2^{-1})(1-a_kz_1^{-1}z_2^{-2})}{(1-z_1^{-1}z_2^{-1})(1-z_1^{-1}z_2^{-2})(1-z_1^{-1})(1-z_1^{-1}z_2^{-3})(1-z_1^{-2}z_2^{-3})}. \end{aligned} \tag{6.7}$$

Then, it follows that

$$\frac{T_{q,z_1}\Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q,z_1}F_-(z)}. \tag{6.8}$$

Proof. By the definition (6.1) of $\Phi(z)$ we have

$$\begin{aligned}\frac{T_{q,z_1}\Phi(z)}{\Phi(z)} =&\frac{(1-(qz_1z_2^3)^{-1})(1-(qz_1^2z_2^3)^{-1})(1-(q^2z_1^2z_2^3)^{-1})}{(1-z_1z_2^3)(1-z_1^2z_2^3)(1-qz_1^2z_2^3)}\\ &\times\frac{(1-(qz_1z_2)^{-1})(1-(qz_1z_2^2)^{-1})(1-(qz_1)^{-1})}{(1-z_1z_2)(1-z_1z_2^2)(1-z_1)}\\ &\times\prod_{k=1}^4\frac{(1-a_kz_1z_2)(1-a_kz_1z_2^2)}{(1-a_k(qz_1z_2)^{-1})(1-a_k(qz_1z_2^2)^{-1})}\\ =&-(qz_1^2z_2^3)^{-1}\frac{(1-(qz_1z_2)^{-1})(1-(qz_1z_2^2)^{-1})}{(1-z_1z_2)(1-z_1z_2^2)}\\ &\times\frac{(1-(qz_1)^{-1})(1-(qz_1z_2^3)^{-1})(1-(q^2z_1^2z_2^3)^{-1})}{(1-z_1)(1-z_1z_2^3)(1-z_1^2z_2^3)}\\ &\times\prod_{k=1}^4\frac{(1-a_kz_1z_2)(1-a_kz_1z_2^2)}{(1-a_k(qz_1z_2)^{-1})(1-a_k(qz_1z_2^2)^{-1})}.\end{aligned}$$

This implies (6.8). □

We denote by $\mathcal{M}((\mathbb{C}^*)^2)$ the $\mathbb{C}$-vector space of meromorphic functions on $(\mathbb{C}^*)^2$, and by $\mathcal{O}((\mathbb{C}^*)^2)$ the $\mathbb{C}$-vector space of holomorphic functions on $(\mathbb{C}^*)^2$. For each function $\varphi(z)\in z_2^{\frac12}\mathcal{M}((\mathbb{C}^*)^2)=\{z_2^{\frac12}f(z)\,|\,f(z)\in\mathcal{M}((\mathbb{C}^*)^2)\}$ we define the function $\nabla\varphi(z)$ by

$$(\nabla\varphi)(z):=F_+(z)T_{q,z_1}^{\frac12}\varphi(z)+F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\in\mathcal{M}((\mathbb{C}^*)^2),$$

and $\nabla_{\mathrm{sym}}\varphi(z)$ by the symmetrization of $\nabla\varphi(z)$:

$$\begin{aligned}(\nabla_{\mathrm{sym}}\varphi)(z):=&\sum_{w\in W}w.(\nabla\varphi(z))\\ =&\sum_{w\in W}w.\Big(F_+(z)T_{q,z_1}^{\frac12}\varphi(z)+F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)\in\mathcal{M}((\mathbb{C}^*)^2)^W,\end{aligned}$$

where $\mathcal{M}((\mathbb{C}^*)^2)^W$ denotes the $\mathbb{C}$-vector space of W-invariant meromorphic functions on $(\mathbb{C}^*)^2$. For $f(z)\in\mathcal{M}((\mathbb{C}^*)^2)$ we use the notation

$$\langle f(z)\rangle=\frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}f(z)\Phi(z)\frac{dz_1}{z_1}\frac{dz_2}{z_2}.$$

In particular, we have $\langle 1\rangle=I(a_1,a_2,a_3,a_4)$ and $\langle a_1^3g(a_1;z)\rangle=I(qa_1,a_2,a_3,a_4)$.

Lemma 6.6 (q-Stokes' theorem)**.** *Suppose that* $|a_k|<1$ $(k=1,2,3,4)$*. Then, for* $\varphi(z)\in z_2^{\frac12}\mathcal{O}((\mathbb{C}^*)^2)$ *we have*

$$\langle\nabla\varphi(z)\rangle=0,$$

and

$$\langle\nabla_{\mathrm{sym}}\varphi(z)\rangle=0.$$

Proof. From (6.8) we have

$$T_{q,z_1}\Big(\Phi(z)F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)=-\Phi(z)F_+(z)T_{q,z_1}^{\frac12}\varphi(z),\tag{6.9}$$

and from (6.7) we have

$$\Phi(z)F_-(z)T_{q,z_1}^{-\frac12}\varphi(z) = z_1 z_2^{\frac32} T_{q,z_1}^{-\frac12}\varphi(z)\Phi_+(z) \times \frac{(z_2^{-1}, qz_1^{-1}z_2^{-1}, qz_1^{-1}z_2^{-2}, qz_1^{-1}, qz_1^{-1}z_2^{-3}, qz_1^{-2}z_2^{-3}; q)_\infty}{\prod_{k=1}^4 (a_k z_2^{-1}, a_k q z_1^{-1} z_2^{-1}, a_k q z_1^{-1} z_2^{-2}; q)_\infty}. \tag{6.10}$$

Since we have $z_1 z_2^{\frac32} T_{q,z_1}^{-\frac12}\varphi(z) \in \mathcal{O}((\mathbb{C}^*)^2)$ from $\varphi(z) \in z_2^{\frac12}\mathcal{O}((\mathbb{C}^*)^2)$, if $|a_k| < 1$ $(k = 1,2,3,4)$ and z_2 is fixed as $|z_2| = 1$, then the right-hand side of (6.10) as a function of z_1 has no poles in the annulus $|q| \le |z_1| \le 1$. Thus, by Cauchy's integral theorem we have

$$\begin{aligned}
&\frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2} T_{q,z_1}\Big(\Phi(z)F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \\
&= \frac{1}{(2\pi\sqrt{-1})^2}\int_{|z_2|=1}\bigg(\int_{|z_1|=|q|}\Phi(z)F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\frac{dz_1}{z_1}\bigg)\frac{dz_2}{z_2} \\
&\qquad\qquad\qquad\qquad\qquad\text{(from variable change } z_1' = qz_1\text{)} \\
&= \frac{1}{(2\pi\sqrt{-1})^2}\int_{|z_2|=1}\bigg(\int_{|z_1|=1}\Phi(z)F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\frac{dz_1}{z_1}\bigg)\frac{dz_2}{z_2} \\
&\qquad\qquad\qquad\qquad\qquad\text{(from Cauchy's integral theorem)} \\
&= \frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\Phi(z)F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\frac{dz_1}{z_1}\frac{dz_2}{z_2}.
\end{aligned} \tag{6.11}$$

Combining (6.9) with (6.11), we obtain

$$\langle\nabla\varphi(z)\rangle = \frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\Phi(z)\Big(F_+(z)T_{q,z_1}^{\frac12}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)\frac{dz_1}{z_1}\frac{dz_2}{z_2} = 0.$$

Moreover, since $\Phi(z)$ and $\frac{dz_1}{z_1}\frac{dz_2}{z_2}$ are W-symmetric, we compute

$$\begin{aligned}
&\langle\nabla_{\mathrm{sym}}\varphi(z)\rangle = \frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\Phi(z)\nabla_{\mathrm{sym}}\varphi(z)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \\
&= \frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\Phi(z)\bigg(\sum_{w\in W} w.\Big(F_+(z)T_{q,z_1}^{\frac12}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)\bigg)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \\
&= \frac{1}{(2\pi\sqrt{-1})^2}\sum_{w\in W}\iint_{\mathbb{T}^2} w.\bigg(\Phi(z)\Big(F_+(z)T_{q,z_1}^{\frac12}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)\frac{dz_1}{z_1}\frac{dz_2}{z_2}\bigg) \\
&= \frac{1}{(2\pi\sqrt{-1})^2}\sum_{w\in W}\iint_{w^{-1}\mathbb{T}^2}\Phi(z)\Big(F_+(z)T_{q,z_1}^{\frac12}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \\
&= \frac{1}{(2\pi\sqrt{-1})^2}\sum_{w\in W}\iint_{\mathbb{T}^2}\Phi(z)\Big(F_+(z)T_{q,z_1}^{\frac12}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac12}\varphi(z)\Big)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \\
&= 0,
\end{aligned}$$

which completes the proof. □

We set $W_0 = \langle s_2, w_0\rangle \subset W$, where $w_0 = (s_1 s_2)^3$ is the longest element of W.

Lemma 6.7. *Suppose that* $\varphi(z)$ *is a* W_0*-invariant function, i.e.,* $s_2.\varphi(z) = \varphi(z)$ *and* $w_0.\varphi(z) = \varphi(z)$ *Then,* $\nabla_{\rm sym}\varphi(z)$ *is expressed as*

$$\nabla_{\rm sym}\varphi(z) = 4\sum_{k=0}^{5} F_k(z)\varphi_k(z), \tag{6.12}$$

where $F_k(z)$ *and* $\varphi_k(z)$ $(k = 0, 1, \ldots, 5)$ *are given as*

$$F_k(z) = (s_1s_2)^k.F_+(z), \quad \varphi_k(z) = (s_1s_2)^k.T_{q,z_1}^{\frac{1}{2}}\varphi(z). \tag{6.13}$$

Proof. From the definition (6.7) of $F_-(z)$, we have

$$F_-(z) = F_+(z^{-1}) = (s_1s_2)^3.F_+(z) = F_3(z).$$

Since $\varphi(z)$ is an invariant with respect to $w_0 = (s_1s_2)^3 \in W$, i.e., $\varphi(z^{-1}) = \varphi(z)$, we have

$$T_{q,z_1}^{-\frac{1}{2}}\varphi(z) = \varphi(q^{-\frac{1}{2}}z_1, z_2) = \varphi(q^{\frac{1}{2}}z_1^{-1}, z_2^{-1}) = \varphi_0(z^{-1}) = (s_1s_2)^3\varphi_0(z) = \varphi_3(z),$$

so that $\nabla\varphi(z)$ is expressed as

$$\nabla\varphi(z) = F_+(z)T_{q,z_1}^{\frac{1}{2}}\varphi(z) + F_-(z)T_{q,z_1}^{-\frac{1}{2}}\varphi(z) = F_0(z)\varphi_0(z) + F_3(z)\varphi_3(z).$$

Thus we obtain

$$\begin{aligned}\nabla_{\rm sym}\varphi(z) &= \sum_{w\in W} w.(\nabla\varphi(z)) = \sum_{w\in W} w.(F_0(z)\varphi_0(z) + F_3(z)\varphi_3(z)) \\ &= 2\sum_{w\in W} w.(F_0(z)\varphi_0(z)).\end{aligned}$$

Then, using the expression (5.1) of W,

$$\nabla_{\rm sym}\varphi(z) = 2\sum_{k=0}^{5}(s_1s_2)^k.\Big(F_0(z)\varphi_0(z) + s_2.(F_0(z)\varphi_0(z))\Big). \tag{6.14}$$

Since $\varphi(z)$ is an invariant with respect to s_2, i.e., $\varphi(z_1, z_2) = \varphi(z_1z_2^3, z_2^{-1})$, we have

$$s_2.\varphi_0(z) = s_2.\varphi(q^{\frac{1}{2}}z_1, z_2) = \varphi(q^{\frac{1}{2}}z_1z_2^3, z_2^{-1}) = \varphi(q^{\frac{1}{2}}z_1, z_2) = \varphi_0(z).$$

Since $F_0(z) = F_+(z)$ is also an invariant with respect to s_2, using this we see that (6.14) implies $\nabla_{\rm sym}\varphi(z) = 4\sum_{k=0}^{5}(s_1s_2)^k.F_0(z)\varphi_0(z)$. □

6.3. q-Difference equations – Proof of Proposition 6.2.

Lemma 6.8. *For* $i = 0, 1$ *it follows that*

$$\frac{\nabla_{\rm sym}(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(z_2 + z_2^{-1})^i}{4} \in \mathbb{C}m_0(z) \oplus \mathbb{C}m_{\varpi_2}(z). \tag{6.15}$$

The explicit expressions of (6.15) *are given by*

$$\frac{\nabla_{\rm sym}(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})}{4} = (E_4 - E_0)\Big[(E_0 + E_4)s_{\varpi_2}(z) - (E_1 + E_2 + E_3)\Big], \tag{6.16}$$

and

$$\frac{\nabla_{\rm sym}(z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(z_2 + z_2^{-1})}{4} = (E_3 - E_1)\Big[(E_0 + E_4)s_{\varpi_2}(z) - (E_1 + E_2 + E_3)\Big], \tag{6.17}$$

where E_r $(r = 0, 1, \ldots, 4)$ *is the* r*th elementary symmetric polynomial of* $a_1, \ldots, a_4$ *given by*

$$E_r = \sum_{1\le i_1<\cdots<i_r\le 4} a_{i_1}a_{i_2}\cdots a_{i_r}, \tag{6.18}$$

i.e., $E_0 = 1$, $E_1 = a_1 + a_2 + a_3 + a_4$, $\ldots$, $E_4 = a_1a_2a_3a_4$.

Proof. The proof follows by direct computation and we omit the details. □

Lemma 6.9. *Let* $e_{\varpi_2}(z)$ *be the function given as* (5.17)*. If we set*

$$\varphi(z) = z_2^{-\frac{1}{2}}(1 + z_2)e(a_1, z_2), \tag{6.19}$$

then, we have

$$\frac{\nabla_{\mathrm{sym}}\varphi(z)}{4} = B_2e_{\varpi_2}(z) + B_1, \tag{6.20}$$

where the coefficients B_1 *and* B_2 *are given as*

$$B_2 = a_1^{-1}(1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4)(1 + a_1a_2a_3a_4), \tag{6.21}$$

and

$$\begin{aligned} B_1 = a_1^{-2}a_2^{-1}&(1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4)\\ &\times (1 + a_1)(1 + a_2)(1 - a_1a_2a_3)(1 - a_1a_2a_4)\end{aligned} \tag{6.22}$$

Proof. Since $\varphi(z)$ is written as

$$\varphi(z) = z_2^{-\frac{1}{2}}(1 + z_2)e(a_1, z_2) = (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(a_1 + a_1^{-1}) - (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})(z_2 + z_2^{-1}),$$

by (6.15) in Lemma 6.8, we have (6.20). Combining $e_{\varpi_2}(z) = s_{\varpi_2}(z) - s_{\varpi_2}(\mathrm{p}_{12})$ with (6.20), we have

$$\frac{\nabla_{\mathrm{sym}}\varphi(z)}{4} = B_2\Big(s_{\varpi_2}(z) - s_{\varpi_2}(\mathrm{p}_{12})\Big) + B_1 = B_2s_{\varpi_2}(z) + \cdots.$$

Comparing both sides of the above equation using (6.16) and (6.17), we obtain

$$\begin{aligned} B_2 &= (E_0 + E_4)(E_0 - E_4)(a_1 + a_1^{-1}) - (E_0 + E_4)(E_3 - E_1)\\ &= (E_0 + E_4)\Big[(E_0 - E_4)(a_1 + a_1^{-1}) - (E_3 - E_1)\Big].\end{aligned}$$

From the factorization

$$(E_0 - E_4)(a_1 + a^{-1}) - (E_3 - E_1) = a_1^{-1}(1 - a_1a_2)(1 - a_1a_3)(1 - a_1a_4),$$

we therefore obtain (6.21).

