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ON THE SECURITY OF ZHANG-TAN'S VARIANTS OF MULTIVARIATE SIGNATURE SCHEMES *

Yasufumi HASHIMOTO

Abstract

Until now, various multivariate public key cryptosystems (MPKCs) have been proposed but some of them are known to be insecure. In IMACC 2015 and Inscript 2015, Zhang and Tan proposed new variants of MPKCs for signatures to enhance the security of the original schemes. However, Zhang-Tan's variants are much less secure than expected. In this paper, we describe an attack on Zhang-Tan's variants to recover a public key of the original scheme.

1 Introduction

After Shor [6] proposed polynomial-time algorithms to factor integers and to solve discrete logarithm problems by quantum computers, constructing cryptosystems secure against quantum attacks is one of big issues in cryptology. *Multivariate public key cryptosystems (MPKCs)*, public key cryptosystems whose public keys are sets of multivariate quadratic forms over finite fields, have been expected to be such cryptosystems. While various MPKCs have been proposed until now, some of them were broken soon after proposed or known to be much less secure than expected.

In IMACC 2015 and Inscript 2015, Zhang and Tan [9, 10] proposed a new idea to repair such insecure MPKCs for signatures. Their idea is adding several variables and terms for the additional variables on the original polynomials, and hiding several equations to eliminate the contributions of additional variables in the process of signature generation. They actually used this idea on already broken schemes MIT [8] and YTS [7, 2] by adding HFE-like polynomials and claimed that this idea enhanced the security drastically. However, the hidden equations can be recovered by sufficiently many signatures and these equations tell us partial information of the secret key. In this paper, we describe how to recover the hidden equations and the secret key partially, and conclude that our attack removes the contributions of the additional variables and recovers a public key of the original scheme.

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2 Multivariate Public Key Cryptosystem

In this section, we describe the general construction of multivariate public key cryptosystems (MPKCs).

Let $n, m \geq 1$ be integers, k a finite field and $q := \#k$. Define a quadratic map $G : k^n \rightarrow k^m$ to be inverted feasibly, i.e. finding $\mathbf{x} \in k^n$ with $G(\mathbf{x}) = \mathbf{y}$ is feasible for any (or most) $\mathbf{y} \in k^m$. The *secret key* is a tuple of three maps (S, G, T) , where $S : k^n \rightarrow k^n$, $T : k^m \rightarrow k^m$ are invertible affine maps and the quadratic map $G : k^n \rightarrow k^m$. The *public key* is the convolution of these three maps

$$F := T \circ G \circ S : k^n \rightarrow k^m.$$

On an encryption scheme, the *cipher-text* $\mathbf{y} \in k^m$ for a given plain-text $\mathbf{x} \in k^n$ is computed by $\mathbf{y} = F(\mathbf{x})$. To *decrypt* \mathbf{y} , find $\mathbf{z} \in k^n$ with $G(\mathbf{z}) = T^{-1}(\mathbf{y})$. Then the plain-text is $\mathbf{x} = S^{-1}(\mathbf{z})$. Since G is constructed to be inverted feasibly, one can decrypt \mathbf{y} feasibly.

On a signature scheme, a signature $\mathbf{x} \in k^n$ for a given message $\mathbf{y} \in k^m$ is generated as follows. Find $\mathbf{z} \in k^n$ with $G(\mathbf{z}) = \mathbf{y}$ and compute $\mathbf{x} = S^{-1}(\mathbf{z})$. The signature $\mathbf{x} \in k^n$ for \mathbf{y} is verified if $\mathbf{y} = F(\mathbf{x})$ holds.

3 Zhang-Tan's variant

In this section, we describe Zhang-Tan's variant [9, 10] on MPKC.

Let $n, n_0, m, l \geq 1$ be integers with $n = n_0 + l$ and $\mathbf{x} = {}^t(x_1, \dots, x_n)$, $\mathbf{x}_1 = {}^t(x_1, \dots, x_{n_0})$, $\mathbf{x}_2 = {}^t(x_{n_0+1}, \dots, x_n)$ are variables. Define the quadratic maps $G : k^n \rightarrow k^m$, $H : k^n \rightarrow k^l$ by

$$\begin{aligned} G(\mathbf{x}) &= {}^t(g_1(\mathbf{x}), \dots, g_m(\mathbf{x})), \\ H(\mathbf{x}) &= {}^t(h_1(\mathbf{x}), \dots, h_l(\mathbf{x})), \end{aligned}$$

where $g_1(\mathbf{x}), \dots, g_m(\mathbf{x})$ are quadratic forms of \mathbf{x}_1 and

$$\begin{aligned} h_i(\mathbf{x}) &= (\text{homogeneous quadratic form of } \mathbf{x}_2) \\ &+ \sum_{1 \leq j \leq l} x_{n_0+j} \cdot (\text{linear form of } \mathbf{x}_1), \quad (1 \leq i \leq l). \end{aligned} \tag{1}$$

Suppose that G, H are inverted feasibly, i.e. finding $\mathbf{x}_1 \in k^{n_0}$ with $G(\mathbf{x}_1) = \mathbf{y}$ is feasible for any (or most) $\mathbf{y} \in k^m$ and finding $\mathbf{x}_2 \in k^l$ with $H(\mathbf{x}_1, \mathbf{x}_2) = 0$ is also feasible for any (or most) $\mathbf{x}_1 \in k^{n_0}$. In [9, 10], G is the central map of MI-T [8] or YTS [7] and H is given by an HFE-like polynomial.

Zhang-Tan's variant is given as follows.

Secret key. Two invertible affine maps $S : k^n \rightarrow k^n$, $T : k^m \rightarrow k^m$, a linear map $T_1 : k^l \rightarrow k^m$ and the quadratic maps $G : k^n \rightarrow k^m$, $H : k^n \rightarrow k^l$ defined above.

Public key. The quadratic map $F : k^n \rightarrow k^m$ defined by

$$F := (T \circ G + T_1 \circ H) \circ S.$$

Signature generation. For a message $\mathbf{y} \in k^m$ to be signed, find $\mathbf{z}_1 \in k^{n_0}$ with $G(\mathbf{z}_1) = T^{-1}(\mathbf{y})$, and $\mathbf{z}_2 \in k^l$ with $H(\mathbf{z}_1, \mathbf{z}_2) = 0$. The signature is $\mathbf{w} = S^{-1}(\mathbf{z}_1, \mathbf{z}_2)$.

Signature verification. The signature \mathbf{w} is verified if $F(\mathbf{w}) = \mathbf{y}$ holds.

Since $G(\mathbf{x})$ is a set of quadratic forms of \mathbf{x}_1 and is constructed to be inverted feasibly, \mathbf{z}_1 is found feasibly. Similarly, finding \mathbf{z}_2 is also feasible. Note that H is constructed such that $H(\mathbf{x}_1, 0) = 0$ for any \mathbf{x}_1 . Then $\mathbf{z}_2 = 0$ is acceptable in the process of signature generation if there are no non-trivial \mathbf{z}_2 for a given \mathbf{z}_1 .

4 On the security of Zhang-Tan's variant

In this section, we propose our attack to reduce the problem of solving $F(\mathbf{x}) = \mathbf{y}$ to the problem of solving $F_0(\mathbf{x}_1) = \mathbf{y}_1$ where $F_0 := T \circ G \circ S_0$ is a public key of the original scheme derived from G , where $S_0 : k^{n_0} \rightarrow k^{n_0}$ is an invertible affine map. For simplicity, we assume that S is a linear map.

First, recall that one finds $\mathbf{z}_1, \mathbf{z}_2$ with $H(\mathbf{z}_1, \mathbf{z}_2) = 0$ in the process of signature generation, and the signature \mathbf{w} satisfies $(\mathbf{z}_1, \mathbf{z}_2) = S(\mathbf{w})$. Then $H \circ S(\mathbf{w}) = 0$ holds for any signature \mathbf{w} generated by the corresponding Zhang-Tan's variant. This means that there are l -linearly independent quadratic forms $u_1(\mathbf{x}), \dots, u_l(\mathbf{x})$ with $u_i(\mathbf{w}) = 0$ for any signature \mathbf{w} and these quadratic forms are linear sums of $h_1(S(\mathbf{x})), \dots, h_l(S(\mathbf{x}))$. Since h_i is a quadratic form of n variables, we can recover $u_1(\mathbf{x}), \dots, u_l(\mathbf{x})$ by $N \gg \frac{1}{2}n(n+1)$ signatures.

By the construction (1) of H , we see that the polynomials $u_i(\mathbf{x})$ are written by

$$u_i(\mathbf{x}) = {}^t \mathbf{x}^t S \begin{pmatrix} 0_{n_0} & * \\ * & *_l \end{pmatrix} S \mathbf{x} + (\text{linear form})$$

for $1 \leq i \leq l$. This is similar to the quadratic form in UOV [5, 3] and then, once $u_1(\mathbf{x}), \dots, u_l(\mathbf{x})$ are given, the attacker can recover an invertible linear map $S_1 : k^n \rightarrow k^n$ with $SS_1 = \begin{pmatrix} *_{n_0} & * \\ 0 & *_l \end{pmatrix}$ by Kipnis-Shamir's attack on UOV [4, 3] in time $O(q^{\max(0, l-n_0)} \cdot (\text{polyn.}))$.

Since $\bar{F} := F \circ S_1 = (T \circ G + T_1 \circ H) \circ (S \circ S_1)$ and $SS_1 = \begin{pmatrix} S_0 & * \\ 0 & *_l \end{pmatrix}$ with some $n_0 \times n_0$ matrix S_0 , we see that $H \circ (S \circ S_0)$ is a set of quadratic forms as given in (1). Then we have $\bar{F}(\mathbf{x}_1, 0) = (T \circ G \circ S_0)(\mathbf{x}_1)$, which is a public key of the original scheme derived from G . We thus conclude that Zhang-Tan's variant does not protect the original scheme strongly.

Our attack is summarized as follows.

Step 1. Let N be an integer sufficiently larger than $\frac{1}{2}n(n+1)$. Choose N messages $\mathbf{y}_1, \dots, \mathbf{y}_N \in k^m$ randomly, and generate signatures $\mathbf{w}_1, \dots, \mathbf{w}_N \in k^n$ for $\mathbf{y}_1, \dots, \mathbf{y}_N \in k^m$ respectively.

Step 2. Find l linearly independent quadratic forms $u_1(\mathbf{x}), \dots, u_l(\mathbf{x})$ with $u_i(\mathbf{w}_j) = 0$ for any $1 \leq j \leq N$.

Step 3. Find an invertible linear map $S_1 : k^n \rightarrow k^n$ with

$$u_i(S_1(\mathbf{x})) = {}^t \mathbf{x} \begin{pmatrix} 0_{n_0} & * \\ * & *_{l} \end{pmatrix} \mathbf{x} + (\text{linear form})$$

for $1 \leq i \leq l$ by Kipnis-Shamir's attack [4, 3].

Step 4. Let $\bar{F} := F \circ S_1$. Then $\bar{F}(\mathbf{x}_1, 0)$ is a public key of the original scheme.

As already explained, the complexity of Step 3 is $O(q^{\max(0, l-n_0)} \cdot (\text{polyn.}))$. It is easy to see that the complexities of other steps of our attack are in polynomial time. Then the total complexity of our attack is $O(q^{\max(0, l-n_0)} \cdot (\text{polyn.}))$, which is much less than $O(q^l \cdot (\text{polyn.}))$ expected by Zhang and Tan [9, 10]. This means that l must be sufficiently larger than n_0 , namely the number n of variables in Zhang-Tan's variant must be sufficiently larger than twice of the number n_0 of the variables in the original scheme. This situation is similar to UOV [3], and we can consider that Zhang-Tan's variant does not have an advantage over UOV. Furthermore, the complexity $O(q^{\max(0, l-n_0)} \cdot (\text{polyn.}))$ might be improved if H has a special structure. For example, when H is given by an HFE-like polynomial [9, 10], the rank attack [1] will reduce the complexity. We thus conclude that Zhang-Tan's variant is not practical at all.

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A locally commentative, deformative learning in the arithmetic noncommutative geometry as flower gardening variation

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*Dedicated to Professor Takashi Ito on his 60th birthday
with gratitude and respect*

Abstract

We would like to study noncommutative geometry, related to Arithmetics in some sense. For this purpose forced, we review and study the lecture notes by Marcolli, on the arithmetic noncommutative geometry.

MSC 2010: Primary 46-01, 46-02, 46L05, 46L51, 46L53, 46L55, 46L80, 46L85, 46L87, 46L89, 19K14, 19K56, 81Q10, 81Q35, 81R15, 81R40, 81R60, 81T75, 11G18, 11M06, 11R37, 11R42.

Keywords: Noncommutative geometry, C^* -algebra, K-theory, spectral triple, zeta function, modular group, modular curve, quantum statistical mechanics, Bost-Connes system, KMS state, class field theory, Schottky, hyperbolic, arithmetic, archimedean, Arakelov geometry, Green function, black hole, L-function, L-factor, algebraic variety, Shimura variety, Cuntz-Krieger algebra.

1 Overture

Preface. Just following, almost along the story of Matilde Marcolli [114], as nothing but a running commentary, as purpose of background learning, we (as beginners) would like to study what is growing and harvested (at that time) on the rather mysterious noncommutative world (as a vast field), related to Arithmetics, as a back to the past for a return to the future (BPRF) flower gardening variation as named. All necessary things are to be considered until the end is to be watched. Our local understanding the details (in part) may be shallow or somewhat limited. With some considerable effort within the time limited for publication, we would like to make some additional or deformed, comments, corrections, descriptions, (partial) proofs, or remarks, on the contents, to be or not to be self-contained.

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In particular, as a note we use the symbols $\diamond \cdots$ (texts) \square as indicating either personal comments, computations to be checked, or etc. As well, some additional remarks starting with **Remark** \cdots (texts) \blacktriangleleft in the text below are also made from suitable related references to be included. All the details could not be contained in these cases. Some notations as well as style are (slightly) changed from the original ones by our taste. As usual, let $i = \sqrt{-1}$. Denote by \mathbb{Q} , \mathbb{R} , and \mathbb{C} the fields of rational, real, and complex numbers, respectively.

Once upon a time and space, there is certainly or simply, a sort of synchronized improvement based on the achievement of Marcolli, like a decorated field on the high splended hill. The variant flower garden is now open!

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Section 1. Let us start by recalling several preliminary notions of noncommutative geometry, following Connes [27].

Section 2. Let us describe how various arithmetic properties of modular curves can be viewed as their noncommutative boundaries as noncommutative spaces or algebras. The main references are Manin-Marcolli [112] and Marcolli [113], [115].