On the other hand, since $e_{\varpi_2}(\mathrm{p}_{12}) = 0$, we have $B_1 = \nabla_{\mathrm{sym}}\varphi(\mathrm{p}_{12})/4$. Using Lemma 6.7 with $F_2(\mathrm{p}_{12}) = F_3(\mathrm{p}_{12}) = F_4(\mathrm{p}_{12}) = F_5(\mathrm{p}_{12}) = 0$ and $\varphi_1(\mathrm{p}_{12}) = 0$, we obtain

$$\begin{aligned} B_1 &= \frac{\nabla_{\mathrm{sym}}\varphi(\mathrm{p}_{12})}{4} = F_0(\mathrm{p}_{12})\varphi_0(\mathrm{p}_{12}) = F_+(\mathrm{p}_{12})\varphi(\mathrm{p}_{12})\\ &= a_1^{-1}a_2^{-\frac{1}{2}}\frac{a_2^{-\frac{1}{2}}(1 + a_2)e(a_1, a_2)\prod_{k=1}^4(1 - a_ka_1)(1 - a_ka_1a_2)}{(1 - a_1)(1 - a_1a_2)(1 - a_1/a_2)(1 - a_1a_2^2)(1 - a_1^2a_2)}\\ &= a_1^{-2}a_2^{-1}\frac{(1 + a_2)(1 - a_1a_2)(1 - a_1/a_2)\prod_{k=1}^4(1 - a_ka_1)(1 - a_ka_1a_2)}{(1 - a_1)(1 - a_1a_2)(1 - a_1/a_2)(1 - a_1a_2^2)(1 - a_1^2a_2)},\end{aligned}$$

which coincides with (6.22). □

Proposition 6.10. *For $e_{\varpi_2}(z)$ it follows that*

$$b_2\langle e_{\varpi_2}(z)\rangle + b_1\langle 1\rangle = 0, \tag{6.23}$$

where the coefficients b_1 and b_2 are given as

$$\begin{aligned} b_2 &= (1 + a_1a_2a_3a_4), \\ b_1 &= a_1^{-1}a_2^{-1}(1+a_1)(1+a_2)(1-a_1a_2a_3)(1-a_1a_2a_4). \end{aligned}$$

Proof. Applying Lemma 6.6 to Lemma 6.9, we immediately have (6.23). Dividing both sides of (6.20) by $a_1^{-1}(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)$, we obtain the explicit forms of the coefficients b_1 and b_2. □

Lemma 6.11. *For $i = 0, 1$ it follows that*

$$\frac{\nabla_{\mathrm{sym}}\big((z_1^2z_2^3)^{-\frac12} + (z_1^2z_2^3)^{\frac12}\big)(z_2 + z_2^{-1})^i}{4} \in \mathcal{F}_2, \tag{6.24}$$

which are expanded as

$$\frac{\nabla_{\mathrm{sym}}\big((z_1^2z_2^3)^{-\frac12} + (z_1^2z_2^3)^{\frac12}\big)}{4} = q^{-\frac12}(E_0 - qE_4^2)s_{\varpi_1}(z) + \cdots, \tag{6.25}$$

$$\begin{aligned} &\frac{\nabla_{\mathrm{sym}}\big((z_1^2z_2^3)^{-\frac12} + (z_1^2z_2^3)^{\frac12}\big)(z_2 + z_2^{-1})}{4} \\ &\qquad = q^{-\frac12}(E_0 - qE_4^2)\Big[s_{2\varpi_2}(z) - s_{\varpi_1}(z)\Big] + \cdots, \end{aligned} \tag{6.26}$$

where E_r $(r = 0, 1, \ldots, 4)$ is the rth elementary symmetric polynomial of $a_1, \ldots, a_4$ given by (6.18).

Proof. The proof follows by direct computation and we omit the details. □

Lemma 6.12. *If we put*

$$\begin{aligned} \varphi(z) &= (z_1^2z_2^3)^{-\frac12}(1 + z_1^2z_2^3)e(a_1, z_2) \\ &= \big((z_1^2z_2^3)^{-\frac12} + (z_1^2z_2^3)^{\frac12}\big)\big((a_1 + a_1^{-1}) - (z_1 + z_2^{-1})\big), \end{aligned} \tag{6.27}$$

then, we have

$$\frac{\nabla_{\mathrm{sym}}\varphi(z)}{4} = C_3 g(a_1; z) + C_2 e_{\varpi_2}(z) + C_1, \tag{6.28}$$

where the coefficients C_1, C_2 and C_3 are given as

$$C_3 = q^{-\frac12}(1 - qa_1^2a_2^2a_3^2a_4^2), \tag{6.29}$$

$$C_2 = q^{-\frac12}a_1^{-2}(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)(1+qa_1^3a_2a_3a_4), \tag{6.30}$$

$$\begin{aligned} C_1 = q^{-\frac12}a_1^{-3}a_2^{-1}&(1-a_1a_2)(1-a_1a_3)(1-a_1a_4) \\ &\times (1-a_1a_2a_3)(1-a_1a_2a_4)(1+a_1)(1+qa_1^2a_2). \end{aligned} \tag{6.31}$$

Proof. Applying (6.25) and (6.26) of Lemma 6.11 to (6.27) we have $\nabla_{\mathrm{sym}}\varphi(z)/4 \in \mathcal{F}_2$. Then, $\nabla_{\mathrm{sym}}\varphi(z)/4$ can be expanded in terms of the basis of $\mathcal{F}_2$ defined in §5.3.1, namely

$$\frac{\nabla_{\mathrm{sym}}\varphi(z)}{4} = C_3 g(a_1; z) + C_{2.5} e_{\varpi_1}(z) + C_2 e_{\varpi_2}(z) + C_1,$$

where C_1, C_2, $C_{2.5}$ and C_3 are some constants independent of z.

We first evaluate C_3 and show $C_{2.5} = 0$. Using expansion (5.13) of $g(a_1; z)$ we have

$$\begin{aligned}\frac{\nabla_{\rm sym}\varphi(z)}{4} &= C_3\Big(-s_{2\varpi_2}(z) + (a_1 + a_1^{-1} + 1)s_{\varpi_1}(z) + \cdots\Big) \\ &\qquad + C_{2.5}\Big(s_{\varpi_1}(z) + \cdots\Big) + C_2 e_{\varpi_2}(z) + C_1 \\ &= -C_3 s_{2\varpi_2}(z) + \Big((a_1 + a_1^{-1} + 1)C_3 + C_{2.5}\Big)s_{\varpi_1}(z) + \cdots .\end{aligned} \tag{6.32}$$

On the other hand, from (6.25) and (6.26) we have an expansion

$$\frac{\nabla_{\rm sym}\varphi(z)}{4} = -q^{-\frac12}(1 - qa_1^2a_2^2a_3^2a_4^2)\Big[s_{2\varpi_2}(z) - \Big((a_1 + a_1^{-1} + 1)\Big)s_{\varpi_1}(z)\Big] + \cdots . \tag{6.33}$$

Comparing coefficients of $s_{\varpi_1}(z)$ and $s_{2\varpi_2}(z)$ in (6.32) and (6.33), we obtain $C_{2.5} = 0$ and $C_3 = q^{-\frac12}(1 - qa_1^2a_2^2a_3^2a_4^2)$.

Next we evaluate C_1. Since $e_{\varpi_2}(\mathrm{p}_{12}) = 0$ and $g(a_1; \mathrm{p}_{12}) = 0$, from (6.28) we have $C_1 = \nabla_{\rm sym}\varphi(\mathrm{p}_{12})/4$. By Lemma 6.7 with $F_2(\mathrm{p}_{12}) = F_3(\mathrm{p}_{12}) = F_4(\mathrm{p}_{12}) = F_5(\mathrm{p}_{12}) = 0$ and $\varphi_1(\mathrm{p}_{12}) = 0$, using

$$\begin{aligned}F_0(z)\varphi_0(z) &= F_+(z)T_{q,z_1}^{\frac12}\varphi(z) \\ &= q^{-\frac12}\frac{(1 + qz_1^2z_2^3)(1 - a_1z_2)(1 - a_1/z_2)\prod_{k=1}^4(1 - a_kz_1z_2)(1 - a_kz_1z_2^2)}{a_1z_1^2z_2^3(1 - z_1z_2)(1 - z_1z_2^2)(1 - z_1)(1 - z_1z_2^3)(1 - z_1^2z_2^3)},\end{aligned}$$

we obtain

$$\begin{aligned}C_1 = \frac{\nabla_{\rm sym}\varphi(\mathrm{p}_{12})}{4} &= F_0(\mathrm{p}_{12})\varphi_0(\mathrm{p}_{12}) = F_+(\mathrm{p}_{12})T_{q,z_1}^{\frac12}\varphi(\mathrm{p}_{12}) \\ &= q^{-\frac12}\frac{(1 + qa_1^2a_2)(1 - a_1a_2)(1 - a_1/a_2)\prod_{k=1}^4(1 - a_ka_1)(1 - a_ka_1a_2)}{a_1^3a_2(1 - a_1)(1 - a_1a_2)(1 - a_1/a_2)(1 - a_1a_2^2)(1 - a_1^2a_2)},\end{aligned}$$

which coincides with (6.31).

Lastly we evaluate C_2. Since $g(a_1; \mathrm{p}_{12}^*) = 0$, from (6.28) we obtain

$$C_2 = \frac{1}{e_{\varpi_2}(\mathrm{p}_{12}^*)}\Big(\frac{\nabla_{\rm sym}\varphi(\mathrm{p}_{12}^*)}{4} - C_1\Big).$$

By Lemma 6.7 with $F_1(\mathrm{p}_{12}^*) = F_2(\mathrm{p}_{12}^*) = F_3(\mathrm{p}_{12}^*) = F_4(\mathrm{p}_{12}^*) = 0$, we obtain

$$\frac{\nabla_{\rm sym}\varphi(\mathrm{p}_{12}^*)}{4} = F_0(\mathrm{p}_{12}^*)\varphi_0(\mathrm{p}_{12}^*) + F_5(\mathrm{p}_{12}^*)\varphi_5(\mathrm{p}_{12}^*).$$

Thus we have

$$C_2 = \frac{1}{e_{\varpi_2}(\mathrm{p}_{12}^*)}\Big(F_0(\mathrm{p}_{12}^*)\varphi_0(\mathrm{p}_{12}^*) + F_5(\mathrm{p}_{12}^*)\varphi_5(\mathrm{p}_{12}^*) - C_1\Big). \tag{6.34}$$

From the explicit form

$$F_5(z)\varphi_5(z) = q^{-\frac12}\frac{(1 + qz_1)(1 - a_1z_1z_2^2)(1 - a_1/z_1z_2^2)\prod_{k=1}^4(1 - a_k/z_2)(1 - a_kz_1z_2)}{a_1z_1(1 - z_2^{-1})(1 - z_1z_2)(1 - 1/z_1z_2^3)(1 - z_1^2z_2^3)(1 - z_1)},$$

we have

$$
\begin{aligned}
&F_5(\mathrm{p}_{12}^*)\varphi_5(\mathrm{p}_{12}^*)\\
&= q^{-\frac{1}{2}} \frac{(1+qa_1^2/a_2)(1-a_1a_2)(1-a_1/a_2)\prod_{k=1}^{4}(1-a_ka_1/a_2)(1-a_ka_1)}{a_1^3a_2^{-1}(1-a_1/a_2)(1-a_1)(1-a_1/a_2^2)(1-a_1a_2)(1-a_1^2/a_2)}\\
&= q^{-\frac{1}{2}}(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)\\
&\quad\times \frac{(1+qa_1^2/a_2)(1-a_1^2)(1-a_1a_3/a_2)(1-a_1a_4/a_2)}{a_1^3a_2^{-1}(1-a_1/a_2^2)},
\end{aligned}
\tag{6.35}
$$

and in the same manner we have

$$
\begin{aligned}
F_0(\mathrm{p}_{12}^*)\varphi_0(\mathrm{p}_{12}^*) = -q^{-\frac{1}{2}}&(1+qa_1a_2)(1+a_1)(1-a_2^2)(1-a_1a_2)\\
&\times \frac{(1-a_1a_3)(1-a_1a_4)(1-a_2a_3)(1-a_2a_4)}{a_1a_2^3(1-a_1/a_2^2)}.
\end{aligned}
\tag{6.36}
$$

Using (6.31), (6.35) and (6.36), a factor $F_0(\mathrm{p}_{12}^*)\varphi_0(\mathrm{p}_{12}^*)+F_5(\mathrm{p}_{12}^*)\varphi_5(\mathrm{p}_{12}^*)-C_1$ in the expression (6.34) of C_2 is computed as

$$
\begin{aligned}
&F_0(\mathrm{p}_{12}^*)\varphi_0(\mathrm{p}_{12}^*)+F_5(\mathrm{p}_{12}^*)\varphi_5(\mathrm{p}_{12}^*)-C_1\\
&= -q^{-\frac{1}{2}} \frac{(1+a_1)(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)}{a_1^3a_2^3(1-a_1/a_2^2)}\\
&\quad\times \Big[a_1^2(1+qa_1a_2)(1-a_2^2)(1-a_2a_3)(1-a_2a_4)\\
&\qquad - a_2^4(1+qa_1^2/a_2)(1-a_1)(1-a_1a_3/a_2)(1-a_1a_4/a_2)\\
&\qquad\quad + a_2^2(1-a_1a_2a_3)(1-a_1a_2a_4)(1+qa_1^2a_2)(1-a_1/a_2^2)\Big]\\
&\qquad\text{(the sum of these three terms is factored into a product of binomials)}\\
&= -q^{-\frac{1}{2}} \frac{(1+a_1)(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)}{a_1^3a_2^3(1-a_1/a_2^2)}\\
&\qquad\times a_2^2(1-a_1)(1-a_2^2)(1-a_1/a_2^2)(1+qa_1^3a_2a_3a_4)\\
&= -q^{-\frac{1}{2}}a_1^{-3}a_2^{-1}(1-a_1^2)(1-a_2^2)(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)(1+qa_1^3a_2a_3a_4).
\end{aligned}
$$

Since $e_{\varpi_2}(\mathrm{p}_{12}^*) = -a_1^{-1}a_2^{-1}(1-a_1^2)(1-a_2^2)$ by (5.18), using the above identity, (6.34) is computed as

$$
\begin{aligned}
C_2 &= (-1)q^{-\frac{1}{2}}a_1^{-3}a_2^{-1}(1-a_1^2)(1-a_2^2)(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)(1+qa_1^3a_2a_3a_4)\\
&\quad\times \frac{-a_1a_2}{(1-a_1^2)(1-a_2^2)},
\end{aligned}
$$

which coincides with (6.30). □

Proposition 6.13. *For $g(a_1;z)$ and $e_{\varpi_2}(z)$ it follows that*

$$
c_3\langle g(a_1;z)\rangle + c_2\langle e_{\varpi_2}(z)\rangle + c_1\langle 1\rangle = 0,
\tag{6.37}
$$

where the coefficient c_1, c_2 and c_3 are given as

$$
\begin{aligned}
c_3 &= (1-qa_1^2a_2^2a_3^2a_4^2),\\
c_2 &= a_1^{-2}(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)(1+qa_1^3a_2a_3a_4),
\end{aligned}
$$

$$\begin{aligned}c_1 = a_1^{-3}a_2^{-1}&(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)\\&\times(1-a_1a_2a_3)(1-a_1a_2a_4)(1+a_1)(1+qa_1^2a_2).\end{aligned}$$

Proof. Applying Lemma 6.6 to Lemma 6.12, we immediately have (6.37). Multiplying both sides of (6.28) by $q^{\frac{1}{2}}$, we obtain the explicit forms of the coefficients c_1, c_2 and c_3. □

We conclude this subsection by proving Proposition 6.2.