Section 3. This section includes an account of the work of Connes and Marcolli [41] on the noncommutative geometry for commensurability classes of \mathbb{Q} -lattices. Also included is the relation of the noncommutative geometry for these classes to the Hilbert 12th problem of explicit class field theory as well as the results of Connes, Marcolli, and Ramachandran [48] on the construction of a quantum statistical mechanical system that recovers the explicit class field theory of imaginary quadratic fields. As well, included is a brief explanation on the real multiplication program of Manin [109], [110] and the problem of real quadratic fields.

Section 4. This section contains the noncommutative geometry for the fibers at arithmetic infinity of varieties over number fields, based on Consani and Marcolli [58], [59], [60], [61], [63]. Also contained is a detailed account of the formula of Manin for the Green function of Arakelov geometry for arithmetic surfaces, based on Manin [106] as well as a proposed physical interpretation of this formula, as in Manin and Marcolli [111].

Noncommutative geometry (**NG**) is developed by Connes, started in the early 1980s as in [22], [24], [27]. The NG extends tools of ordinary geometry to treat spaces obtained as certain quotients by some equivalence relations, for which the usual ring of functions on quotients defined as functions invariant with respect to the equivalence relation becomes too trivial to capture the information on the structure of points of the quotient space, and so to define a noncommutative algebra or space with generators corresponding to coordinates of the space, remembering the structure of the equivalence relation, and analogous to noncommuting variables in quantum mechanics. These quantum spaces defined so extend the Gel'fand-Naimark correspondence between locally compact, Hausdorff spaces and commutative C^* -algebras of all continuous, complex-valued functions on the spaces (vanishing at infinity) by dropping the commutativity hypothesis. Namely, a noncommutative (**NC**) (topological) space is defined to be a noncommutative C^* -algebra.

Such quotient spaces are abundant in nature. For instance, they arise by foliations as equivalence relations. As well, noncommutative spaces arise naturally in number theory and arithmetic geometry. The first connection between noncommutative geometry and number theory is obtained by Bost and Connes [13], in which exhibited is an interesting noncommutative space or algebra with remarkable arithmetic properties related to the class field theory. This also involves the phenomena as spontaneous symmetry breaking in quantum statistical mechanics related to the Galois theory. That space can be viewed as the NC space of 1-dimensional \mathbb{Q} -lattices up to scale, modulo the equivalence relation of commensurability (Connes-Marcolli [41]). As well, the NC space is closely

related to the noncommutative space by Connes [29] to obtain a spectral realization of zeros of the Riemann zeta function. In fact, this NC space is that of commensurability classes of 1-dimensional \mathbb{Q} -lattices, but with the scale factor also taken into account.

The deeper connections between noncommutative geometry and number theory are obtained by Connes and Moscovici [54], [55] on the modular Hecke algebras. It is shown that the Rankin-Cohen brackets as an important algebraic structure on modular forms (Zagier [159]) have a natural interpretation in terms of the Hopf algebra of the transverse geometry of foliations with codimension 1, in the sense of NG. The modular Hecke algebras, in which products and action of Hecke operators on modular forms are naturally combined, can be viewed as the holomorphic part of the NC algebra of coordinates on the space of commensurability classes of 2-dimensional \mathbb{Q} -lattices, constructed by Connes and Marcolli [41].

On the other hand, the classification of noncommutative 3-spheres is obtained by Connes and Dubois-Violette [35], [36], [37], in which the corresponding moduli space has a ramified cover by a noncommutative nilmanifold, where the NC analog of the Jacobian of this covering map is expressed naturally in terms of the Dedekind eta function, as an interesting case of number theory within NG. As another such case, calculated by Connes [31] the explicit cyclic cohomology Chern character of a spectral triple on $SU_q(2)$, also defined by Chakraborty and Pal [17].

More other instances of NG spaces that arise in the context of number theory and arithmetic geometry can be found in the noncommutative compactification of modular curves, as Connes, Douglas, and Schwarz [34], Manin and Marcolli [112]. This NC space is again related to the NG of \mathbb{Q} -lattices. In fact, it can be seen as a stratum in the compactification of the space of commensurability classes of 2-dimensional \mathbb{Q} -lattices (CM [41]).

Another context in which NG provides a useful tool for arithmetic geometry is given by the description of the totally degenerate fibers at arithmetic infinity of arithmetic varieties over number fields, analyzed by Katia Consani and Marcolli [58], [59], [60], [61].

The present text is based on the lecture notes [114] on the lectures given by Marcolli, in 2002 to 2004, may be used later on.

In the lectures, contained as the main focus is the noncommutative geometry of modular curves (MM [112]) and of the archimedean fibers of arithmetic varieties ([61]). Also included is a theory of the NG space of commensurability classes of 2-dimensional \mathbb{Q} -lattices ([41]). In particular, conveyed is the general picture less than the details of the proofs of the specific results. Though proof are almost not included in the (original) text, the reader may find references to the relevant literature, such as [41], [48], [61], and [112].

From the foreword by Manin as backward. (Edited, changed for short.) Let us be imagining C^* -algebras as coordinate rings, and making bridges to commutative geometry via crossed products of C^* -algebras, to involve noncommutative Riemannian geometry, with algebraic tools as K-theory and cyclic

cohomology for C^* -algebras and their subalgebras.

Let us be studying various aspects of uniformization by the classical modular group as well as Schottky groups. The modular group acts on the complex half plane, partially compactified by cusps as rational points of the boundary projective line over \mathbb{C} . The action at irrational points is viewed as the noncommutative boundary as NC spaces, entering NG. Moreover, as contents mentioned as follows:

- ⊙ Schottky uniformization provides a visualization of Arakelov geometry at arithmetic infinity.
- ⊙ Gauss characterising regular polygons that can be constructed using only ruler and compass.
- ⊙ In the complex plane, roots of unity of degree n form vertices of a regular n -gon.
- ⊙ Being constructible as the maximal Galois extension $\overline{\mathbb{Q}}$ of \mathbb{Q} , to calculate the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- ⊙ Gauss Galois group $\text{Gal}^{Ab}(\overline{\mathbb{Q}}/\mathbb{Q})$ in Bost-Connes symmetry breaking and Gauss statistics of continued fractions in the chaotic cosmology models.

1.1 The space-algebra-geometry dictionary

There is a dictionary relating concepts of ordinary geometry to the corresponding counterparts in noncommutative geometry (cf. [27] as well as [154]).

Table 2: The S-A-G overview dictionary, as translated

Space as Geometry	Algebra as Geometry
<p>Measure spaces: $(X, \mathfrak{B}$ Borel structure) Dynamical system $(X, \mathfrak{B}, \Gamma)$ by a Borel action of a group Γ</p>	<p>von Neumann algebras: Classical $L^\infty(X, \mathfrak{B})$ Quantum vN crossed product as $\mathfrak{M} = L^\infty(X) \rtimes \Gamma$ by automorphisms</p>
<p>Topological spaces: $(X, \mathfrak{D}$ topology) $\subset (X, \mathfrak{B})$ (compact or not); Dynamical system $(X, \mathfrak{D}, \Gamma)$ by homeomorphisms</p>	<p>C^*-algebras: Classical $C(X)$ (or $C_0(X)$) in $L^\infty(X)$ (unital or not); Quantum C^*-crossed product as $\mathfrak{A} = C(X) \rtimes \Gamma \subset \mathfrak{M}$</p>
<p>Differential geometry Smooth (compact) manifold M; Dynamical system (M, Γ)</p>	<p>Smooth dense $*$-subalgebras: Classical $C^\infty(M) \subset C(M)$; Quantum $\mathcal{A} = C^\infty(M) \rtimes \Gamma \subset \mathfrak{A}$</p>
<p>Riemannian geometry Riemann metric d with (Dirac) differential operator D</p>	<p>Noncommutative geometry Spectral triple $(\mathfrak{A}$ with $\mathcal{A}, H, D)$ on Hilbert space H with D derivation</p>
<p>More specified geometry with more fine structure</p>	<p>More analogous algebras with more suitable characters</p>

1.2 Noncommutative spaces encountered

The way to assign the algebra of coordinates to a quotient space $X = Y/\sim$ of a space Y by an equivalence relation \sim may be explained as follows.