Proof of Proposition 6.2. Eliminating $\langle e_{\varpi_2}(z)\rangle$ from (6.23) and (6.37), i.e.,

$$0 = b_2c_3\langle g(a_1;z)\rangle + b_2c_2\langle e_{\varpi_2}(z)\rangle + b_2c_1\langle 1\rangle = b_2c_3\langle g(a_1;z)\rangle + (-b_1c_2+b_2c_1)\langle 1\rangle. \tag{6.38}$$

we obtain

$$\langle g(a_1;z)\rangle = \frac{b_1c_2-b_2c_1}{b_2c_3}\langle 1\rangle$$

The numerator of the coefficient is computed as

$$\begin{aligned}
&b_1c_2-b_2c_1\\
&= a_1^{-1}a_2^{-1}(1+a_1)(1+a_2)(1-a_1a_2a_3)(1-a_1a_2a_4)\\
&\qquad\qquad\times a_1^{-2}(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)(1+qa_1^3a_2a_3a_4)\\
&\quad -(1+a_1a_2a_3a_4)\times a_1^{-3}a_2^{-1}(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)\\
&\qquad\qquad\times(1-a_1a_2a_3)(1-a_1a_2a_4)(1+a_1)(1+qa_1^2a_2)\\
&= a_1^{-3}a_2^{-1}(1+a_1)(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)(1-a_1a_2a_3)(1-a_1a_2a_4)\\
&\qquad\times\Big[(1+a_2)(1+qa_1^3a_2a_3a_4)-(1+a_1a_2a_3a_4)(1+qa_1^2a_2)\Big]\\
&= a_1^{-3}a_2^{-1}(1+a_1)(1-a_1a_2)(1-a_1a_3)(1-a_1a_4)(1-a_1a_2a_3)(1-a_1a_2a_4)\\
&\qquad\times a_2(1-a_1a_3a_4)(1-qa_1^2)\\
&= a_1^{-3}(1+a_1)(1-qa_1^2)\prod_{k=2}^{4}(1-a_1a_k)\prod_{2\le i<j<\le 4}(1-a_1a_ia_j),
\end{aligned}$$

and the denominator of the coefficient is $b_2c_3=(1+a_1a_2a_3a_4)(1-qa_1^2a_2^2a_3^2a_4^2)$. Therefore we obtain

$$\langle g(a_1;z)\rangle = \frac{a_1^{-3}(1+a_1)(1-qa_1^2)\prod_{k=2}^{4}(1-a_1a_k)\prod_{2\le i<j<\le 4}(1-a_1a_ia_j)}{(1+a_1a_2a_3a_4)(1-qa_1^2a_2^2a_3^2a_4^2)}\langle 1\rangle.$$

Since we have $I(qa_1,a_2,a_3,a_4)=\langle a_1^3g(a_1;z)\rangle$ and $I(a_1,a_2,a_3,a_4)=\langle 1\rangle$ by definition of the integral $I(a_1,a_2,a_3,a_4)$, the above relation between $\langle g(a_1;z)\rangle$ and $\langle 1\rangle$ implies the q-difference equation (6.3) in Proposition 6.2. □

6.4. **Special value of the integral – Proof of Lemma 6.4.**

Lemma 6.14.

$$\begin{aligned}
&\frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}(z_1,z_1z_2^3,z_1^2z_2^3;q)_\infty(z_1^{-1},z_1^{-1}z_2^{-3},z_1^{-2}z_2^{-3};q)_\infty\frac{dz_1}{z_1}\frac{dz_2}{z_2}\\
&=\frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}(z_2,z_1z_2,z_1z_2^2;q)_\infty(z_2^{-1},z_1^{-1}z_2^{-1},z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\frac{dz_2}{z_2}.
\end{aligned}\tag{6.39}$$

Proof. This is confirmed by variable change $w = z_2^3$ for the left-hand side. $\square$

The claim of Lemma 6.4 is that the left-hand side of (6.39) coincides with $6/(q;q)_\infty^2$. Instead of this it suffices to prove that the right-hand side of (6.39) is equal to $6/(q;q)_\infty^2$. Writing the right-hand side of (6.39) as an iterated integral, we have

$$\begin{aligned}(6.40)\quad &\frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}(z_2, z_1z_2, z_1z_2^2;q)_\infty(z_2^{-1}, z_1^{-1}z_2^{-1}, z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\frac{dz_2}{z_2}\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_2, z_2^{-1};q)_\infty\\ &\qquad\times\left(\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_1z_2, z_1^{-1}z_2^{-1}, z_1z_2^2, z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\right)\frac{dz_2}{z_2}.\end{aligned}$$

We first compute the integral of z_1-variable.

$$\begin{aligned}&\frac{(q;q)_\infty^2}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_1z_2, z_1^{-1}z_2^{-1}, z_1z_2^2, z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(1-z_1^{-1}z_2^{-1})(1-z_1^{-1}z_2^{-2})z_1^{-1}\vartheta(z_1z_2;q)\vartheta(z_1z_2^2;q)dz_1.\end{aligned}$$

The integrand as a function of z_1 has the unique essential singularity at $z_1 = 0$. Using the triple product (2.5) of Jacobi's theta function, the integrand is expanded as

$$\begin{aligned}&(1-z_1^{-1}z_2^{-1})(1-z_1^{-1}z_2^{-2})z_1^{-1}\vartheta(z_1z_2;q)\vartheta(z_1z_2^2;q)\\ &= \left(z_1^{-1} - z_1^{-2}(z_2^{-1}+z_2^{-2}) + z_1^{-3}z_2^{-3}\right)\sum_{m,n=-\infty}^{\infty}(-z_1z_2)^nq^{\binom{n}{2}}(-z_1z_2^2)^mq^{\binom{m}{2}},\end{aligned}$$

so that the residue (the coefficient of z_1^{-1}) is equal to the sum of A, B, C written below.

$$\begin{aligned}A &= \sum_{n+m=0} z_2^{n+2m}q^{\binom{n}{2}+\binom{m}{2}} = \sum_{m=-\infty}^{\infty} z_2^m q^{\binom{-m}{2}+\binom{m}{2}} = \sum_{m=-\infty}^{\infty} z_2^m q^{m^2},\\ B &= -(z_2^{-1}+z_2^{-2})\sum_{n+m=1}(-1)z_2^{n+2m}q^{\binom{n}{2}+\binom{m}{2}} = \sum_{n+m=1}(z_2^{n+2m-1}+z_2^{n+2m-2})q^{\binom{n}{2}+\binom{m}{2}}\\ &= \sum_{m=-\infty}^{\infty}(z_2^m+z_2^{m-1})q^{\binom{1-m}{2}+\binom{m}{2}} = \sum_{m=-\infty}^{\infty}(z_2^m+z_2^{m-1})q^{m(m-1)},\\ C &= z_2^{-3}\sum_{n+m=2} z_2^{n+2m}q^{\binom{n}{2}+\binom{m}{2}} = \sum_{n+m=2} z_2^{n+2m-3}q^{\binom{n}{2}+\binom{m}{2}} = \sum_{m=-\infty}^{\infty} z_2^{m-1}q^{\binom{2-m}{2}+\binom{m}{2}}\\ &= \sum_{m=-\infty}^{\infty} z_2^{m-1}q^{(m-1)^2}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(6.41)\quad &\frac{(q;q)_\infty^2}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_1z_2, z_1^{-1}z_2^{-1}, z_1z_2^2, z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\\ &= 2\sum_{m=-\infty}^{\infty} z_2^m q^{m^2} + \sum_{m=-\infty}^{\infty}(z_2^m+z_2^{m-1})q^{m(m-1)}.\end{aligned}$$

From (6.40), we have

$$
\begin{aligned}
&\frac{(q;q)_\infty^3}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}(z_2,z_1z_2,z_1z_2^2;q)_\infty(z_2^{-1},z_1^{-1}z_2^{-1},z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\frac{dz_2}{z_2}\\
&=\frac{(q;q)_\infty}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_2,z_2^{-1};q)_\infty\\
&\qquad\qquad\times\left(\frac{(q;q)_\infty^2}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_1z_2,z_1^{-1}z_2^{-1},z_1z_2^2,z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\right)\frac{dz_2}{z_2}\\
&=\frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_2^{-1}-z_2^{-2})\vartheta(z_2;q)\\
&\qquad\qquad\times\left(\frac{(q;q)_\infty^2}{2\pi\sqrt{-1}}\int_{\mathbb{T}}(z_1z_2,z_1^{-1}z_2^{-1},z_1z_2^2,z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\right)dz_2.
\end{aligned}
\tag{6.42}
$$

From (6.41), the above integrand as a function of z_2 is expanded as

$$
\begin{aligned}
&(z_2^{-1}-z_2^{-2})\sum_{n=-\infty}^{\infty}(-z_2)^nq^{\binom{n}{2}}\left(2\sum_{m=-\infty}^{\infty}z_2^mq^{m^2}+\sum_{m=-\infty}^{\infty}(z_2^m+z_2^{m-1})q^{m(m-1)}\right)\\
&=\sum_{n=-\infty}^{\infty}(-z_2)^nq^{\binom{n}{2}}\left(2\sum_{m=-\infty}^{\infty}(z_2^{m-1}-z_2^{m-2})q^{m^2}+\sum_{m=-\infty}^{\infty}(z_2^{m-1}-z_2^{m-3})q^{m(m-1)}\right),
\end{aligned}
$$

whose residue (the coefficient of z_2^{-1}) is computed as

$$
\begin{aligned}
&2\sum_{n+m=0}(-1)^nq^{\binom{n}{2}+m^2}+2\sum_{n+m=1}(-1)^{n+1}q^{\binom{n}{2}+m^2}\\
&\quad+\sum_{n+m=0}(-1)^nq^{\binom{n}{2}+m(m-1)}+\sum_{n+m=2}(-1)^{n+1}q^{\binom{n}{2}+m(m-1)}.
\end{aligned}
$$

Using the triple product (2.5) again, we obtain

$$
\begin{aligned}
\sum_{n+m=0}(-1)^nq^{\binom{n}{2}+m^2}&=\sum_{n=-\infty}^{\infty}(-1)^nq^{\binom{n}{2}+(-n)^2}=\sum_{n=-\infty}^{\infty}(-q)^nq^{3\binom{n}{2}}\\
&=\vartheta(q;q^3)=(q,q^2,q^3;q^3)_\infty=(q;q)_\infty,\\
\sum_{n+m=1}(-1)^{n+1}q^{\binom{n}{2}+m^2}&=\sum_{m=-\infty}^{\infty}(-1)^{2-m}q^{\binom{1-m}{2}+m^2}=\sum_{m=-\infty}^{\infty}(-1)^mq^{\binom{m}{2}+m^2}\\
&=(q;q)_\infty,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n+m=0}(-1)^nq^{\binom{n}{2}+m(m-1)}&=\sum_{n=-\infty}^{\infty}(-1)^nq^{\binom{n}{2}+n(n+1)}=\sum_{n=-\infty}^{\infty}(-q^2)^nq^{3\binom{n}{2}}\\
&=\vartheta(q^2;q^3)=(q^2,q,q^3;q^3)_\infty=(q;q)_\infty,\\
\sum_{n+m=2}(-1)^{n+1}q^{\binom{n}{2}+m(m-1)}&=\sum_{n=-\infty}^{\infty}(-1)^{n+1}q^{\binom{n}{2}+(n-1)(n-2)}\\
&=\sum_{n=-\infty}^{\infty}(-q)^{n-1}q^{3\binom{n-1}{2}}=\vartheta(q;q^3)=(q;q)_\infty.
\end{aligned}
$$

Consequently, the left-hand side of (6.42) is written as

$$\frac{(q;q)_\infty^3}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}(z_2,z_1z_2,z_1z_2^2;q)_\infty(z_2^{-1},z_1^{-1}z_2^{-1},z_1^{-1}z_2^{-2};q)_\infty\frac{dz_1}{z_1}\frac{dz_2}{z_2}=6(q;q)_\infty,$$

which is our claim of this subsection.

7. Nassrallah–Rahman integral of type G_2

In this section we define the Nassrallah–Rahman integral of type G_2 extending the Askey–Wilson integral of type G_2 (Gustafson's integral), and discuss two q-difference equations for the Nassrallah–Rahman integral of type G_2.

7.1. Nassrallah–Rahman q-hypergeometric integral of type G_2 – Definition and Results.

Let $\Phi(z)$ be function of $z=(z_1,z_2)\in(\mathbb{C}^*)^2$ defined by

$$\Phi(z)=\Phi_+(z)\Phi_+(z^{-1}), \tag{7.1}$$

where $z^{-1}=(z_1^{-1},z_2^{-1})$ and

$$\Phi_+(z)=\frac{(z_2,z_1z_2,z_1z_2^2,z_1,z_1z_2^3,z_1^2z_2^3;q)_\infty}{\prod_{k=1}^5(a_kz_2,a_kz_1z_2,a_kz_1z_2^2;q)_\infty}(qa_6^{-1}z_2,qa_6^{-1}z_1z_2,qa_6^{-1}z_1z_2^2;q)_\infty.$$

We define $J(a_1,a_2,a_3,a_4,a_5,a_6)$ as

$$J(a_1,a_2,a_3,a_4,a_5,a_6)=\frac{1}{(2\pi\sqrt{-1})^2}\iint_{\mathbb{T}^2}\Phi(z)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$$

One of the main result is the following.

Theorem 7.1. *Under the condition $a_1a_2\cdots a_6=-1$, the integral $J(a_1,\dots,a_6)$ satisfies*

$$\begin{aligned}C_1J(qa_1,a_2,a_3,a_4,a_5,a_6)+C_5J(a_1,a_2,a_3,a_4,qa_5,a_6)\\+C_6J(a_1,a_2,a_3,a_4,a_5,qa_6)=0\end{aligned} \tag{7.2}$$

where the coefficient C_1, C_5, C_6 is given as

$$C_1=a_1^2(1-a_1)(a_5^2-a_6^2)\prod_{k=1}^6\frac{1}{1-a_ka_1}\prod_{k=2}^4(1-a_ka_5a_6),$$

$$C_5=a_5^2(1-a_5)(a_6^2-a_1^2)\prod_{k=1}^6\frac{1}{1-a_ka_5}\prod_{k=2}^4(1-a_ka_1a_6),$$

$$C_6=a_6^8(1-a_6)(a_1^2-a_5^2)\prod_{k=1}^6\frac{1}{1-a_ka_6}\prod_{k=2}^4(1-a_ka_1a_5).$$

Proof. The proof will be given in §7.3. □

The other result of this paper is the following:

Theorem 7.2. *Under the condition $a_1a_2\cdots a_6q=-1$, the integral $J(a_1,\dots,a_6)$ satisfies*

$$\begin{aligned}C_{56}J(a_1,a_2,a_3,a_4,qa_5,qa_6)+C_{16}J(qa_1,a_2,a_3,a_4,a_5,qa_6)\\+C_{15}J(qa_1,a_2,a_3,a_4,qa_5,a_6)=0\end{aligned} \tag{7.3}$$

where the coefficient C_{56}, C_{16}, C_{15} is given as

$$
\begin{aligned}
C_{56} &= a_1\frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)}\frac{(1-qa_1a_5)(1-qa_1a_6)}{(1-a_1^2/a_5^2)(1-a_1^2/a_6^2)}\\
&\qquad\times\prod_{k=2}^{4}(1-a_1a_k)\prod_{2\le i<j\le 4}(1-a_1a_ia_j),\\
C_{16} &= a_5\frac{(1-a_5^2)(1-qa_5^2)}{(1-a_5)}\frac{(1-qa_5a_1)(1-qa_5a_6)}{(1-a_5^2/a_1^2)(1-a_5^2/a_6^2)}\\
&\qquad\times\prod_{k=2}^{4}(1-a_5a_k)\prod_{2\le i<j\le 4}(1-a_5a_ia_j),\\
C_{15} &= \frac{(1-a_6^2)(1-qa_6^2)}{a_6^5(1-a_6)}\frac{(1-qa_6a_1)(1-qa_6a_5)}{(1-a_6^2/a_1^2)(1-a_6^2/a_5^2)}\\
&\qquad\times\prod_{k=2}^{4}(1-a_6a_k)\prod_{2\le i<j\le 4}(1-a_6a_ia_j).
\end{aligned}
$$

Proof. The proof will be given in §7.4. □

Remark. As a corollary of Theorem 7.2, when we consider the limit $a_5 \to 0$ ($a_6^{-1} \to 0$), the q-difference equation (6.3) in Proposition 6.2 for Gustafson integral $I(a_1, a_2, a_3, a_4)$ can be obtained from (7.3). Here we confirm this fact. Under the condition $a_1a_2\cdots a_6q = -1$, if $a_5 \to 0$, then we have the limits

$$
\begin{aligned}
J(a_1,a_2,a_3,a_4,qa_5,qa_6) &= J(a_1,a_2,a_3,a_4,qa_5,-a_1^{-1}\cdots a_5^{-1}) \to I(a_1,a_2,a_3,a_4),\\
J(qa_1,a_2,a_3,a_4,a_5,qa_6) &= J(qa_1,a_2,a_3,a_4,qa_5,-a_1^{-1}\cdots a_5^{-1}) \to I(qa_1,a_2,a_3,a_4),\\
J(qa_1,a_2,a_3,a_4,qa_5,a_6) &= J(qa_1,a_2,a_3,a_4,qa_5,-q^{-1}a_1^{-1}\cdots a_5^{-1}) \to I(qa_1,a_2,a_3,a_4),
\end{aligned}
$$

so that we have

$$
a_6C_{56}I(a_1,a_2,a_3,a_4) + a_6(C_{16}+C_{15})I(qa_1,a_2,a_3,a_4) = 0.
$$

where the coefficients a_6C_{56}, a_6C_{16}, a_6C_{15} are computed as follows.

$$
\begin{aligned}
a_6C_{56} &= a_1a_6\frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)}\frac{(1-qa_1a_5)(1-qa_1a_6)}{(1-a_1^2/a_5^2)(1-a_1^2/a_6^2)}\\
&\qquad\times\prod_{k=2}^{4}(1-a_1a_k)\prod_{2\le i<j\le 4}(1-a_1a_ia_j)\\
&= qa_5^2a_6^2\frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)}\frac{(1-qa_1a_5)(1-1/qa_1a_6)}{(1-a_5^2/a_1^2)(1-a_1^2/a_6^2)}\\
&\qquad\times\prod_{k=2}^{4}(1-a_1a_k)\prod_{2\le i<j\le 4}(1-a_1a_ia_j)\\
&\to qa_5^2a_6^2\frac{(1-a_1^2)(1-qa_1^2)}{(1-a_1)}\prod_{k=2}^{4}(1-a_1a_k)\prod_{2\le i<j\le 4}(1-a_1a_ia_j)
\end{aligned}
$$

$$= \frac{(1+a_1)(1-qa_1^2)}{qa_1^2a_2^2a_3^2a_4^2} \prod_{k=2}^{4}(1-a_1a_k) \prod_{2\le i<j\le 4}(1-a_1a_ia_j),$$

$$\begin{aligned}
a_6C_{16} &= a_5a_6\frac{(1-a_5^2)(1-qa_5^2)}{(1-a_5)}\frac{(1-qa_5a_1)(1-qa_5a_6)}{(1-a_5^2/a_1^2)(1-a_5^2/a_6^2)} \\
&\quad\times \prod_{k=2}^{4}(1-a_5a_k) \prod_{2\le i<j\le 4}(1-a_5a_ia_j) \\
&= -qa_5^2a_6^2\frac{(1-a_5^2)(1-qa_5^2)}{(1-a_5)}\frac{(1-qa_5a_1)(1-1/qa_5a_6)}{(1-a_5^2/a_1^2)(1-a_5^2/a_6^2)} \\
&\quad\times \prod_{k=2}^{4}(1-a_5a_k) \prod_{2\le i<j\le 4}(1-a_5a_ia_j) \\
&\to -qa_5^2a_6^2(1-1/qa_5a_6) = -\frac{(1+a_1a_2a_3a_4)}{qa_1^2a_2^2a_3^2a_4^2},
\end{aligned}$$

$$\begin{aligned}
a_6C_{15} &= \frac{(1-a_6^2)(1-qa_6^2)}{a_6^4(1-a_6)}\frac{(1-qa_6a_1)(1-qa_6a_5)}{(1-a_6^2/a_1^2)(1-a_6^2/a_5^2)} \\
&\quad\times \prod_{k=2}^{4}(1-a_6a_k) \prod_{2\le i<j\le 4}(1-a_6a_ia_j) \\
&= \frac{q^3a_1a_5a_6^6}{-a_6^9/a_1^2a_5^2}\frac{(1-1/a_6^2)(1-1/qa_6^2)}{(1-1/a_6)}\frac{(1-1/qa_6a_1)(1-1/qa_6a_5)}{(1-a_1^2/a_6^2)(1-a_5^2/a_6^2)} \\
&\quad\times a_6^6a_2^3a_3^3a_4^3 \prod_{k=2}^{4}(1-1/a_6a_k) \prod_{2\le i<j\le 4}(1-1/a_6a_ia_j) \\
&= -q^3a_1^3a_2^3a_3^3a_4^3a_5^3a_6^3\frac{(1-1/a_6^2)(1-1/qa_6^2)}{(1-1/a_6)}\frac{(1-1/qa_6a_1)(1-1/qa_6a_5)}{(1-a_1^2/a_6^2)(1-a_5^2/a_6^2)} \\
&\quad\times \prod_{k=2}^{4}(1-1/a_6a_k) \prod_{2\le i<j\le 4}(1-1/a_6a_ia_j) \\
&= \frac{(1-1/a_6^2)(1-1/qa_6^2)}{(1-1/a_6)}\frac{(1-1/qa_6a_1)(1-1/qa_6a_5)}{(1-a_1^2/a_6^2)(1-a_5^2/a_6^2)} \\
&\quad\times \prod_{k=2}^{4}(1-1/a_6a_k) \prod_{2\le i<j\le 4}(1-1/a_6a_ia_j) \\
&\to (1-1/qa_5a_6) = (1+a_1a_2a_3a_4).
\end{aligned}$$

Therefore, if $a_5 \to 0$, consequently we have

$$\begin{aligned}
&a_6C_{56} \to \frac{(1+a_1)(1-qa_1^2)}{qa_1^2a_2^2a_3^2a_4^2} \prod_{k=2}^{4}(1-a_1a_k) \prod_{2\le i<j\le 4}(1-a_1a_ia_j), \\
&a_6C_{16} \to -\frac{(1+a_1a_2a_3a_4)}{qa_1^2a_2^2a_3^2a_4^2}, \\
&a_6C_{15} \to (1+a_1a_2a_3a_4),
\end{aligned}$$

$$a_6(C_{16}+C_{15}) \to (1-qa_5^2a_6^2)(1-1/qa_5a_6) = -qa_5^2a_6^2(1-1/qa_5^2a_6^2)(1-1/qa_5a_6)$$
$$= -\frac{(1+a_1a_2a_3a_4)(1-qa_1^2a_2^2a_3^2a_4^2)}{qa_1^2a_2^2a_3^2a_4^2}.$$

Therefore we had the following from

$$\frac{I(qa_1,a_2,a_3,a_4)}{I(a_1,a_2,a_3,a_4)} = \frac{(1+a_1)(1-qa_1^2)}{(1+a_1a_2a_3a_4)(1-qa_1^2a_2^2a_3^2a_4^2)} \prod_{i=2}^{4}(1-a_1a_i) \prod_{2\le j<k\le 4}(1-a_1a_ja_k),$$

which coincides with the q-difference equation (6.3) in Proposition 6.2. □

Under the condition $a_1a_2\cdots a_6 = -q$, we want to find the system of q-difference equations for $J(a_1,a_2,a_3,a_4,a_5,a_6)$. Since a_6 is written as $a_6 = -q(a_1a_2\cdots a_5)^{-1}$ $J(a_1,a_2,a_3,a_4,a_5,a_6)$ is equal to

$$\bar{J} = \bar{J}(a_1,a_2,a_3,a_4,a_5) := J(a_1,a_2,a_3,a_4,a_5,-q(a_1a_2\cdots a_5)^{-1}),$$

which we regard $\bar{J}$ as a function of (a_1,a_2,a_3,a_4,a_5). If we consider the result in Theorem 7.1 under $a_6 \to q^{-1}a_6$, and that in Theorem 7.2 under $a_6 \to q^{-2}a_6$, then we immediately have the following:

Corollary 7.3. *Under the condition $a_1a_2\cdots a_6 = -q$, the integral $\bar{J}(a_1,a_2,a_3,a_4,a_5)$ satisfies the system of first-order simultaneous q-difference equations of rank 2 as*

$$T_{q,a_1}(\bar{J}, T_{q,a_5}\bar{J}) = (\bar{J}, T_{q,a_5}\bar{J})A.$$

where the 2×2 matrix A is given in terms of Gauss matrix decomposition as

$$A = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & \delta \end{pmatrix}^{-1}$$

Here the entries α, β, γ and δ are given as $\alpha = -T_{q,a_6}^{-1}(C_6/C_1)$, $\beta = -T_{q,a_6}^{-1}(C_5/C_1)$, $\gamma = -T_{q^2,a_6}^{-1}(C_{16}/C_{56})$ and $\delta = -T_{q^2,a_6}^{-1}(C_{15}/C_{56})$, respectively.

Remark 1. The determinant of the coefficient matrix A is given as

$$\det A = \alpha/\delta = \frac{T_{q,a_6}^{-1}(C_6/C_1)}{T_{q^2,a_6}^{-1}(C_{15}/C_{56})}.$$

Remark 2. Eliminating $T_{q,a_5}\bar{J}$ from this system, the second order q-difference equation for $\bar{J}$ can be obtained.

7.2. **q-Stokes formula.**

Lemma 7.4. *Let $F_+(z)$ and $F_-(z)$ be functions specified as*

$$F_+(z) = (z_1^2z_2^3)^{-\frac{3}{2}} \frac{\prod_{k=1}^{6}(1-a_kz_1z_2)(1-a_kz_1z_2^2)}{(1-z_1z_2)(1-z_1z_2^2)(1-z_1)(1-z_1z_2^3)(1-z_1^2z_2^3)}. \tag{7.4}$$

Then, it follows that

$$\frac{T_{q,z_1}\Phi(z)}{\Phi(z)} = -\frac{F_+(z)}{T_{q,z_1}F_-(z)}, \qquad F_-(z) = F_+(z^{-1}).$$

Proof. By the definition (7.1) of $\Phi(z)$ we have

$$\begin{aligned}
\frac{T_{q,z_1}\Phi(z)}{\Phi(z)} &= \frac{(1-(qz_1z_2)^{-1})(1-(qz_1z_2^2)^{-1})(1-(qz_1)^{-1})}{(1-z_1z_2)(1-z_1z_2^2)(1-z_1)} \\
&\quad\times \frac{(1-(qz_1z_2^3)^{-1})(1-(qz_1^2z_2^3)^{-1})(1-(q^2z_1^2z_2^3)^{-1})}{(1-z_1z_2^3)(1-z_1^2z_2^3)(1-qz_1^2z_2^3)} \\
&\quad\times \frac{(1-qa_6^{-1}(qz_1z_2)^{-1})(1-qa_6^{-1}(qz_1z_2^2)^{-1})}{(1-qa_6^{-1}z_1z_2)(1-qa_6^{-1}z_1z_2^2)} \\
&\quad\times \prod_{k=1}^{5} \frac{(1-a_kz_1z_2)(1-a_kz_1z_2^2)}{(1-a_k(qz_1z_2)^{-1})(1-a_k(qz_1z_2^2)^{-1})} \\
&= -(qz_1^2z_2^3)^{-3}\frac{(1-(qz_1z_2)^{-1})(1-(qz_1z_2^2)^{-1})}{(1-z_1z_2)(1-z_1z_2^2)} \\
&\quad\times \frac{(1-(qz_1)^{-1})(1-(qz_1z_2^3)^{-1})(1-(q^2z_1^2z_2^3)^{-1})}{(1-z_1)(1-z_1z_2^3)(1-z_1^2z_2^3)} \\
&\quad\times \prod_{k=1}^{6} \frac{(1-a_kz_1z_2)(1-a_kz_1z_2^2)}{(1-a_k(qz_1z_2)^{-1})(1-a_k(qz_1z_2^2)^{-1})}
\end{aligned}$$

This implies Lemma 7.4. □

We remark that if we change the definitions of $\Phi(z)$ and $F_+(z)$ in §6.2 to (7.1) and (7.4) in this section, respectively, then the same arguments as q-Stokes' theorem (Lemma 6.6) and Lemma 6.7 hold.

7.3. **q-Difference equation – Proof of Theorem 7.1.** From the definition (7.1) of $\Phi(z)$, we have

$$\frac{T_{q,a_k}\Phi(z)}{\Phi(z)} = a_k^3\, g(a_k;z) \quad (k=1,2,3,4,5), \qquad \frac{T_{q,a_6}\Phi(z)}{\Phi(z)} = a_6^{-3}\, g(a_6;z),$$

so that

$$\begin{aligned}
J(qa_1,a_2,a_3,a_4,a_5,a_6) &= a_1^3\langle g(a_1;z)\rangle, \\
J(a_1,a_2,a_3,a_4,qa_5,a_6) &= a_5^3\langle g(a_5;z)\rangle, \\
J(a_1,a_2,a_3,a_4,a_5,qa_6) &= a_6^{-3}\langle g(a_6;z)\rangle.
\end{aligned}$$

Taking account of symmetry of parameters, Theorem 7.1 is equivalent to the following:

Proposition 7.5. *Under the condition $a_1a_2\cdots a_6=-1$, the integral $J(a_1,\dots,a_6)$ satisfies*

$$C_1\langle g(a_1;z)\rangle + C_2\langle g(a_2;z)\rangle + C_3\langle g(a_3;z)\rangle = 0, \tag{7.5}$$

where

$$C_1 = \frac{a_1^5(1-a_1)(a_2^2-a_3^2)}{(1-a_2^2a_3)(1-a_2a_3^2)}\prod_{k=1}^{6}\frac{1-a_ka_2a_3}{1-a_ka_1}, \tag{7.6}$$

$$C_2 = \frac{a_2^5(1-a_2)(a_3^2-a_1^2)}{(1-a_1^2a_3)(1-a_1a_3^2)}\prod_{k=1}^{6}\frac{1-a_ka_1a_3}{1-a_ka_2}, \tag{7.7}$$

$$(7.8) \qquad C_3 = \frac{a_3^5(1-a_3)(a_1^2-a_2^2)}{(1-a_1^2a_2)(1-a_1a_2^2)} \prod_{k=1}^{6} \frac{1-a_ka_1a_2}{1-a_ka_3}.$$

The aim of this subsection is to prove Proposition 7.5. Before proving it we show two lemmas.

Lemma 7.6. *For $i=0,1,2$ it follows that*

$$(7.9) \qquad \frac{\nabla_{\mathrm{sym}}(z_2+z_2^{-1})^i(z_2^{-\frac{1}{2}}+z_2^{\frac{1}{2}})}{4} \in \mathcal{F}_3$$

The explicit expressions of (7.9) *are given by*

$$(7.10) \qquad \frac{\nabla_{\mathrm{sym}}(z_2^{-\frac{1}{2}}+z_2^{\frac{1}{2}})}{4} = (S_0+S_6)(S_0-S_6)s_{\varpi_1+\varpi_2}(z)+\cdots,$$

$$(7.11) \qquad \frac{\nabla_{\mathrm{sym}}(z_2+z_2^{-1})(z_2^{-\frac{1}{2}}+z_2^{\frac{1}{2}})}{4} = (S_0+S_6)(S_0-S_6)s_{3\varpi_2}(z)+0s_{\varpi_1+\varpi_2}(z)+\cdots,$$

$$(7.12) \qquad \frac{\nabla_{\mathrm{sym}}(z_2+z_2^{-1})^2(z_2^{-\frac{1}{2}}+z_2^{\frac{1}{2}})}{4} = (S_0+S_6)\Big[(S_1-S_5)s_{3\varpi_2}(z)+(S_4-S_2+S_0-S_6)\Big]s_{\varpi_1+\varpi_2}(z)+\cdots,$$

where S_r $(r=0,1,\ldots,6)$ is the rth elementary symmetric polynomial of $a_1,\ldots,a_6$ given by

$$(7.13) \qquad S_r = \sum_{1\le i_1<\cdots<i_r\le 6} a_{i_1}a_{i_2}\cdots a_{i_r}$$

i.e., $S_0=1$, $S_1=a_1+\cdots+a_6$, $\ldots$, $S_6=a_1\cdots a_6$. In particular, under the condition

$$a_1a_2\cdots a_6=-1$$

it follows that

$$(7.14) \qquad \frac{\nabla_{\mathrm{sym}}(z_2+z_2^{-1})^i(z_2^{-\frac{1}{2}}+z_2^{\frac{1}{2}})}{4} \in \mathcal{F}_2 \quad (i=0,1,2).$$

Proof. We can confirm (7.10)–(7.12) by direct calculation. Since all of the coefficients of $s_{3\varpi_2}(z)$ and $s_{\varpi_1+\varpi_2}(z)$ in the right-hand sides of (7.10)–(7.12) have the factor S_0+S_6, under the condition $a_1a_2\cdots a_6=-1$, i.e., $S_0+S_6=0$, we obtain (7.14). □

Lemma 7.7. *Suppose that $a_1a_2\cdots a_6=-1$. Put*

$$\begin{aligned}\varphi(z) &= z_2^{-\frac{1}{2}}(1+z_2)e(a_1;z_2)e(a_2;z_2)\\ &= z_2^{-\frac{1}{2}}(1+z_2)(1-a_1z_2)(1-a_1z_2^{-1})(1-a_2z_2)(1-a_2z_2^{-1})a_1^{-1}a_2^{-1}\end{aligned}$$

Then, $\nabla_{\mathrm{sym}}\varphi(z)/4$ is expressed as

$$(7.15) \qquad \frac{\nabla_{\mathrm{sym}}\varphi(z)}{4} = c_1g(a_1;z)+c_2g(a_2;z)+c_{12}G(z) \in \mathcal{F}_2,$$

where the coefficients c_1, c_2, c_{12} are given as

$$(7.16) \qquad c_1 = a_1^2a_2^{-4}a_3^{-2}\frac{(1+a_2)(1+a_3)(1-a_2a_3)}{(1-a_1/a_2)(1-a_1/a_2a_3)}\prod_{k=4}^{6}(1-a_ka_2)(1-a_ka_2a_3),$$