- One way to consider functions f on Y with $f(a) = f(b)$ for $a \sim b$ in Y .
 - ◊ If each equivalence class as a subset of Y is dense in Y and if f is continuous on Y and constant c on an equivalence class, then $f \equiv c1$ on Y . \square
- The other way to consider functions on the graph of the equivalence relation.

A noncommutative algebra is obtained by the second description, with a convolution product determined by the groupoid law of the equivalence relation, like the usual product of the C^* -algebra of $n \times n$ matrices over \mathbb{C} as

$$f * g = fg = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{1n} & \cdots & f_{nn} \end{pmatrix} (g_{ij})_{i,j=1}^n = \left(\sum_{k=1}^n f_{ik}g_{kj} \right)_{i,j=1}^n \in M_n(\mathbb{C}).$$

Even though the two functional notions for quotients are not the same, but they may be related by Morita equivalence, as a suitable notion as a (stable) isomorphism between NC spaces. However, the second notion is the only one to allow to make sense of (NC) geometry for the general quotient spaces.

Example 1.1. Refer to [30]. Let $Y = [0, 1] \times \{0, 1\}$ with the equivalence relation as points $(x, 0) \sim (x, 1)$ for any $x \in (0, 1)$ the open interval. Then $X = Y/\sim$ is identified with a topological union with the quotient topology as

$$\{(0, 0), (0, 1)\} \cup (0, 1) \cup \{(1, 0), (0, 1)\}.$$

The first way implies $C([0, 1])$ (not \mathbb{C} , corrected), but the commutative C^* -algebra of all continuous, complex-valued functions on the closed interval $[0, 1]$. On the other hand, the second way implies the continuous C^* -algebra bundle over $[0, 1]$ with fibers $M_2(\mathbb{C})$ on $(0, 1)$ and \mathbb{C}^2 at 0 and 1, which can be written as a C^* -algebra extension E as a NC algebra

$$0 \rightarrow SM_2(\mathbb{C}) = C_0((0, 1)) \otimes M_2(\mathbb{C}) \rightarrow E \rightarrow \mathbb{C}^2 \oplus \mathbb{C}^2 \rightarrow 0,$$

where each \mathbb{C}^2 is identified with the diagonal part of $M_2(\mathbb{C})$. \blacktriangleleft

Note that there is analogy between the idea of preserving the information of the equivalence relation in the description of quotient spaces and the Grothendieck theory of stacks in algebraic geometry.

Morita equivalence. In NC geometry, the proper notion as isomorphisms of C^* -algebras is provided by Morita equivalence of C^* -algebras, or (equivalent) their stable isomorphisms. Namely, for (separable) C^* -algebras \mathfrak{A} and \mathfrak{B} , they are **stably** isomorphic if $\mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K}$, where $\mathbb{K} = \mathbb{K}(H)$ is the C^* -algebras of all compact operators on a (separable) Hilbert space H .

C^* -algebras \mathfrak{A} and \mathfrak{B} are (strongly) **Morita** equivalent if there is a \mathfrak{A} - \mathfrak{B} bimodule \mathfrak{M} , which is a left Hilbert \mathfrak{A} -module with an \mathfrak{A} -valued inner product

$\langle \cdot, \cdot \rangle_{\mathfrak{A}}$, and is a right Hilbert \mathfrak{B} -module with an \mathfrak{B} -valued inner product $\langle \cdot, \cdot \rangle_{\mathfrak{B}}$, such that (corrected)

- (1) $\mathfrak{A} = \overline{\langle \mathfrak{M}, \mathfrak{M} \rangle_{\mathfrak{A}}}$ and $\mathfrak{B} = \overline{\langle \mathfrak{M}, \mathfrak{M} \rangle_{\mathfrak{B}}}$,
- (2) $\langle \xi_1, \xi_2 \rangle_{\mathfrak{A}} \xi_3 = \xi_1 \langle \xi_2, \xi_3 \rangle_{\mathfrak{B}}$, $\xi_1, \xi_2, \xi_3 \in \mathfrak{M}$,
- (3) $\langle a\xi, a\xi \rangle_{\mathfrak{B}} \leq \|a\|^2 \langle \xi, \xi \rangle_{\mathfrak{B}}$, $\langle \xi b, \xi b \rangle_{\mathfrak{A}} \leq \|b\|^2 \langle \xi, \xi \rangle_{\mathfrak{A}}$,

for $a \in \mathfrak{A}, b \in \mathfrak{B}, \xi \in \mathfrak{M}$, so that \mathfrak{A} and \mathfrak{B} act on \mathfrak{M} from the left and right, as bounded operators.

◊ We may assume that $\mathfrak{M} = \overline{\mathfrak{A}\mathfrak{C}\mathfrak{B}}$ for a C^* -algebra \mathfrak{C} such that

$$\mathfrak{A}\mathfrak{C}\mathfrak{A} \subset \mathfrak{A} \subset \mathfrak{C} \quad \text{and} \quad \mathfrak{B}\mathfrak{C}\mathfrak{B} \subset \mathfrak{B} \subset \mathfrak{C}.$$

Then $\mathfrak{A}\mathfrak{M} \subset \mathfrak{M}$ and $\mathfrak{M}\mathfrak{B} \subset \mathfrak{M}$. Also, we may assume that $\langle x, y \rangle_{\mathfrak{A}} = xy^* \in \mathfrak{A}$ and $\langle x, y \rangle_{\mathfrak{B}} = x^*y \in \mathfrak{B}$ for $x, y \in \mathfrak{M}$. Because

$$\mathfrak{A}\mathfrak{C}\mathfrak{B}\mathfrak{B}^*\mathfrak{C}^*\mathfrak{A}^* \subset \mathfrak{M}\mathfrak{M}^* \subset \mathfrak{A} \quad \text{and} \quad \mathfrak{B}^*\mathfrak{C}^*\mathfrak{A}^*\mathfrak{A}\mathfrak{C}\mathfrak{B} \subset \mathfrak{M}^*\mathfrak{M} \subset \mathfrak{B}.$$

Then, for any $x, y, z \in \mathfrak{M}$,

$$\langle x, y \rangle_{\mathfrak{A}} z = xy^*z = x\langle y, z \rangle_{\mathfrak{B}}.$$

As well, for any $a \in \mathfrak{A}, b \in \mathfrak{B}, \xi \in \mathfrak{M}$,

$$\begin{aligned} \langle a\xi, a\xi \rangle_{\mathfrak{B}} &= \xi^* a^* a \xi \leq \xi^* \|a^* a\| 1 \xi = \|a\|^2 \langle \xi, \xi \rangle_{\mathfrak{B}} \quad \text{in } \mathfrak{B}_+, \\ \langle \xi b, \xi b \rangle_{\mathfrak{A}} &= \xi b b^* \xi^* \leq \xi \|b b^*\| 1 \xi^* = \|b\|^2 \langle \xi, \xi \rangle_{\mathfrak{A}}, \quad \text{in } \mathfrak{A}_+. \end{aligned}$$

For instance, we may let

$$\begin{aligned} \mathfrak{A}\mathfrak{C}\mathfrak{B} &\equiv \begin{pmatrix} \mathfrak{A}' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{C}' & \mathfrak{C}' \\ \mathfrak{C}' & \mathfrak{C}' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{B}' \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathfrak{A}'\mathfrak{C}'\mathfrak{B}' \equiv \mathfrak{M}' \\ 0 & 0 \end{pmatrix} \equiv \mathfrak{M}. \end{aligned}$$

Then

$$\mathfrak{M}\mathfrak{M}^* = \begin{pmatrix} \mathfrak{M}'(\mathfrak{M}')^* & 0 \\ 0 & 0 \end{pmatrix} \subset \mathfrak{A} \quad \text{and} \quad \mathfrak{M}^*\mathfrak{M} = \begin{pmatrix} 0 & 0 \\ 0 & (\mathfrak{M}')^*\mathfrak{M}' \end{pmatrix} \subset \mathfrak{B}.$$

This picture quite explains the meaning of Morita equivalence. Namely the triangle corners!