(7.17) $$c_2 = a_1^{-4}a_2^2a_3^{-2}\frac{(1+a_1)(1+a_3)(1-a_1a_3)}{(1-a_2/a_1)(1-a_2/a_1a_3)}\prod_{k=4}^{6}(1-a_ka_1)(1-a_ka_1a_3),$$

(7.18) $$c_{12} = a_1^{-2}a_2^{-3}(1+a_2/a_1)(1-a_1a_2)(1-a_1a_2a_3)(1-a_1a_2/a_3) \times \prod_{k=3}^{6}(1-a_ka_1)(1-a_ka_2).$$

Proof. We set $\widetilde{\varphi}(z) = \nabla_{\rm sym}\varphi(z)/4$. Since $\varphi(z)$ is expanded as

$$\varphi(z) = (z_2^{-\frac{1}{2}} + z_2^{\frac{1}{2}})\Big[(z_2+z_2^{-1})^2 - (A_1+A_2)(z_2+z_2^{-1}) + A_1A_2 + 2)\Big], \quad A_i = a_i + a_i^{-1}$$

From (7.14) of Lemma 7.7, under the condition $a_1a_2\cdots a_6 = -1$, we have $\widetilde{\varphi}(z) \in \mathcal{F}_2$, namely we have the expansion as

$$\widetilde{\varphi}(z) = c_1g(a_1;z) + c_2g(a_2;z) + c_3g(a_3;z) + c_{12}G(z),$$

where c_1, c_2, c_3 and c_{12} are the constants independent of z. From the vanishing properties of $g(a_i, z)$ and $G(z)$, we immediately have

$$c_1 = \frac{\widetilde{\varphi}(\mathrm{p}_{23})}{g(a_1;\mathrm{p}_{23})},\ c_2 = \frac{\widetilde{\varphi}(\mathrm{p}_{13})}{g(a_2;\mathrm{p}_{13})},\ c_3 = \frac{\widetilde{\varphi}(\mathrm{p}_{12})}{g(a_3;\mathrm{p}_{12})},\ c_{12} = \frac{\widetilde{\varphi}(\mathrm{p}_{12}^*) - c_3g(a_3;\mathrm{p}_{12}^*)}{G(\mathrm{p}_{12}^*)}$$

We first evaluate c_1, c_2, c_3. From Lemma 6.7 $\widetilde{\varphi}(z)$ is written as

$$\widetilde{\varphi}(z) = \frac{\nabla_{\rm sym}\varphi(z)}{4} = \sum_{k=0}^{5} F_k(z)\varphi_k(z).$$

If $z = \mathrm{p}_{ij}$ (i.e., $z_2 = a_j$, $z_1z_2 = a_i$, $z_1z_2^2 = a_ia_j$), then $F_2(\mathrm{p}_{ij}) = F_3(\mathrm{p}_{ij}) = F_4(\mathrm{p}_{ij}) = F_5(\mathrm{p}_{ij}) = 0$, so that we have

$$\widetilde{\varphi}(\mathrm{p}_{ij}) = F_0(\mathrm{p}_{ij})\varphi_0(\mathrm{p}_{ij}) + F_1(\mathrm{p}_{ij})\varphi_1(\mathrm{p}_{ij}).$$

Since $\varphi(z)$ does not include z_1, we have

$$\varphi_0(z) = \varphi(z) = z_2^{-\frac{1}{2}}(1+z_2)e(a_1;z_2)e(a_2;z_2),$$
$$\varphi_1(z) = (z_1z_2)^{-\frac{1}{2}}(1+z_1z_2)e(a_1;z_1z_2)e(a_2;z_1z_2),$$

so that we have

$$\varphi_0(\mathrm{p}_{ij}) = a_j^{-\frac{1}{2}}(1+a_j)e(a_1;a_j)e(a_2;a_j),$$
$$\varphi_1(\mathrm{p}_{ij}) = a_i^{-\frac{1}{2}}(1+a_i)e(a_1;a_i)e(a_2;a_i).$$

In particular $\varphi_1(\mathrm{p}_{1j}) = 0$ and

$$\begin{aligned}\varphi_0(\mathrm{p}_{1j}) &= a_j^{-\frac{1}{2}}(1+a_j)e(a_1;a_j)e(a_2;a_j)\\ &= a_j^{-\frac{1}{2}}(1+a_j)\frac{(1-a_1a_j)(1-a_1/a_j)(1-a_2a_j)(1-a_2/a_j)}{a_1a_2}.\end{aligned}$$

Therefore we obtain

$$\widetilde{\varphi}(\mathrm{p}_{1j}) = F_0(\mathrm{p}_{1j})\varphi_0(\mathrm{p}_{1j}) = F_+(\mathrm{p}_{1j})\varphi(\mathrm{p}_{1j}).$$

Since $\varphi(\mathrm{p}_{12}) = 0$, we obtain

$$c_3 = \frac{\widetilde{\varphi}(\mathrm{p}_{12})}{g(a_3;\mathrm{p}_{12})} = \frac{F_+(\mathrm{p}_{12})\varphi(\mathrm{p}_{12})}{g(a_3;\mathrm{p}_{12})} = 0.$$

Moreover, we can calculate

$$
\begin{aligned}
c_2 &= \frac{\widetilde{\varphi}(\mathrm{p}_{13})}{g(a_2;\mathrm{p}_{13})} = \frac{F_+(\mathrm{p}_{13})\varphi(\mathrm{p}_{13})}{g(a_2;\mathrm{p}_{13})} \\
&= (a_1^2a_3)^{-\frac{3}{2}} \frac{\prod_{k=1}^6(1-a_ka_1)(1-a_ka_1a_3)}{(1-a_1)(1-a_1a_3)(1-a_1/a_3)(1-a_1a_3^2)(1-a_1^2a_3)} \\
&\quad \times a_3^{-\frac{1}{2}}(1+a_3)\frac{(1-a_1a_3)(1-a_1/a_3)(1-a_2a_3)(1-a_2/a_3)}{a_1a_2} \\
&\quad \times \frac{a_2^3}{(1-a_2a_1)(1-a_2/a_1)(1-a_2a_3)(1-a_2/a_3)(1-a_1a_2a_3)(1-a_2/a_1a_3)} \\
&= a_1^{-4}a_2^2a_3^{-2}\frac{(1+a_1)(1+a_3)(1-a_1a_3)}{(1-a_2/a_1)(1-a_2/a_1a_3)}\prod_{k=4}^6(1-a_ka_1)(1-a_ka_1a_3),
\end{aligned}
$$

which is (7.17). From symmetry of the equations c_1 is obtained from c_2 changing $a_1 \leftrightarrow a_2$. Thus we have

$$
c_1 = a_1^2a_2^{-4}a_3^{-2}\frac{(1+a_2)(1+a_3)(1-a_2a_3)}{(1-a_1/a_2)(1-a_1/a_2a_3)}\prod_{k=4}^6(1-a_ka_2)(1-a_ka_2a_3),
$$

which coincides with (7.16).

We next evaluate c_{12}. From Lemma 6.7, if $z = \mathrm{p}_{ij}^*$ (i.e., $z_2 = a_j/a_i$, $z_1z_2 = a_i$, $z_1z_2^2 = a_j$), then $F_2(\mathrm{p}_{ij}^*) = F_3(\mathrm{p}_{ij}^*) = F_4(\mathrm{p}_{ij}^*) = 0$, so that we have

$$
\widetilde{\varphi}(\mathrm{p}_{ij}^*) = F_0(\mathrm{p}_{ij}^*)\varphi_0(\mathrm{p}_{ij}^*) + F_1(\mathrm{p}_{ij}^*)\varphi_1(\mathrm{p}_{ij}^*) + F_5(\mathrm{p}_{ij}^*)\varphi_5(\mathrm{p}_{ij}^*).
$$

Since $\varphi(z)$ does not include z_1, we have

$$
\begin{aligned}
\varphi_0(z) &= \varphi(z) = z_2^{-\frac{1}{2}}(1+z_2)e(a_1;z_2)e(a_2;z_2), \\
\varphi_1(z) &= (z_1z_2)^{-\frac{1}{2}}(1+z_1z_2)e(a_1;z_1z_2)e(a_2;z_1z_2), \\
\varphi_5(z) &= (z_1z_2^2)^{-\frac{1}{2}}(1+z_1z_2^2)e(a_1;z_1z_2^2)e(a_2;z_1z_2^2),
\end{aligned}
$$

so that we have

$$
\begin{aligned}
\varphi_0(\mathrm{p}_{ij}^*) &= (a_j/a_i)^{-\frac{1}{2}}(1+a_j/a_i)e(a_1;a_j/a_i)e(a_2;a_j/a_i), \\
\varphi_1(\mathrm{p}_{ij}^*) &= a_i^{-\frac{1}{2}}(1+a_i)e(a_1;a_i)e(a_2;a_i), \\
\varphi_5(\mathrm{p}_{ij}^*) &= a_j^{-\frac{1}{2}}(1+a_j)e(a_1;a_j)e(a_2;a_j).
\end{aligned}
$$

From $\varphi_1(\mathrm{p}_{12}^*) = \varphi_5(\mathrm{p}_{12}^*) = 0$, we have

$$
\begin{aligned}
\varphi_0(\mathrm{p}_{12}^*) &= (a_2/a_1)^{-\frac{1}{2}}(1+a_2/a_1)e(a_1;a_2/a_1)e(a_2;a_2/a_1) \\
&= (a_2/a_1)^{-\frac{1}{2}}(1+a_2/a_1)(1-a_2)(1-a_1^2/a_2)(1-a_2^2/a_1)(1-a_1)a_1^{-1}a_2^{-1},
\end{aligned}
$$

so that we have

$$
\begin{aligned}
\widetilde{\varphi}(\mathrm{p}_{12}^*) &= F_0(\mathrm{p}_{12}^*)\varphi_0(\mathrm{p}_{12}^*) = F_+(\mathrm{p}_{12}^*)\varphi(\mathrm{p}_{12}^*) \\
&= (a_1a_2)^{-\frac{3}{2}}\frac{\prod_{k=1}^6(1-a_ka_1)(1-a_ka_2)}{(1-a_1)(1-a_2)(1-a_1^2/a_2)(1-a_2^2/a_1)(1-a_1a_2)} \\
&\quad \times (a_2/a_1)^{-\frac{1}{2}}(1+a_2/a_1)(1-a_2)(1-a_1^2/a_2)(1-a_2^2/a_1)(1-a_1)a_1^{-1}a_2^{-1}
\end{aligned}
$$

$$= \frac{(1+a_2/a_1)\prod_{k=1}^{6}(1-a_ka_1)(1-a_ka_2)}{a_1^2a_2^3(1-a_1a_2)}.$$

Using the fact that $c_3 = 0$, we obtain

$$\begin{aligned} c_{12} &= \frac{\widetilde{\varphi}(\mathrm{p}_{12}^*) - c_3 g(a_3;\mathrm{p}_{12}^*)}{G(\mathrm{p}_{12}^*)} = \frac{\widetilde{\varphi}(\mathrm{p}_{12}^*)}{G(\mathrm{p}_{12}^*)} \\ &= \frac{(1+a_2/a_1)\prod_{k=1}^{6}(1-a_ka_1)(1-a_ka_2)}{a_1^2a_2^3(1-a_1a_2)} \times \frac{a_1a_2\,e(a_1a_2,a_3)}{(1-a_1^2)(1-a_2^2)} \\ &= a_1^{-2}a_2^{-3}(1+a_2/a_1)(1-a_1a_2)(1-a_1a_2a_3)(1-a_1a_2/a_3) \\ &\quad \times \prod_{k=3}^{6}(1-a_ka_1)(1-a_ka_2), \end{aligned}$$

which coincides with (7.18). □

Proof of Proposition 7.5. Using Lemma 6.6, (7.15) of Lemma 7.7 implies

$$c_1\langle g(a_1;z)\rangle + c_2\langle g(a_2;z)\rangle + c_{12}\langle G(z)\rangle = 0, \tag{7.19}$$

where c_1, c_2, c_{12} are given as (7.16)–(7.18). If we consider (7.15) of Lemma 7.7 with $a_2 \leftrightarrow a_3$, we obtain

$$d_1\langle g(a_1;z)\rangle + d_3\langle g(a_3;z)\rangle + d_{13}\langle G(z)\rangle = 0 \tag{7.20}$$

where, d_1, d_3, d_{13} are given as (7.16)–(7.18) with $a_2 \leftrightarrow a_3$, i.e.,

$$\begin{aligned} d_1 &= a_1^2a_3^{-4}a_2^{-2}\frac{(1+a_2)(1+a_3)(1-a_2a_3)}{(1-a_1/a_3)(1-a_1/a_2a_3)}\prod_{k=4}^{6}(1-a_ka_3)(1-a_ka_2a_3), \\ d_3 &= a_1^{-4}a_3^2a_2^{-2}\frac{(1+a_1)(1+a_2)(1-a_1a_2)}{(1-a_3/a_1)(1-a_3/a_1a_2)}\prod_{k=4}^{6}(1-a_ka_1)(1-a_ka_1a_2), \\ d_{13} &= a_1^{-2}a_3^{-3}(1+a_3/a_1)(1-a_1a_2)(1-a_1a_3)(1-a_2a_3)(1-a_1a_2a_3)(1-a_1a_3/a_2) \\ &\quad \times \prod_{k=4}^{6}(1-a_ka_1)(1-a_ka_3). \end{aligned}$$

Eliminating $\langle G(z)\rangle$ from (7.19) and (7.20), we have the relation (7.5). □

7.4. q-Difference equation – Proof of Theorem 7.2.

By definition we have

$$\begin{aligned} J(a_1,a_2,a_3,a_4,qa_5,qa_6) &= a_5^3a_6^{-3}\langle g(a_5;z)g(a_6;z)\rangle, \\ J(qa_1,a_2,a_3,a_4,qa_5,a_6) &= a_1^3a_5^3\langle g(a_1;z)g(a_5;z)\rangle, \\ J(qa_1,a_2,a_3,a_4,a_5,qa_6) &= a_1^3a_6^{-3}\langle g(a_1;z)g(a_6;z)\rangle. \end{aligned}$$

Taking account of symmetry of the parameters, Theorem 7.2 is equivalent to the following.

Proposition 7.8. *Under the condition $a_1a_2\cdots a_6q = -1$, it follows that*

$$C_{23}\langle g(a_2;z)g(a_3;z)\rangle + C_{13}\langle g(a_1;z)g(a_3;z)\rangle + C_{12}\langle g(a_1;z)g(a_2;z)\rangle = 0, \tag{7.21}$$

where

$$C_{23}=\frac{(1-a_1^2)(1-qa_1^2)}{a_1^2(1-a_1)}\frac{(1-qa_1a_2)(1-qa_1a_3)}{(1-a_1^2/a_2^2)(1-a_1^2/a_3^2)}\prod_{k=4}^{6}(1-a_1a_k)\prod_{4\le i<j\le 6}(1-a_1a_ia_j),$$

$$C_{13}=\frac{(1-a_2^2)(1-qa_2^2)}{a_2^2(1-a_2)}\frac{(1-qa_2a_1)(1-qa_2a_3)}{(1-a_2^2/a_1^2)(1-a_2^2/a_3^2)}\prod_{k=4}^{6}(1-a_2a_k)\prod_{4\le i<j\le 6}(1-a_2a_ia_j),$$

$$C_{12}=\frac{(1-a_3^2)(1-qa_3^2)}{a_3^2(1-a_3)}\frac{(1-qa_3a_1)(1-qa_3a_2)}{(1-a_3^2/a_1^2)(1-a_3^2/a_2^2)}\prod_{k=4}^{6}(1-a_3a_k)\prod_{4\le i<j\le 6}(1-a_3a_ia_j).$$

The aim of this subsection is to prove Proposition 7.8. Before proving it we show four lemmas.