In particular, $\mathbb{C} \cong \mathbb{C} \oplus 0_n = \mathfrak{A}$ and $M_n(\mathbb{C}) \cong 0 \oplus M_n(\mathbb{C}) = \mathfrak{B}$ as diagonal sums in $M_{n+1}(\mathbb{C})$ are Morita equivalent since

$$\begin{aligned} \mathfrak{A}\mathfrak{C}\mathfrak{B} &= (\mathbb{C} \oplus 0_n) M_{n+1}(\mathbb{C}) (0 \oplus M_n(\mathbb{C})) \\ &= \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0_n \end{pmatrix} \begin{pmatrix} \mathbb{C} & \mathbb{C}^n \\ (\mathbb{C}^n)^t & M_n(\mathbb{C}) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & M_n(\mathbb{C}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbb{C}^n \\ 0 & 0_n \end{pmatrix} = \mathfrak{M}, \end{aligned}$$

$$\mathfrak{M}\mathfrak{M}^* \cong \mathbb{C}^n (\mathbb{C}^n)^t \cong \mathbb{C} \quad \text{and} \quad \mathfrak{M}^*\mathfrak{M} \cong (\mathbb{C}^n)^* \mathbb{C}^n \cong M_n(\mathbb{C}),$$

with \mathbb{C}^n as a row vector space and $(\mathbb{C}^n)^t$ as a column vector space. \square

The tools de noncommutative geometry.

Table 3: The tool box

Name (tool as invariant)	Use (limited)
K-theory, Kasparov KK-theory	Topological spaces, Vector bundles, C^* -algebras, Extensions
Hochschild (co)homology, Connes (cyclic) cohomology	Topological spaces, Function algebras, C^* -algebras, subalgebras
Homotopy theory as quotients	Topological spaces, Baum-Connes assembly maps
Metric structure, Differential structure	Manifolds, Dirac operator, Spectral triples
Real, Spin structure, Geometric, Analytic structure	Real, Spin manifolds, Characteristic classes, Zeta functions

As an important note, please handle these tools with the greatest care!

1.3 Spectral triples appeared

We are going to take a (moment) look at the analogue in the NC world of Riemann geometry, provided by Connes. Spectral triples as a generalization of the structure of a Riemannian manifold play a crucial role in noncommutative geometry. In the usual context of Riemann geometry, the definition of the infinitesimal element ds on a smooth spin manifold can be expressed in terms of the inverse D^{-1} of the classical Dirac operator D . The theory of spectral triples (ST) is motivated by this key remark. In particular, the geodesic distance between two points of the manifold is defined in terms of D^{-1} (cf. [27, VI]).

Example 1.2. (Added). The **geodesic** distance formula for a Riemann spin manifold M is given by

$$d(x, y) = \sup\{|f(x) - f(y)| \mid f \in C^\infty(M), \|[D, f]\| \leq 1\},$$

where D is the **Dirac** operator, defined as $\frac{d}{ds}$, so that $[D, f] = M_{Df}$ the multiplication operator on the Hilbert space $L^2(M)$, because

$$\begin{aligned} [D, f]g &= [D, M_f]g = (DM_f - M_fD)g \\ &= D(fg) - fDg \\ &= (Df)g + fDg - fDg = M_{Df}g, \quad f, g \in C^\infty(M), \end{aligned}$$

and $C^\infty(M)$ is dense in $L^2(M)$. For more details, may refer to [150].

The triple

$$(C(M), L^2(M), D = \frac{d}{ds})$$

is the typical example of spectral triples in the commutative case. Indeed, the C^* -algebra $C(M)$ of all continuous, complex-valued functions on a compact (Riemann spin) manifold M contains the norm dense $*$ -subalgebra $C^\infty(M)$ of all smooth differentiable functions on M , on which as a dense domain in $C(M)$, the (Dirac) differential operator $\frac{d}{ds}$ is certainly defined. The representation of $C(M)$ on $L^2(M)$ is given as multiplication operators. ◀

As a generalization of the commutative case to the noncommutative case, given by Connes is the following (cf. [27], [28], [50]):

Definition 1.3. A (Connes) **spectral triple** (SpTr) is defined to be a triple (\mathfrak{A}, H, D) of a C^* -algebra \mathfrak{A} , a C^* -algebra representation $\rho : \mathfrak{A} \rightarrow \mathbb{B}(H)$ the C^* -algebra of all bounded operators on a separable Hilbert space H , and a (unbounded, Dirac) operator D , such that

- (1) $D = D^*$ as self-adjointness, with its spectrum in \mathbb{R} .
- (2) For any $\lambda \notin \mathbb{R}$, the resolvent (inverse) operator $(D - \lambda 1)^{-1}$ is a compact operator on H .
- (3) For any $a \in \mathcal{A} = \mathfrak{A}^\infty$ a dense involutive (or $*$ -) subalgebra \mathfrak{A} , the commutator $[D, \rho(a)]$ is a bounded operator on H .

Remark. (Edited). The above property (2) can be regarded as a generalization of the ellipticity property of the standard Dirac operator on a compact manifold. In the classical case of Riemann manifolds, the property (3) is equivalent to the Lipschitz condition, which is weaker than the smoothness condition. ◀

◊ Indeed, the geodesic distance formula for M is converted as the **Lipschitz** norm for $f \in C^\infty(M)$ as

$$L(f) \equiv \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)} = \left\| \frac{df}{ds} \right\|_\infty = \|M_D f\| = \|[D, f]\|.$$

Then the boundedness of $L(f)$ with $\|f\|_\infty \leq 1$ is equivalent to that of the operator $[D, \cdot]$. ◻

The basic geometric structure encoded by the theory of spectral triples is Riemann geometry. But in more refined cases such as Kähler geometry, the additional structure can be encoded as additional symmetries. For instance, in the case where the algebra involves the action of the Lefschetz operator of a compact Kähler manifold, can be encoded the information on the Kähler form, at the cohomological level (cf. [61], [63]).

As the relations among NG, arithmetic geometry (AG) and number theory (NT), the interesting feature of spectral triples (SpTr) is that we have an associated family of zeta functions (defined as the trace for the operator $|D|^{-z}$) and the theory of volumes and integration, which is related to special values of these zeta functions.