Lemma 7.9. *Let $\alpha_i(z)$, $\beta_i(z)$, $\gamma_i(z)$ be functions specified by*

$$\alpha_i(z):=\frac{1}{4}\nabla_{\mathrm{sym}}\Big((z_2+z_2^{-1})^i(z_2^{-\frac12}+z_2^{\frac12})\Big),$$

$$\beta_i(z):=\frac{1}{4}\nabla_{\mathrm{sym}}\Big((z_2+z_2^{-1})^i(z_2^{-\frac12}+z_2^{\frac12})\big((z_1z_2)^{-1}+z_1z_2+(z_1z_2^2)^{-1}+z_1z_2^2\big)\Big),$$

$$\gamma_i(z):=\frac{1}{4}\nabla_{\mathrm{sym}}\Big((z_2+z_2^{-1})^i(z_2^{-\frac12}+z_2^{\frac12})\big((z_1z_2)^{-1}+z_1z_2\big)\big((z_1z_2^2)^{-1}+z_1z_2^2\big)\Big).$$

Then, we have

$$\alpha_i(z)\in\mathcal{F}_4\quad(i=0,1,2,3),\qquad \beta_i(z)\in\mathcal{F}_4\quad(i=0,1,2)\tag{7.22}$$

and

$$\beta_3(z)\in\mathcal{F}_5,\qquad \gamma_i(z)\in\mathcal{F}_5\quad(i=0,1,2,3).$$

$\beta_3(z)\in\mathcal{F}_5$ and $\gamma_i(z)\in\mathcal{F}_5$ $(i=0,1,2,3)$ are expressed as

$$\begin{aligned}\beta_3(z)&=q^{-\frac12}(S_0+qS_6)\\&\quad\times\Big[(S_6-S_0)s_{5\varpi_2}(z)+(S_1-S_5)s_{\varpi_1+3\varpi_2}(z)+(S_4-S_2)s_{2\varpi_1+\varpi_2}(z)\Big]+\cdots\end{aligned}$$

and

$$\begin{aligned}\gamma_0(z)&=q^{-1}(S_0+qS_6)(S_0-qS_6)s_{2\varpi_1+\varpi_2}(z)+\cdots,\\
\gamma_1(z)&=q^{-1}(S_0+qS_6)(S_0-qS_6)\Big[s_{\varpi_1+3\varpi_2}(z)+0s_{2\varpi_1+\varpi_2}(z)\Big]+\cdots,\\
\gamma_2(z)&=q^{-1}(S_0+qS_6)(S_0-qS_6)\Big[s_{5\varpi_2}(z)+0s_{\varpi_1+3\varpi_2}(z)+s_{2\varpi_1+\varpi_2}(z)\Big]+\cdots,\\
\gamma_3(z)&=q^{-1}(S_0+qS_6)\Big[(S_1-qS_5)s_{5\varpi_2}(z)\\
&\qquad+\Big(qS_4-S_2+2(S_0-qS_6)+(S_6-qS_0)\Big)s_{\varpi_1+3\varpi_2}(z)\\
&\qquad+\Big((1-q)S_3+qS_1-S_5\Big)s_{2\varpi_1+\varpi_2}(z)\Big]+\cdots,\end{aligned}$$

where S_r $(r=0,1,\dots,6)$ is the rth elementary symmetric polynomial of $a_1,\dots,a_6$ given by (7.13). In particular, under the condition

$$a_1a_2\cdots a_6q=-1$$

we have

$$\beta_3(z) \in \mathcal{F}_4, \quad \gamma_i(z) \in \mathcal{F}_4 \quad (i=0,1,2,3). \tag{7.23}$$

Proof. The proof follows by direct computation and we omit the details. □

Lemma 7.10. *Suppose that* $a_1a_2\cdots a_6q=-1$. *Let* $\varphi_i(z)$ $(i=1,2,3)$ *be functions specified by*

$$\begin{aligned}\varphi_i(z) &= z_2^{-\frac{1}{2}}(1+z_2)e(a_1,z_2)e(a_2,z_2)e(a_3,z_2)e(c_i,z_1z_2)e(c_i,z_1z_2^2)\\ &= z_2^{-\frac{1}{2}}(1+z_2)(1-a_1z_2)(1-a_1z_2^{-1})(1-a_2z_2)\\ &\quad\times(1-a_2z_2^{-1})(1-a_3z_2)(1-a_3z_2^{-1})a_1^{-1}a_2^{-1}a_3^{-1}\\ &\quad\times(1-c_iz_1z_2)(1-c_i/z_1z_2)(1-c_iz_1z_2^2)(1-c_i/z_1z_2^2)c_i^{-2},\end{aligned}$$

and $g_{ij}(z)$, $G_i(z)$ *be the functions defined as* (5.26), (5.28), *respectively. If* $c_i=qa_i$ $(i=1,2,3)$, *then* $\nabla_{\rm sym}\varphi_i(z)/4$ *are expanded as*

$$\begin{aligned}\frac{\nabla_{\rm sym}\varphi_1(z)}{4} &= c_{13}^{(1)}g_{13}(z)+c_{12}^{(1)}g_{12}(z)+c_1^{(1)}G_1(z),\\ \frac{\nabla_{\rm sym}\varphi_2(z)}{4} &= c_{23}^{(2)}g_{23}(z)+c_{12}^{(2)}g_{12}(z)+c_2^{(2)}G_2(z),\\ \frac{\nabla_{\rm sym}\varphi_3(z)}{4} &= c_{23}^{(3)}g_{23}(z)+c_{13}^{(3)}g_{13}(z)+c_3^{(3)}G_3(z),\end{aligned}$$

where the coefficients are given as

$$\begin{aligned}c_1^{(1)} &= q^{-1}a_2^{-2}a_3^{-2}(a_2+a_3)(1-a_2a_3)(1-qa_1a_2)(1-qa_1a_3)e(a_2a_3,a_4)\\ &\quad\times\prod_{k=4}^{6}(1-a_ka_2)(1-a_ka_3),\\ c_2^{(2)} &= q^{-1}a_1^{-2}a_3^{-2}(a_1+a_3)(1-a_1a_3)(1-qa_1a_2)(1-qa_2a_3)e(a_1a_3,a_4)\\ &\quad\times\prod_{k=4}^{6}(1-a_ka_1)(1-a_ka_3),\\ c_3^{(3)} &= q^{-1}a_1^{-2}a_2^{-2}(a_1+a_2)(1-a_1a_2)(1-qa_1a_3)(1-qa_2a_3)e(a_1a_2,a_4)\\ &\quad\times\prod_{k=4}^{6}(1-a_ka_1)(1-a_ka_2),\end{aligned} \tag{7.24}$$

$$\begin{aligned}c_{12}^{(i)} = c_{21}^{(i)} &= \frac{(1+a_3)(1+a_4)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^{6}(1-a_ka_3)}{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-a_3/a_1)(1-a_3/a_2)}\\ &\quad\times e(q^{\frac{1}{2}}a_i,q^{\frac{1}{2}}a_3)e(q^{\frac{1}{2}}a_i,q^{\frac{1}{2}}a_3a_4),\\ c_{13}^{(i)} = c_{31}^{(i)} &= \frac{(1+a_2)(1+a_4)(1-a_2a_4a_5)(1-a_2a_4a_6)\prod_{k=4}^{6}(1-a_ka_2)}{(1-a_2a_4/a_1)(1-a_2a_4/a_3)(1-a_2/a_1)(1-a_2/a_3)}\\ &\quad\times e(q^{\frac{1}{2}}a_i,q^{\frac{1}{2}}a_2)e(q^{\frac{1}{2}}a_i,q^{\frac{1}{2}}a_2a_4),\\ c_{23}^{(i)} = c_{32}^{(i)} &= \frac{(1+a_1)(1+a_4)(1-a_1a_4a_5)(1-a_1a_4a_6)\prod_{k=4}^{6}(1-a_ka_1)}{(1-a_1a_4/a_2)(1-a_1a_4/a_3)(1-a_1/a_2)(1-a_1/a_3)}\end{aligned} \tag{7.25}$$

$$\times\, e(q^{\frac12}a_i, q^{\frac12}a_1)e(q^{\frac12}a_i, q^{\frac12}a_1a_4).$$

Proof. We set $\widetilde{\varphi}_i(z) = \nabla_{\rm sym}\varphi_i(z)/4$. From expansion of $\varphi_i(z)$, $\widetilde{\varphi}_i(z)$ can be expressed as a linear combination of $\alpha_i(z)$, $\beta_i(z)$, $\gamma_i(z)$ $(i = 0, 1, 2, 3)$ in Lemma 7.9. This implies $\widetilde{\varphi}_i(z) \in \mathcal{F}_5$. Under the condition $a_1a_2\cdots a_6q = -1$, from (7.22) and (7.23) we have $\widetilde{\varphi}_i(z) \in \mathcal{F}_4$. By Lemma 5.6, using basis $\{g_{ij}(z)\,|\,1 \le i < j \le 4\}\cup\{G_1(z), G_2(z), G_3(z)\}$ of $\mathcal{F}_4$, $\widetilde{\varphi}_i(z)$ can be expanded as

$$\widetilde{\varphi}_i(z) = \sum_{1\le j<k\le 4} c^{(i)}_{jk} g_{jk}(z) + \sum_{k=1}^{3} c^{(i)}_k G_k(z). \tag{7.26}$$

We first show $c^{(i)}_{j4} = 0$ $(j = 1, 2, 3)$ and evaluate $c^{(i)}_{jk}$ $(1 \le j < k \le 3)$. From the symmetry of parameters, without loss of generality, it suffices to show $c^{(i)}_{14} = 0$ and the explicit form of $c^{(i)}_{12}$.

We consider (7.26) with $z = \mathrm{p}_{23}$. From (5.29) we see $c^{(i)}_{14}g_{14}(\mathrm{p}_{23}) = \widetilde{\varphi}_i(\mathrm{p}_{23})$. By Lemma 6.7 and $F_2(\mathrm{p}_{23}) = F_3(\mathrm{p}_{23}) = F_4(\mathrm{p}_{23}) = F_5(\mathrm{p}_{23}) = 0$ and $(\varphi_i)_0(\mathrm{p}_{23}) = (\varphi_i)_1(\mathrm{p}_{23}) = 0$, we have $\widetilde{\varphi}_i(\mathrm{p}_{23}) = 0$. Thus we obtain

$$c^{(i)}_{14} = \frac{\widetilde{\varphi}_i(\mathrm{p}_{23})}{g_{14}(\mathrm{p}_{23})} = 0.$$

We next consider (7.26) with $z = \mathrm{p}_{34}$. From (5.29) we have $c^{(i)}_{12}g_{12}(\mathrm{p}_{34}) = \widetilde{\varphi}_i(\mathrm{p}_{34})$. On the other hand By Lemma 6.7 and $F_2(\mathrm{p}_{34}) = F_3(\mathrm{p}_{34}) = F_4(\mathrm{p}_{34}) = F_5(\mathrm{p}_{34}) = 0$ and $(\varphi_i)_1(\mathrm{p}_{34}) = 0$, we obtain $\widetilde{\varphi}_i(\mathrm{p}_{34}) = F_0(\mathrm{p}_{34})(\varphi_i)_0(\mathrm{p}_{34}) = F_+(\mathrm{p}_{34})\varphi_i(q^{\frac12}a_3/a_4, a_4)$. Thus we can compute

$$\begin{aligned}
c^{(i)}_{12} &= \frac{F_+(\mathrm{p}_{34})\varphi_i(q^{\frac12}a_3/a_4, a_4)}{g_{12}(\mathrm{p}_{34})} \\
&= (a_3^2a_4)^{-\frac32}\frac{\prod_{k=1}^6(1-a_ka_3)(1-a_ka_3a_4)}{(1-a_3)(1-a_3a_4)(1-a_3/a_4)(1-a_3a_4^2)(1-a_3^2a_4)} \\
&\quad\times\frac{a_4^{-\frac12}(1+a_4)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4)e(c_i,q^{\frac12}a_3)e(c_i,q^{\frac12}a_3a_4)}{e(a_1,a_3)e(a_1,a_4)e(a_1,a_3a_4)e(a_2,a_3)e(a_2,a_4)e(a_2,a_3a_4)} \\
&= (a_3^2a_4)^{-\frac32}\frac{\prod_{k=1}^6(1-a_ka_3)(1-a_ka_3a_4)}{(1-a_3)a_3e(a_3,a_4)(1-a_3a_4^2)(1-a_3^2a_4)} \\
&\quad\times\frac{a_4^{-\frac12}(1+a_4)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4)e(c_i,q^{\frac12}a_3)e(c_i,q^{\frac12}a_3a_4)}{e(a_1,a_3)e(a_1,a_4)e(a_1,a_3a_4)e(a_2,a_3)e(a_2,a_4)e(a_2,a_3a_4)} \\
&= a_3^{-4}a_4^{-2}\frac{(1+a_3)(1+a_4)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^6(1-a_ka_3)}{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-a_3/a_1)(1-a_3/a_2)a_3^{-4}a_4^{-2}} \\
&\quad\times e(c_i,q^{\frac12}a_3)e(c_i,q^{\frac12}a_3a_4) \\
&= \frac{(1+a_3)(1+a_4)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^6(1-a_ka_3)}{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-a_3/a_1)(1-a_3/a_2)} \\
&\quad\times e(q^{\frac12}a_i,q^{\frac12}a_3)e(q^{\frac12}a_i,q^{\frac12}a_3a_4),
\end{aligned} \tag{7.27}$$

which coincides with (7.25).

We next evaluate $c_k^{(i)}$ $(k = 1,2,3)$. From the symmetry of parameters, without loss of generality, it suffices to compute $c_1^{(i)}$. We consider (7.26) with $z = \mathrm{p}^*_{23}$. From (5.29) we have

$$\widetilde{\varphi}_i(\mathrm{p}^*_{23}) = c_1^{(i)} G_1(\mathrm{p}^*_{23}) + c_{14}^{(i)} g_{14}(\mathrm{p}^*_{23}) = c_1^{(i)} G_1(\mathrm{p}^*_{23}). \tag{7.28}$$

(We have already shown $c_{14}^{(i)} = 0$.) On the other hand, by Lemma 6.7 and $F_2(\mathrm{p}^*_{23}) = F_3(\mathrm{p}^*_{23}) = F_4(\mathrm{p}^*_{23}) = 0$ and $(\varphi_i)_1(\mathrm{p}^*_{23}) = (\varphi_i)_5(\mathrm{p}^*_{23}) = 0$, we have

$$\widetilde{\varphi}_i(\mathrm{p}^*_{23}) = F_0(\mathrm{p}^*_{23})(\varphi_i)_0(\mathrm{p}^*_{23}) = F_+(\mathrm{p}^*_{23})\varphi_i(q^{\frac12}a_2^2/a_3, a_3/a_2). \tag{7.29}$$

Comparing (7.28) with (7.29), $c_1^{(i)}$ is computed as

$$\begin{aligned}
c_1^{(i)} &= \frac{F_+(\mathrm{p}^*_{23})\varphi_i(q^{\frac12}a_2^2/a_3, a_3/a_2)}{G_1(\mathrm{p}^*_{23})} \\
&= (a_2a_3)^{-\frac32}\frac{\prod_{k=1}^6(1-a_ka_2)(1-a_ka_3)}{(1-a_2)(1-a_3)(1-a_2^2/a_3)(1-a_3^2/a_2)(1-a_2a_3)} \\
&\quad \times (a_3/a_2)^{-\frac12}(1+a_3/a_2)e(a_1,a_3/a_2)e(a_2,a_3/a_2)e(a_3,a_3/a_2) \\
&\quad \times e(c_i,q^{\frac12}a_2)e(c_i,q^{\frac12}a_3) \\
&\quad \times \frac{a_2a_3\, e(a_2a_3,a_4)}{(1-a_2^2)(1-a_3^2)e(a_1,a_3/a_2)e(a_1,a_2)e(a_1,a_3)} \\
&= (a_2a_3)^{-\frac32}\frac{\prod_{k=1}^6(1-a_ka_2)(1-a_ka_3)}{a_2a_3e(a_2,a_3/a_2)e(a_3,a_3/a_2)(1-a_2a_3)} \\
&\quad \times (a_3/a_2)^{-\frac12}(1+a_3/a_2)e(a_2,a_3/a_2)e(a_3,a_3/a_2) \\
&\quad \times e(c_i,q^{\frac12}a_2)e(c_i,q^{\frac12}a_3) \\
&\quad \times \frac{a_2a_3\, e(a_2a_3,a_4)}{(1-a_2^2)(1-a_3^2)e(a_1,a_2)e(a_1,a_3)} \\
&= a_2^{-1}a_3^{-2}(1-a_1a_2)(1-a_1a_3)(1-a_2a_3)\prod_{k=4}^6(1-a_ka_2)(1-a_ka_3) \\
&\quad \times (1+a_3/a_2)e(q^{\frac12}a_i,q^{\frac12}a_2)e(q^{\frac12}a_i,q^{\frac12}a_3)\frac{e(a_2a_3,a_4)}{e(a_1,a_2)e(a_1,a_3)}.
\end{aligned} \tag{7.30}$$