Volume form. Based on [27], [28]. May refer to [150] as well. A Connes spectral triple (\mathfrak{A}, H, D) is said to be n -**summable**, or of dimension n if $|D|^{-n}$ is an infinitesimal (or a compact operator) of order one, in the sense that the eigenvalues $\lambda_k(|D|^{-n})$ of $|D|^{-n}$ satisfy the estimate $\lambda_k(|D|^{-n}) = O(k^{-1})$ ($k \rightarrow$

∞), which means $0 \leq \lambda_k(|D|^{-n}) \leq \frac{M}{k}$ for some constant $M > 0$ and any k large enough.

Consider a positive compact operator T with decreasing eigenvalues $\lambda_j(T) \geq 0$ with multiplicities counted such that $\sum_{j=0}^{k-1} \lambda_j(T) = O(\log k)$ ($k \rightarrow \infty$) as a (logarithmic) **divergence**, equivalently, with the sequence boundedness $\frac{1}{\log k} \sum_{j=0}^{k-1} \lambda_j(T) \leq M$ for some $M > 0$ and any k large enough, as the (inverse) logarithmic (possible) divergence. The **Dixmier** trace $\text{tr}_\omega(T)$ of such $T \in \mathfrak{L}^{1,\infty}(H) \subset \mathbb{B}(H)$, containing a **trace** class operator $T \in \mathfrak{L}^1(H)$ such that

$$\|T\|_1 = \text{tr}(|T|) = \sum_{j=0}^{\infty} \lambda_j(|T|) < \infty,$$

is defined to be

$$\text{tr}_\omega(T) = \lim_{\omega} \frac{1}{\log k} \sum_{j=0}^{k-1} \lambda_j(T) \equiv \omega\left(\left\{\frac{1}{\log k} \sum_{j=0}^{k-1} \lambda_j(T)\right\}_k\right),$$

where ω is a certainly chosen functional in $C^b(\mathbb{N})^*$ the dual space of $C^b(\mathbb{N})$ the C^* -algebra of all bounded complex sequences, with several properties, extending the usual limit for convergent sequences.

◊ Namely, $\lim \leq \lim_\omega = \omega$ as the limit extension. Note that functionals (on NC spaces or algebras) are viewed as (noncommutative) integrations. \square

The Dixmier trace is also defined on compact operators with order one as infinitesimals.

◊ The reason for this is implied by that the **Euler** constant γ as the limit

$$\lim_{k \rightarrow \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \log k\right) = \gamma$$

exists, with $\frac{1}{2} \leq \gamma < 1$. \square

An operator T in the definition domain of the Dixmier trace(s) is said to be **measurable** if the D-trace $\text{tr}_\omega(T)$ does not depend on the choice of the limit functional $\omega = \lim_\omega$.

The operator $|D|^{-n}$ on a noncommutative space \mathfrak{A} generalizes the notion of a volume form on a usual space. The (noncommutative) (total) **volume** for \mathfrak{A} or \mathcal{A} (n -summable) is defined to be

$$V = \text{tr}_\omega(|D|^{-n}) \equiv |\mathfrak{A}| \text{ or } |\mathcal{A}|.$$

More generally, for an element a of the algebra \mathfrak{D} generated by \mathcal{A} and $[D, \mathcal{A}]$, its (normalized) **integration** with respect to the volume form $|D|^{-n}$ is defined as

$$\frac{1}{V} \int a = \frac{1}{V} \text{tr}_\omega(a|D|^{-n}).$$

The usual notion of integration on a Riemann spin manifold M can be recovered in this context by the formula

$$\int_M f dv = 2^{n - [\frac{n}{2}] - 1} \pi^{\frac{n}{2}} n \Gamma\left(\frac{n}{2}\right) \text{tr}_\omega(M_f |D|^{-n}),$$

where D is the classical Dirac operator on M associated to the metric that determines the volume form dv , and M_f is the multiplication operator acting on the Hilbert space of all square integrable (L^2) spinors (or sections of a certain bundle) on M (cf. [27], [97]).

Zeta functions. For a spectral triple (SpTr) (\mathfrak{A}, H, D) , the **zeta** function associated to the Dirac operator D is defined to be

$$\zeta_D(z) = \mathrm{tr}(|D|^{-z}) = \sum_{\lambda \in \sigma_p(|D|)} \mathrm{tr}(p(\lambda, |D|)) \frac{1}{\lambda^z},$$

where $p(\lambda, |D|)$ denotes the orthogonal projection on the eigenspace for an eigenvalue $\lambda \in \sigma_p(|D|)$ the point spectrum in the spectrum $\sigma(|D|)$ of $|D|$. If $|D|$ is compact, then $\sigma(|D|) = \sigma_p(|D|)$.

As an important result in the theory of SpTr ([27, IV Proposition 4]), the total volume is related to the **residue** of the zeta function at $s = 1$ by the formula

$$V = \mathrm{res}_{s=1} \mathrm{tr}(|D|^{-s}) = \lim_{s \rightarrow 1+0} (s-1) \zeta_D(s).$$

Moreover, there is a family of (**extended**) zeta functions associated to a spectral triple (\mathfrak{A}, H, D) , containing ζ_D . For any $a \in \mathfrak{D}$ the domain of D , the two zeta functions with respect to a and D are defined as

$$\begin{aligned} \zeta_{a,D}(z) &= \mathrm{tr}(a|D|^{-z}) = \sum_{\lambda \in \sigma_p(|D|)} \mathrm{tr}(a \cdot p(\lambda, |D|)) \frac{1}{\lambda^z}, \quad \text{and} \\ \zeta_{a,D}(s, z) &= \sum_{\lambda \in \sigma_p(|D|)} \mathrm{tr}(a p(\lambda, |D|)) \frac{1}{(s-\lambda)^z}. \end{aligned}$$

These zeta functions are related to the heat kernel $\exp(-t|D|)$ by **Mellin** transform as

$$\begin{aligned} \zeta_{a,D}(z) &= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \theta_{a,D}(t) dt, \quad \text{and} \\ \zeta_{a,D}(s, z) &= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \theta_{a,D,s}(t) dt, \end{aligned}$$

where

$$\begin{aligned} \theta_{a,D}(t) &= \sum_{\lambda \in \sigma_p(|D|)} \mathrm{tr}(a p(\lambda, |D|)) \frac{1}{e^{t\lambda}} = \mathrm{tr}(a e^{-t|D|}) \\ \theta_{a,D,s}(t) &= \sum_{\lambda \in \sigma_p(|D|)} \mathrm{tr}(a p(\lambda, |D|)) \frac{1}{e^{t(s-\lambda)}}. \end{aligned}$$

Under suitable hypothesis on the asymptotic expansion of $\theta_{a,D,s}$ (cf. [107, 2 Theorems 2.7 and 2.8]), the two zeta functions admit a unique analytic continuation (cf. [50]). There is an associated regularized **determinant** in the sense

of Ray-Singer (cf. [138]):

$$\det_{\infty,a,D}(s) = \exp\left(-\frac{d}{dz}\zeta_{a,D}(s,z)|_{z=0}\right).$$

The family of zeta functions $\zeta_{a,D}(z)$ also provides a refined notion of dimension for a spectral triple (\mathfrak{A}, H, D) , which is called its **dimension** spectrum and is defined to be a subset $\Sigma = \Sigma(\mathfrak{A}, H, D)$ of \mathbb{C} such that all the zeta functions $\zeta_{a,D}(z)$ for any $a \in \mathfrak{D}$ extend holomorphically to $\mathbb{C} \setminus \Sigma$.

Examples of SpTr with dimension spectrum not contained in \mathbb{R} can be constructed by Cantor sets.