Therefore, we have $c_1^{(2)} = c_1^{(3)} = 0$ and $c_1^{(1)}$ is evaluated as

$$\begin{aligned}
c_1^{(1)} &= a_2^{-1}a_3^{-2}(1-a_1a_2)(1-a_1a_3)(1-a_2a_3)\prod_{k=4}^6(1-a_ka_2)(1-a_ka_3) \\
&\quad \times \frac{(1+a_3/a_2)q^{-1}a_2^{-1}a_3^{-1}(1-qa_1a_2)(1-a_2/a_1)(1-qa_1a_3)(1-a_3/a_1)e(a_2a_3,a_4)}{a_2^{-1}a_3^{-1}(1-a_1a_2)(1-a_2/a_1)(1-a_1a_3)(1-a_3/a_1)} \\
&= q^{-1}a_2^{-1}a_3^{-2}e(a_2a_3,a_4)(1+a_3/a_2)(1-a_2a_3)(1-qa_1a_2)(1-qa_1a_3) \\
&\quad \times \prod_{k=4}^6(1-a_ka_2)(1-a_ka_3),
\end{aligned}$$

which coincides with (7.24). □

Remark. Applying Lemma 6.6 to Lemma 7.10, we have

$$\begin{pmatrix} c_1^{(1)} & 0 & 0 \\ 0 & c_2^{(2)} & 0 \\ 0 & 0 & c_3^{(3)} \end{pmatrix} \begin{pmatrix} \langle G_1(z)\rangle \\ \langle G_2(z)\rangle \\ \langle G_3(z)\rangle \end{pmatrix} = - \begin{pmatrix} 0 & c_{13}^{(1)} & c_{12}^{(1)} \\ c_{23}^{(2)} & 0 & c_{12}^{(2)} \\ c_{23}^{(3)} & c_{13}^{(3)} & 0 \end{pmatrix} \begin{pmatrix} \langle g_{23}(z)\rangle \\ \langle g_{13}(z)\rangle \\ \langle g_{12}(z)\rangle \end{pmatrix}.$$

In other words, $\langle G_i(z)\rangle$ can be expressed as a linear combination of $\langle g_{12}(z)\rangle$, $\langle g_{13}(z)\rangle$ and $\langle g_{23}(z)\rangle$, i.e.,

$$\begin{pmatrix} \langle G_1(z)\rangle \\ \langle G_2(z)\rangle \\ \langle G_3(z)\rangle \end{pmatrix} = - \begin{pmatrix} 0 & c_{13}^{(1)}/c_1^{(1)} & c_{12}^{(1)}/c_1^{(1)} \\ c_{23}^{(2)}/c_2^{(2)} & 0 & c_{12}^{(2)}/c_2^{(2)} \\ c_{23}^{(3)}/c_3^{(3)} & c_{13}^{(3)}/c_3^{(3)} & 0 \end{pmatrix} \begin{pmatrix} \langle g_{23}(z)\rangle \\ \langle g_{13}(z)\rangle \\ \langle g_{12}(z)\rangle \end{pmatrix}. \tag{7.31}$$

Lemma 7.11. *Let $h_i(z)$ be function specified by*

$$h_i(z) := \frac{1}{4}\nabla_{\mathrm{sym}}\Big((z_2+z_2^{-1})^i((z_1z_2)^{-\frac12}+(z_1z_2)^{\frac12})((z_1z_2^2)^{-\frac12}+(z_1z_2^2)^{\frac12})\Big).$$

If $i=0,1,2,3$, then $h_i(z)\in\mathcal{F}_4$. The explicit expressions of $h_i(z)$ $(i=0,1,2,3)$ are given as

$$\begin{aligned}
h_0(z) &= q^{-\frac12}(S_0+q^{\frac12}S_6)(S_0-q^{\frac12}S_6)s_{2\varpi_1}(z)+\cdots\in\mathcal{F}_4,\\
h_1(z) &= q^{-\frac12}(S_0+q^{\frac12}S_6)(S_0-q^{\frac12}S_6)\Big[s_{\varpi_1+2\varpi_2}(z)-s_{2\varpi_1}(z)\Big]+\cdots\in\mathcal{F}_4,\\
h_2(z) &= q^{-\frac12}(S_0+q^{\frac12}S_6)(S_0-q^{\frac12}S_6)\Big[s_{4\varpi_2}(z)-s_{\varpi_1+2\varpi_2}(z)+2s_{2\varpi_1}(z)\Big]+\cdots\in\mathcal{F}_4,\\
h_3(z) &= q^{-\frac12}(S_0+q^{\frac12}S_6)\Big[\Big(q^{\frac12}(S_6-S_5-S_0)+(S_6+S_1-S_0)\Big)s_{4\varpi_2}(z)\\
&\qquad -\Big(q^{\frac12}(3S_6-S_4-S_1)+(S_5+S_2-3S_0)\Big)s_{\varpi_1+2\varpi_2}(z)\\
&\qquad +\Big(q^{\frac12}(3S_6-S_3-S_2)+(S_4+S_3-3S_0)\Big)s_{2\varpi_1}(z)\Big]+\cdots\in\mathcal{F}_4,
\end{aligned}$$

where S_r $(r=0,1,\ldots,6)$ is the rth elementary symmetric polynomial of $a_1,\ldots,a_6$ given by (7.13).

Proof. The proof follows by direct computation and we omit the details. □

Lemma 7.12. *Let $\phi(z)$ be function specified by*

$$\begin{aligned}
\phi(z) &= (z_1^2z_2^3)^{-\frac12}(1+z_1z_2)(1+z_1z_2^2)e(a_1,z_2)e(a_2,z_2)e(a_3,z_2)\\
&= (z_1^2z_2^3)^{-\frac12}(1+z_1z_2)(1+z_1z_2^2)\\
&\quad\times(1-a_1z_2)(1-a_1z_2^{-1})(1-a_2z_2)\\
&\quad\times(1-a_2z_2^{-1})(1-a_3z_2)(1-a_3z_2^{-1})a_1^{-1}a_2^{-1}a_3^{-1},
\end{aligned}$$

and $g_{ij}(z)$, $G_i(z)$ be functions defined by (5.26), (5.28), *respectively. Then $\nabla_{\mathrm{sym}}\phi(z)/4$ is expanded as*

$$\frac{\nabla_{\mathrm{sym}}\phi(z)}{4} = d_{23}g_{23}(z)+d_{13}g_{13}(z)+d_{12}g_{12}(z)+d_1G_1(z)+d_2G_2(z)+d_3G_3(z), \tag{7.32}$$

where the coefficients are given as

$$d_{23} = q^{-\frac{1}{2}} \frac{(1+q^{\frac{1}{2}}a_1)(1+q^{\frac{1}{2}}a_1a_4)(1+a_1)(1-a_1a_4a_5)(1-a_1a_4a_6)}{a_1(1-a_1/a_2)(1-a_1/a_3)(1-a_1a_4/a_2)(1-a_1a_4/a_3)} \times \prod_{k=4}^{6}(1-a_ka_1),$$

$$d_{13} = q^{-\frac{1}{2}} \frac{(1+q^{\frac{1}{2}}a_2)(1+q^{\frac{1}{2}}a_2a_4)(1+a_2)(1-a_2a_4a_5)(1-a_2a_4a_6)}{a_2(1-a_2/a_1)(1-a_2/a_3)(1-a_2a_4/a_1)(1-a_2a_4/a_3)} \times \prod_{k=4}^{6}(1-a_ka_2),$$

$$d_{12} = q^{-\frac{1}{2}} \frac{(1+q^{\frac{1}{2}}a_3)(1+q^{\frac{1}{2}}a_3a_4)(1+a_3)(1-a_3a_4a_5)(1-a_3a_4a_6)}{a_3(1-a_3/a_1)(1-a_3/a_2)(1-a_3a_4/a_1)(1-a_3a_4/a_2)} \times \prod_{k=4}^{6}(1-a_ka_3). \tag{7.33}$$

$$d_1 = q^{-\frac{1}{2}} \frac{(1+q^{\frac{1}{2}}a_2)(1+q^{\frac{1}{2}}a_3)(1-a_2a_3)(1-a_2a_3a_4)(1-a_2a_3/a_4)}{a_2^2a_3^2(1-a_2/a_1)(1-a_3/a_1)} \times \prod_{k=4}^{6}(1-a_ka_2)(1-a_ka_3), \tag{7.34}$$

$$d_2 = q^{-\frac{1}{2}} \frac{(1+q^{\frac{1}{2}}a_1)(1+q^{\frac{1}{2}}a_3)(1-a_1a_3)(1-a_1a_3a_4)(1-a_1a_3/a_4)}{a_1^2a_3^2(1-a_1/a_2)(1-a_3/a_2)} \times \prod_{k=4}^{6}(1-a_ka_1)(1-a_ka_3),$$

$$d_3 = q^{-\frac{1}{2}} \frac{(1+q^{\frac{1}{2}}a_1)(1+q^{\frac{1}{2}}a_2)(1-a_1a_2)(1-a_1a_2a_4)(1-a_1a_2/a_4)}{a_1^2a_2^2(1-a_1/a_3)(1-a_2/a_3)} \times \prod_{k=4}^{6}(1-a_ka_1)(1-a_ka_2).$$

Remark. This lemma does not need the balancing condition like $a_1 \cdots a_6 = -1$ in Lemma 7.7 or $a_1 \cdots a_6 q = -1$ in Lemma 7.10.

Proof. We set $\widetilde{\phi}(z) = \nabla_{\rm sym}\phi(z)/4$. From expansion of $\phi(z)$ $\widetilde{\phi}(z)$ can be expressed as a linear combination of $h_i(z)$ $(i = 0,1,2,3)$ in Lemma 7.11. This implies $\widetilde{\phi}(z) \in \mathcal{F}_4$. By Lemma 5.6, using basis $\{g_{ij}(z) \,|\, 1 \le i < j \le 4\} \cup \{G_1(z), G_2(z), G_3(z)\}$ of $\mathcal{F}_4$, $\widetilde{\varphi}_i(z)$ can be expanded as

$$\widetilde{\phi}_i(z) = \sum_{1\le j<k\le 4} d_{jk}g_{jk}(z) + \sum_{k=1}^{3} d_kG_k(z) \tag{7.35}$$

We first show that $d_{j4} = 0$ $(j = 1,2,3)$ and that d_{jk} $(1 \le j < k \le 3)$ is given as (7.33). From symmetry of parameters, without loss of generality, it suffices to show $d_{14} = 0$ and the explicit form of d_{12}.

We consider (7.35) with $z=\mathrm{p}_{23}$. From (5.29) we see that $d_{14}g_{14}(\mathrm{p}_{23})=\widetilde{\phi}(\mathrm{p}_{23})$. By Lemma 6.7 and $F_2(\mathrm{p}_{23})=F_3(\mathrm{p}_{23})=F_4(\mathrm{p}_{23})=F_5(\mathrm{p}_{23})=0$ and $\phi_0(\mathrm{p}_{23})=\phi_1(\mathrm{p}_{23})=0$, we have $\widetilde{\phi}(\mathrm{p}_{23})=0$. Thus we obtain $d_{14}=\widetilde{\phi}(\mathrm{p}_{23})/g_{14}(\mathrm{p}_{23})=0$.

We next consider (7.35) with $z=\mathrm{p}_{34}$. From (5.29), we have $d_{12}g_{12}(\mathrm{p}_{34})=\widetilde{\phi}(\mathrm{p}_{34})$. On the other hand, By Lemma 6.7 and $F_2(\mathrm{p}_{34})=F_3(\mathrm{p}_{34})=F_4(\mathrm{p}_{34})=F_5(\mathrm{p}_{34})=0$ and $\phi_1(\mathrm{p}_{34})=0$, we have $\widetilde{\phi}(\mathrm{p}_{34})=F_0(\mathrm{p}_{34})\phi_0(\mathrm{p}_{34})=F_+(\mathrm{p}_{34})\phi(q^{\frac{1}{2}}a_3/a_4,a_4)$. Thus we can compute

$$d_{12}=\frac{F_+(\mathrm{p}_{34})\phi(q^{\frac{1}{2}}a_3/a_4,a_4)}{g_{12}(\mathrm{p}_{34})}, \tag{7.36}$$

which coincides with (7.33).

We next evaluate d_k $(k=1,2,3)$. From symmetry of parameters, without loss of generality, it suffices to compute d_1. We consider (7.35) with $z=\mathrm{p}^*_{23}$. From (5.29) we have

$$\widetilde{\phi}(\mathrm{p}^*_{23})=d_1G_1(\mathrm{p}^*_{23})+d_{14}g_{14}(\mathrm{p}^*_{23})=d_1G_1(\mathrm{p}^*_{23}). \tag{7.37}$$

(We have already shown $d_{14}=0$.) On the other hand, by Lemma 6.7 and $F_2(\mathrm{p}^*_{23})=F_3(\mathrm{p}^*_{23})=F_4(\mathrm{p}^*_{23})=0$ and $\phi_1(\mathrm{p}^*_{23})=\phi_5(\mathrm{p}^*_{23})=0$, we have

$$\widetilde{\phi}(\mathrm{p}^*_{23})=F_0(\mathrm{p}^*_{23})\phi_0(\mathrm{p}^*_{23})=F_+(\mathrm{p}^*_{23})\phi(q^{\frac{1}{2}}a_2^2/a_3,a_3/a_2). \tag{7.38}$$

Comparing (7.37) with (7.38), d_1 is computed as

$$d_1=\frac{F_+(\mathrm{p}^*_{23})\phi(q^{\frac{1}{2}}a_2^2/a_3,a_3/a_2)}{G_1(\mathrm{p}^*_{23})}, \tag{7.39}$$

which coincides with (7.34). □

Proof of Proposition 7.8. Applying Lemma 6.6 to Lemma 7.12, we have

$$(d_{23},d_{13},d_{12})\begin{pmatrix}\langle g_{23}(z)\rangle\\ \langle g_{13}(z)\rangle\\ \langle g_{12}(z)\rangle\end{pmatrix}+(d_1,d_2,d_3)\begin{pmatrix}\langle G_1(z)\rangle\\ \langle G_2(z)\rangle\\ \langle G_3(z)\rangle\end{pmatrix}=0.$$

Combining this with (7.31), we can eliminate $\langle G_i(z)\rangle$ $(i=1,2,3)$, namely

$$\left[(d_{23},d_{13},d_{12})-(d_1,d_2,d_3)\begin{pmatrix}0 & c^{(1)}_{13}/c^{(1)}_1 & c^{(1)}_{12}/c^{(1)}_1\\ c^{(2)}_{23}/c^{(2)}_2 & 0 & c^{(2)}_{12}/c^{(2)}_2\\ c^{(3)}_{23}/c^{(3)}_3 & c^{(3)}_{13}/c^{(3)}_3 & 0\end{pmatrix}\right]\begin{pmatrix}\langle g_{23}(z)\rangle\\ \langle g_{13}(z)\rangle\\ \langle g_{12}(z)\rangle\end{pmatrix}=0,$$

so that

$$\left(d_{23}-\frac{d_2c^{(2)}_{23}}{c^{(2)}_2}-\frac{d_3c^{(3)}_{23}}{c^{(3)}_3},\ d_{13}-\frac{d_1c^{(1)}_{13}}{c^{(1)}_1}-\frac{d_3c^{(3)}_{13}}{c^{(3)}_3},\ d_{12}-\frac{d_1c^{(1)}_{12}}{c^{(1)}_1}-\frac{d_2c^{(2)}_{12}}{c^{(2)}_2}\right)\begin{pmatrix}\langle g_{23}(z)\rangle\\ \langle g_{13}(z)\rangle\\ \langle g_{12}(z)\rangle\end{pmatrix}=0.$$

Therefore, putting

$$C'_{ij}=d_{ij}-c^{(i)}_{ij}\frac{d_i}{c^{(i)}_i}-c^{(j)}_{ij}\frac{d_j}{c^{(j)}_j}\quad(1\le i<j\le 3), \tag{7.40}$$

we obtain

$$C'_{23}\langle g_{23}(z)\rangle + C'_{13}\langle g_{13}(z)\rangle + C'_{12}\langle g_{12}(z)\rangle = 0. \tag{7.41}$$

From direct calculation of (7.40), we have

$$C'_{23} = \frac{(1-a_1^2)(1-qa_1^2)\prod_{k=4}^{6}(1-a_ka_1)\prod_{4\le i<j\le 6}(1-a_1a_ia_j)}{q^{\frac12}a_1^2a_4a_5a_6(1-a_1)(1-a_1^2/a_2^2)(1-a_1^2/a_3^2)(1-qa_2a_3)}, \tag{7.42}$$

$$C'_{13} = \frac{(1-a_2^2)(1-qa_2^2)\prod_{k=4}^{6}(1-a_ka_2)\prod_{4\le i<j\le 6}(1-a_2a_ia_j)}{q^{\frac12}a_2^2a_4a_5a_6(1-a_2)(1-a_2^2/a_1^2)(1-a_2^2/a_3^2)(1-qa_1a_3)}, \tag{7.43}$$

$$C'_{12} = \frac{(1-a_3^2)(1-qa_3^2)\prod_{k=4}^{6}(1-a_ka_3)\prod_{4\le i<j\le 6}(1-a_3a_ia_j)}{q^{\frac12}a_3^2a_4a_5a_6(1-a_3)(1-a_3^2/a_1^2)(1-a_3^2/a_2^2)(1-qa_1a_2)}, \tag{7.44}$$

which will be confirmed just below. If we put

$$C_{ij} = C'_{ij}(q^{\frac12}a_4a_5a_6)(1-qa_1a_2)(1-qa_1a_3)(1-qa_2a_3),$$

then, from (7.41) we can obtain our conclusion (7.21).