Index map. As well, spectral triples (\mathfrak{A}, H, D) determine an index map. In fact, the $D = D^*$ self-adjoint has a polar decomposition $D = F|D|$, where $|D| = \sqrt{D^*D} \geq 0$ positive and F is a sign operator with $F^2 = \text{id}$ the identity map on H . Following [27], [28], define a cyclic **cocycle** τ as

$$\tau(a_0, a_1, \dots, a_n) = \text{tr}(a_0[F, a_1] \cdots [F, a_n]),$$

where n is the dimension of the SpTr . For n even, the trace $\text{tr}(\cdot)$ should be replaced by a super (or Dixmier) trace, as usual in index theory. This cocycle has pairs with the K-theory groups of \mathfrak{A} , with $K_0(\mathfrak{A})$ in the even case and with $K_1(\mathfrak{A})$ in the odd case, and defines a class of τ in the cyclic cohomology $HC^n(\mathfrak{A})$ for \mathfrak{A} . That class is said to be the **Chern** character of the $\text{SpTr}(\mathfrak{A}, H, D)$.

In the case where the spectral triple (\mathfrak{A}, H, D) has discrete dimension spectrum Σ , there is a local formula for the cyclic cohomology Chern character, analogous to the local formula for the index in the space case. This is obtained by producing **Hochschild** representatives as (cf. [28], [50]):

$$\varphi(a_0, a_1, \dots, a_n) = \text{tr}_\omega(a_0[D, a_1] \cdots [D, a_n]|D|^{-n}).$$

Infinite dimensional geometry. The main difficulty in constructing specific examples of spectral triples is in producing a (Dirac-like) operator D with bounded commutators with elements of \mathfrak{A} , and also a non-trivial index map.

In general, noncommutative spaces do not admit finitely summable spectral triples. For instance, the operator $|D|^{-p}$ is of trace class for some $p > 0$. Obstructions of such spectral triples to exist are analyzed by Connes [26]. It is then useful to consider a weaker notion. Namely, the **θ -summable** spectral triples come up and satisfies the property as $\text{tr}(e^{-tD^2}) < \infty$ for any $t > 0$. Such spectral triples may be thought of as infinite dimensional noncommutative geometry. There are some examples of spectral triples that are not finitely summable but θ -summable, because of the growth rate of multiplicities of eigenvalues of D or D^2 .

Spectral triples and Morita equivalence. Let (\mathfrak{A}, H, D) be a spectral triple. Assume that \mathfrak{A} is Morita equivalent to a C^* -algebra \mathfrak{B} , implemented by a \mathfrak{B} - \mathfrak{A} bimodule \mathfrak{M} , which is a finite projective (left and right) Hilbert \mathfrak{B} - \mathfrak{A} module. The spectral triple (\mathfrak{A}, D, H) can be transferred to another by using this Morita equivalence as follows.

First consider the (left) \mathfrak{A} -module (corrected) Ω_D^1 generated by the elements $a_1[D, a_2]$ for any $a_1, a_2 \in \mathfrak{A}$. Define a **connection** as $\nabla : \mathfrak{M} \rightarrow \mathfrak{M} \otimes_{\mathfrak{A}} \Omega_D^1$ a linear mapping by requiring that

$$\nabla(\xi a) = (\nabla \xi)a + \xi \otimes [D, a], \quad \xi \in \mathfrak{M}, a \in \mathfrak{A},$$

as a derivation rule. Also require that

$$\langle \xi_1, \nabla \xi_2 \rangle_{\mathfrak{A}} - \langle \nabla \xi_1, \xi_2 \rangle_{\mathfrak{A}} = [D, \langle \xi_1, \xi_2 \rangle_{\mathfrak{A}}]$$

as compatibility of ∇ with $[D, \cdot]$.

◇ For instance, as a possible computation, for $\xi_1 = b_1 c_1 a_1$ and $\xi_2 = b_2 c_2 a_2$ in $\mathfrak{M} = \overline{\mathfrak{B}\mathfrak{C}\mathfrak{A}}$ for a C^* -algebra \mathfrak{C} ,

$$\begin{aligned} \langle \xi_1, \nabla \xi_2 \rangle_{\mathfrak{A}} - \langle \nabla \xi_1, \xi_2 \rangle_{\mathfrak{A}} &= \langle b_1 c_1 a_1, b_2 c_2 a_2 \otimes a_1''[D, a_2''] \rangle - \langle b_1 c_1 a_1 \otimes a_1'[D, a_2'], b_2 c_2 a_2 \rangle \\ &= a_1^* c_1^* b_1^* b_2 c_2 a_2 \otimes a_1''[D, a_2''] - [D, a_2']^* (a_1')^* \otimes a_1^* c_1^* b_1^* b_2 c_2 a_2 \\ &= \langle \xi_1, \xi_2 \rangle_{\mathfrak{A}} a_1''[D, a_2''] - [D, a_2']^* (a_1')^* \langle \xi_1, \xi_2 \rangle_{\mathfrak{A}} \\ &= D \langle \xi_1, \xi_2 \rangle_{\mathfrak{A}} - \langle \xi_1, \xi_2 \rangle_{\mathfrak{A}} D = [D, \langle \xi_1, \xi_2 \rangle_{\mathfrak{A}}], \end{aligned}$$

where the compatibility requires those equalities by choosing those dashed elements. \square

There is an **induced** spectral triple

$$(\mathfrak{B}, \mathfrak{M} \otimes_{\mathfrak{A}} H, \nabla \otimes^{\sim} D),$$

where the action of \mathfrak{B} on $\mathfrak{M} \otimes_{\mathfrak{A}} H$ a Hilbert space is defined as

$$b(\xi \otimes_{\mathfrak{A}} \eta) = (b\xi) \otimes_{\mathfrak{A}} \eta, \quad b \in \mathfrak{B}, \xi \in \mathfrak{M}, \eta \in H,$$

and the induced Dirac operator $\nabla \otimes^{\sim} D$ by the connection ∇ is defined as

$$(\nabla \otimes^{\sim} D)(\xi \otimes_{\mathfrak{A}} \eta) = \xi \otimes D(\eta) + (\nabla \xi) \eta \in \mathfrak{M} \otimes_{\mathfrak{A}} H.$$

Note that we need to assume that the connection ∇ is Hermitian, because commutators $[D, a]$ for $a \in \mathfrak{A}$ may be non-trivial, and hence $1 \otimes D$ would not be well defined on $\mathfrak{M} \otimes_{\mathfrak{A}} H$.

K-theory for C^* -algebras. The K-theory groups for C^* -algebras, as in a homology theory for C^* -algebras, are important invariants that capture information on the topology of C^* -algebras as noncommutative spaces, as like that the topological K-theory groups for spaces, as a cohomology theory for space, capture information on the topological of spaces (cf. [11], [154]).

The K_0 -group $K_0(\mathcal{A})$ of a (untal) algebra \mathcal{A} is defined to be an abelian group of suitable equivalence classes of idempotents of matrix algebras over \mathcal{A} with suitable operations. For a (unital) C^* -algebra \mathfrak{A} , $K_0(\mathfrak{A})$ is defined similarly by self-adjoint idempotents, i.e., projections.

Two projections p and q of a C^* -algebra \mathfrak{A} are (Murray-) **von Neumann** (vN) equivalent, denoted as $p \sim q$ if there is a partial isometry $v \in \mathfrak{A}$ such that $p = v^*v$ and $q = vv^*$.

Any idempotent $p = p^2$ in \mathfrak{A} may be replaced with a projection $pp^*(1 - (p - p^*)^2)^{-1}$ (?), preserving the von Neumann equivalence.