Lastly, we confirm the explicit forms of C'_{ij} in (7.42)–(7.44) Without loss of generality, we confirm (7.44) of C'_{12}. Since $c_{12}^{(i)}$, $c_1^{(i)}$, d_{12}, d_1 are given as (7.27), (7.30), (7.36), (7.39), respectively, using

$$\frac{d_1}{c_1^{(1)}} = \frac{\phi(q^{\frac12}a_2^2/a_3, a_3/a_2)}{\varphi_1(q^{\frac12}a_2^2/a_3, a_3/a_2)}, \quad \frac{d_2}{c_2^{(2)}} = \frac{\phi(q^{\frac12}a_1^2/a_3, a_3/a_1)}{\varphi_2(q^{\frac12}a_1^2/a_3, a_3/a_1)}$$

(7.40) of C'_{12} is written as

$$C'_{12} = d_{12} - c_{12}^{(1)}\frac{d_1}{c_1^{(1)}} - c_{12}^{(2)}\frac{d_2}{c_2^{(2)}} \tag{7.45}$$

$$= \frac{F_+(\mathrm{p}_{34})}{g_{12}(\mathrm{p}_{34})}\Bigg[\phi(q^{\frac12}a_3/a_4, a_4) \tag{7.46}$$

$$- \varphi_1(q^{\frac12}a_3/a_4, a_4)\frac{\phi(q^{\frac12}a_2^2/a_3, a_3/a_2)}{\varphi_1(q^{\frac12}a_2^2/a_3, a_3/a_2)} \tag{7.47}$$

$$- \varphi_2(q^{\frac12}a_3/a_4, a_4)\frac{\phi(q^{\frac12}a_1^2/a_3, a_3/a_1)}{\varphi_2(q^{\frac12}a_1^2/a_3, a_3/a_1)}\Bigg]. \tag{7.48}$$

We first compute the first term of C'_{12} corresponding to (7.46) as

$$\begin{aligned}
&\phi(q^{\frac12}a_3/a_4, a_4)\\
&= (qa_3^2a_4)^{-\frac12}(1+q^{\frac12}a_3)(1+q^{\frac12}a_3a_4)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4)\\
&= q^{-\frac12}a_3^{-1}(1+q^{\frac12}a_3)(1+q^{\frac12}a_3a_4)\,a_4^{-\frac12}e(a_1,a_4)e(a_2,a_4)e(a_3,a_4).
\end{aligned} \tag{7.49}$$

We next compute the second term of C'_{12} corresponding to (7.47). Using

$$\begin{aligned}
\frac{\phi(z)}{\varphi_i(z)} &= \frac{(z_1^2z_2^3)^{-\frac12}(1+z_1z_2)(1+z_1z_2^2)e(a_1,z_2)e(a_2,z_2)e(a_3,z_2)}{z_2^{-\frac12}(1+z_2)e(a_1,z_2)e(a_2,z_2)e(a_3,z_2)e(q^{\frac12}a_i,z_1z_2)e(q^{\frac12}a_i,z_1z_2^2)}\\
&= \frac{z_1^{-1}z_2^{-1}(1+z_1z_2)(1+z_1z_2^2)}{(1+z_2)e(q^{\frac12}a_i,z_1z_2)e(q^{\frac12}a_i,z_1z_2^2)},
\end{aligned}$$

we can compute the second term as

$$
\begin{aligned}
& -\varphi_1(q^{\frac12}a_3/a_4,a_4)\frac{\phi(q^{\frac12}a_2^2/a_3,a_3/a_2)}{\varphi_1(q^{\frac12}a_2^2/a_3,a_3/a_2)} \\
&\qquad = -\frac{q^{-\frac12}a_2^{-1}(1+q^{\frac12}a_2)(1+q^{\frac12}a_3)}{(1+a_3/a_2)e(q^{\frac12}a_1,q^{\frac12}a_2)e(q^{\frac12}a_1,q^{\frac12}a_3)} \\
&\qquad\qquad \times e(q^{\frac12}a_1,q^{\frac12}a_3)e(q^{\frac12}a_1,q^{\frac12}a_3a_4) \\
&\qquad\qquad \times a_4^{-\frac12}(1+a_4)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4) \\
&\qquad = -\frac{(1+q^{\frac12}a_2)(1+q^{\frac12}a_3)e(q^{\frac12}a_1,q^{\frac12}a_3a_4)}{q^{\frac12}a_2(1+a_3/a_2)e(q^{\frac12}a_1,q^{\frac12}a_2)} \\
&\qquad\qquad \times a_4^{-\frac12}(1+a_4)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4).
\end{aligned}
\tag{7.50}
$$

In the same manner, we obtain the third term of C'_{12} corresponding to (7.48) as

$$
\begin{aligned}
& -\varphi_2(q^{\frac12}a_3/a_4,a_4)\frac{\phi(q^{\frac12}a_1^2/a_3,a_3/a_1)}{\varphi_2(q^{\frac12}a_1^2/a_3,a_3/a_1)} \\
&\qquad = -\frac{(1+q^{\frac12}a_1)(1+q^{\frac12}a_3)e(q^{\frac12}a_2,q^{\frac12}a_3a_4)}{q^{\frac12}a_1(1+a_3/a_1)e(q^{\frac12}a_2,q^{\frac12}a_1)} \\
&\qquad\qquad \times a_4^{-\frac12}(1+a_4)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4).
\end{aligned}
\tag{7.51}
$$

Combining (7.45) with (7.49)–(7.51), we can compute C'_{12} as follows.

$$
\begin{aligned}
C'_{12} &= \frac{F_+(\mathrm{p}_{34})}{g_{12}(\mathrm{p}_{34})}q^{-\frac12}a_4^{-\frac12}e(a_1,a_4)e(a_2,a_4)e(a_3,a_4) \\
&\quad \times \Bigg[a_3^{-1}(1+q^{\frac12}a_3)(1+q^{\frac12}a_3a_4) \\
&\qquad - \frac{(1+q^{\frac12}a_2)(1+q^{\frac12}a_3)(1+a_4)e(q^{\frac12}a_1,q^{\frac12}a_3a_4)}{a_2(1+a_3/a_2)e(q^{\frac12}a_1,q^{\frac12}a_2)} \\
&\qquad - \frac{(1+q^{\frac12}a_1)(1+q^{\frac12}a_3)(1+a_4)e(q^{\frac12}a_2,q^{\frac12}a_3a_4)}{a_1(1+a_3/a_1)e(q^{\frac12}a_2,q^{\frac12}a_1)}\Bigg] \\
&= \frac{F_+(\mathrm{p}_{34})}{g_{12}(\mathrm{p}_{34})}q^{-\frac12}a_4^{-\frac12}\frac{(1+q^{\frac12}a_3)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4)}{a_1a_2a_3(1+a_3/a_1)(1+a_3/a_2)e(q^{\frac12}a_1,q^{\frac12}a_2)} \\
&\quad \times \Bigg[a_1a_2(1+q^{\frac12}a_3a_4)(1+a_3/a_1)(1+a_3/a_2)e(q^{\frac12}a_1,q^{\frac12}a_2) \\
&\qquad - a_1a_3(1+a_4)(1+a_3/a_1)(1+q^{\frac12}a_2)e(q^{\frac12}a_1,q^{\frac12}a_3a_4) \\
&\qquad + a_2a_3(1+a_4)(1+a_3/a_2)(1+q^{\frac12}a_1)e(q^{\frac12}a_2,q^{\frac12}a_3a_4)\Bigg]
\end{aligned}
\tag{7.52}
$$

[[computation here, see the identity (A.1) of 4 terms in §A]]

$$
= \frac{F_+(\mathrm{p}_{34})}{g_{12}(\mathrm{p}_{34})}q^{-\frac12}a_4^{-\frac12}\frac{(1+q^{\frac12}a_3)e(a_1,a_4)e(a_2,a_4)e(a_3,a_4)}{a_1a_2a_3(1+a_3/a_1)(1+a_3/a_2)e(q^{\frac12}a_1,q^{\frac12}a_2)}
$$

$$
\begin{aligned}
&\quad \times (-1) q^{-\frac{1}{2}} a_3 (1-a_2/a_1)(1-a_3a_4/a_2)(1-a_1/a_3a_4)(1-q^{\frac{1}{2}}a_3)(1+qa_1a_2a_4) \\
&= \frac{F_+(\mathrm{p}_{34})}{g_{12}(\mathrm{p}_{34})} a_4^{-\frac{1}{2}} e(a_1,a_4)e(a_2,a_4)e(a_3,a_4) \\
&\quad \times (-1) \frac{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-qa_3^2)(1+qa_1a_2a_4)}{q^{\frac{1}{2}} a_3a_4(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
&= (a_3^2a_4)^{-\frac{3}{2}} \frac{\prod_{k=1}^6 (1-a_ka_3)(1-a_ka_3a_4)}{(1-a_3)(1-a_3a_4)(1-a_3/a_4)(1-a_3a_4^2)(1-a_3^2a_4)} \\
&\quad \times \frac{a_4^{-\frac{1}{2}} e(a_1,a_4)e(a_2,a_4)e(a_3,a_4)}{e(a_1,a_3)e(a_1,a_4)e(a_1,a_3a_4)e(a_2,a_3)e(a_2,a_4)e(a_2,a_3a_4)} \\
&\quad \times (-1) \frac{(1-a_3a_4/a_1)(1-a_3a_4/a_2)(1-qa_3^2)(1+qa_1a_2a_4)}{q^{\frac{1}{2}} a_3a_4(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
&= (a_3^2a_4)^{-\frac{3}{2}} \frac{(1-a_1a_3)(1-a_2a_3)(1-a_3^2)(1-a_3a_4a_5)(1-a_3a_4a_6)}{(1-a_3)a_3e(a_3,a_4)} \\
&\quad \times \frac{a_4^{-\frac{1}{2}} e(a_1,a_4)e(a_2,a_4)e(a_3,a_4)\prod_{k=4}^6(1-a_ka_3)}{e(a_1,a_3)e(a_1,a_4)e(a_1,a_3a_4)e(a_2,a_3)e(a_2,a_4)e(a_2,a_3a_4)} \\
&\quad \times (-1) \frac{a_3^2a_4^2 e(a_3a_4,a_1)e(a_3a_4,a_2)(1-qa_3^2)(1+qa_1a_2a_4)}{q^{\frac{1}{2}} a_3a_4(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
&= \frac{(1-a_1a_3)(1-a_2a_3)(1-a_3^2)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^6(1-a_ka_3)}{(1-a_3)} \\
&\quad \times \frac{a_3^{-3}a_4^{-1}}{e(a_1,a_3)e(a_2,a_3)} \times (-1) \frac{(1-qa_3^2)(1-a_3^{-1}a_5^{-1}a_6^{-1})}{q^{\frac{1}{2}}(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)}
\end{aligned}
$$

[[used the balancing condition only here for $1+qa_1a_2a_4 = 1-a_3^{-1}a_5^{-1}a_6^{-1}$]]

$$
\begin{aligned}
&= \frac{(1-a_1a_3)(1-a_2a_3)(1-a_3^2)(1-a_3a_4a_5)(1-a_3a_4a_6)\prod_{k=4}^6(1-a_ka_3)}{(1-a_3)} \\
&\quad \times \frac{a_3^{-4}a_4^{-1}a_5^{-1}a_6^{-1}}{e(a_1,a_3)e(a_2,a_3)} \times \frac{(1-qa_3^2)(1-a_3a_5a_6)}{q^{\frac{1}{2}}(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
&= (1-a_1a_3)(1-a_2a_3)\prod_{k=4}^6(1-a_ka_3)\prod_{4\le i<j\le 6}(1-a_3a_ia_j) \\
&\quad \times \frac{a_3^{-4}a_4^{-1}a_5^{-1}a_6^{-1}}{a_3^{-2}(1-a_1a_3)(1-a_3/a_1)(1-a_2a_3)(1-a_3/a_2)} \\
&\quad \times \frac{(1-a_3^2)(1-qa_3^2)}{q^{\frac{1}{2}}(1-a_3)(1+a_3/a_1)(1+a_3/a_2)(1-qa_1a_2)} \\
&= \frac{(1-a_3^2)(1-qa_3^2)\prod_{k=4}^6(1-a_ka_3)\prod_{4\le i<j\le 6}(1-a_3a_ia_j)}{q^{\frac{1}{2}}a_3^2a_4a_5a_6(1-a_3)(1-a_3^2/a_1^2)(1-a_3^2/a_2^2)(1-qa_1a_2)},
\end{aligned}
$$

which coincide with (7.44). □

Appendix A. Four-term identity

In this appendix we show an identity of four terms, which is used in (7.52).

Lemma A.1. *The following identity holds for x_1, x_2, x_3, a:*

$$\begin{aligned} & x_1(1-x_1)(a-x_2)(a-x_3)(1-x_2x_3)(x_3-x_2) \\ & \quad + x_2(1-x_2)(a-x_1)(a-x_3)(1-x_1x_3)(x_1-x_3) \\ & \quad\quad + x_3(1-x_3)(a-x_1)(a-x_2)(1-x_1x_2)(x_2-x_1) \\ & = (x_1-x_2)(x_2-x_3)(x_3-x_1)(1-a)(a-x_1x_2x_3), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & a(1-x_1)(1-x_2/a)(1-x_3/a)e(x_2,x_3) \\ & \quad + a(1-x_2)(1-x_1/a)(1-x_3/a)e(x_3,x_1) \\ & \quad\quad + a(1-x_3)(1-x_1/a)(1-x_2/a)e(x_1,x_2) \\ & = (1-x_1/x_2)(1-x_2/x_3)(1-x_3/x_1)(1-a)(1-x_1x_2x_3/a). \end{aligned} \tag{A.1}$$

Proof. The proof follows by direct computation. □

Remark. In the computation (7.52) we used (A.1) with

$$x_1 = -q^{\frac{1}{2}}a_1, \quad x_2 = -q^{\frac{1}{2}}a_2, \quad x_3 = -q^{\frac{1}{2}}a_3a_4, \quad a = q^{\frac{1}{2}}a_3.$$

Remark. The above identity is a special case ($q \to 0$) of the following four-term identity satisfied by $\vartheta(x;q)$:

$$\begin{aligned} & ax_1x_3\vartheta(-q^{\frac{1}{2}}x_1;q)\vartheta(-q^{\frac{1}{2}}x_1/a;q)\vartheta(x_1;q)\vartheta(x_2/a;q)\vartheta(x_3/a;q)\vartheta(x_2x_3;q)\vartheta(x_2/x_3;q) \\ & \quad + ax_1x_2\vartheta(-q^{\frac{1}{2}}x_2;q)\vartheta(-q^{\frac{1}{2}}x_2/a;q)\vartheta(x_2;q)\vartheta(x_1/a;q)\vartheta(x_3/a;q)\vartheta(x_1x_3;q)\vartheta(x_3/x_1;q) \\ & \quad + ax_2x_3\vartheta(-q^{\frac{1}{2}}x_3;q)\vartheta(-q^{\frac{1}{2}}x_3/a;q)\vartheta(x_3;q)\vartheta(x_1/a;q)\vartheta(x_2/a;q)\vartheta(x_1x_2;q)\vartheta(x_1/x_2;q) \\ & = x_1x_2x_3\theta(-q^{\frac{1}{2}};q)\theta(-q^{\frac{1}{2}}a;q)\vartheta(x_1/x_2;q)\vartheta(x_2/x_3;q)\vartheta(x_3/x_1;q)\vartheta(a;q)\vartheta(x_1x_2x_3/a;q) \end{aligned}$$

We omit the details.

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