◇ For, $z \in \mathbb{C}$, $q(z) = |z|^2(1 - |z - \bar{z}|^2)^{-1} = \overline{q(z)}$ is a projection? If so, for $z \in \mathbb{R}$, $|z|^2 = |z|^4$. Thus, $z = 0$ or 1 . Hence, $q(z)$ is a projection on $\{0, 1\}$. But the functional calculus can be applied to what? As another possible candidate, as an attempt, may set $q = -(p - p^*)^2 = (p - p^*)^*(p - p^*)$. Then $q = q^*$, and

$$\begin{aligned} q^2 &= (p - p^*)^4 = (p^2 - pp^* - p^*p + (p^*)^2)^2 \\ &= (p(1 - p^*) + p^*(1 - p))^2 = (p(1 - p^*))^2 + (p^*(1 - p))^2 \\ &= (p(p^* - 1))^2 + (p^*(p - 1))^2 = p(p^* - 1) + p^*(p - 1) = q, \end{aligned}$$

provided that $(p(p^* - 1))^2 = p(p^* - 1)$ and $(p^*(p - 1))^2 = p^*(p - 1)$. This holds if $(pp^*p - 2p)(1 - p^*) = 0$ and $(p^*pp^* - 2p^*)(1 - p) = 0$, which holds if p is a partial isometry with $p(1 - p^*) = 0$ as being orthogonal. Note as well that $1 + q = 1 + (p - p^*)^*(p - p^*) \geq 1$, so that $1 + q$ is invertible. \square

For p, q projections of matrix algebras over \mathfrak{A} , they are **stably** equivalent if there is a projection r of a matrix algebra over \mathfrak{A} such that diagonal sums $p \oplus r \sim q \oplus r$. Define $K_0(\mathfrak{A})_+$ to be the monoid (or additive semi-group) of (stably or vN) equivalent classes of projections of matrix algebras over \mathfrak{A} (unital), with operation as $[p] \oplus [q] = [p \oplus q]$. Let $K_0(\mathfrak{A})$ be the Grothendieck group of $K_0(\mathfrak{A})_+$

The K_1 -group $K_1(\mathfrak{A})$ of a unital (algebra or) C^* -algebra \mathfrak{A} is defined to be the inductive limit of quotient groups

$$\begin{aligned} \varinjlim GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0, \quad \text{induced by} \\ GL_n(\mathfrak{A}) \rightarrow GL_{n+1}(\mathfrak{A}), \quad x \mapsto x \oplus 1 \text{ the diagonal sum,} \end{aligned}$$

where $GL_n(\mathfrak{A})$ is the group of invertible $n \times n$ matrices over \mathfrak{A} and $GL_n(\mathfrak{A})_0$ is the connected component of $GL_n(\mathfrak{A})$ with the identity matrix, with the multiplicative operation as $[u] \cdot [v] = [uv] = [u \oplus v]$, so that $K_1(\mathfrak{A})$ is abelian even if the quotients $GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$ are not.

However, the K-theory groups for C^* -algebras are not always computable or determined, but certainly computed in many important cases. On the other hand, there are two fundamental developments in noncommutative geometry, as faced with difficulty in determining K-theory groups at certain cases such as irrational rotation C^* -algebras (cf. [140]).

The first development is the theory of cyclic cohomology, introduced by Connes (cf. [24], [25], [27]), to provide cocycles that pairs with K-theory groups as Chern characters. The second is the K-theory geometrically refined by Baum-Connes (BC2000) [9], to produce the assembly map

$$\mu : K^*(X, G) \rightarrow K_*(C_0(X) \rtimes G)$$

for $* = 0, 1$, from the geometric K-theory for a space X (non-compact, or not) with an action by a group G to the analytic K-theory for the C^* -algebra crossed product $C_0(X) \rtimes G$ (which may be viewed as the G -equivariant K-theory $K_*^G(C_0(X))$, for instance, when G is a compact group). The Baum-Connes conjecture says that the assembly map is an isomorphism between the (topological) K-theory for spaces and the (topological) K-theory for C^* -algebras.

◊ It says that at the topological level, such spaces or dynamical systems may be identified with the corresponding crossed product C^* -algebras as NC spaces. \square

For more details on the BC conjecture, for instance, may refer to [149].

1.4 Why NCG?

Table 4: The reasonable, sensible dictionary for why NCG?

Sense	Space geometry	Noncommutative geometry
1	Untouchable boundary in (quotient) spaces	Part in one, complete quantum boundary as C^* -algebras
2	Dynamical systems by homeomorphisms	Crossed product C^* -algebras by automorphisms (or endo-)
3	Coordinates as variables	Operators as NC variables
4	Space invariants such as K-theory and (co)homology with Chern character	C^* -algebra analogues such as $K(K)$ -theory and (cyclic) cohomology with Connes character
5	Classical mechanics	Quantum (statistical) mechanics
6	Visual type in a sense	Deformed type in some sense

The specific reasons are explained as in the following.

As one direction in which noncommutative geometry is applied to number theory, so featured is the explicit class field theory. It is shown by Bost and Connes [13] that the Galois theory of the cyclotomic field \mathbb{Q}^{Ab} is live naturally as symmetries of zero temperature equilibrium states of a quantum statistical mechanical system (QSMS). Moreover, it is shown by Connes and Marcolli (CM) that the similar noncommutative space with a natural time evolution gives rise to zero temperature equilibrium states whose symmetries are given by the automorphisms of the modular field. Furthermore, it is shown by CM and Ramachandran that these constructions fit into a possible theory of noncommutative Shimura varieties, and exhibited is a related QSMS that recovers the explicit class field theory for imaginary quadratic fields. It opens up a concrete possibility that noncommutative geometry may be able to deal with the real quadratic fields or other cases where the Hilbert 12th problem of explicit class field theory (ECFT) is not yet fully understood from a number theoretic point of view. The connections between the H12 problem for real quadratic fields and NG are considered by Manin [109] and [110]. Described later on are these results as well as perspectives.

As another direction in which noncommutative geometry may yield contributions to number theory and arithmetic geometry, so featured are the L -functions. It is Connes [29] who first brings its possibility to light on the spec-

tral realization of zeros of the Riemann zeta function and of L -functions with Grossencharakter. This work provides a new approach to the generalized Riemann hypothesis through a Selberg trace formula for the action of the idèle class group on a noncommutative space obtained as a quotient of the space of adèles. The noncommutative space involved in this construction is closely related to the Bost-Connes system, up to a duality.

As a different direction, it is shown by Consani and Marcolli that the techniques in NG may help describing the geometry of the fibers at infinity of arithmetic varieties and the corresponding Gamma factors contributing to the L -functions. Described later on are these results.

These approaches may lead to some other results on L -functions of arithmetic varieties.

As another striking example as the NG leading to new conceptual explanations or viewing on classical number theoretic objects, Connes and Moscovici [54], [55] give the modular Hecke algebras that recover and extend the structures like the Rankin-Cohen brackets of modular forms [159], in terms of Hopf algebra symmetries of noncommutative spaces and their Hopf cyclic cohomology. In the similar spirit, the results of Manin and Marcolli discussed soon later recover arithmetic properties of modular curves from the corresponding noncommutative boundary, which opens the possibility of similar constructions on the boundary of other moduli spaces of arithmetic significance.

This subject may be still a building site in rapid development. The results collected and the perspectives may present a unified picture that outlines the shape of the emerging landscape, so as to be called arithmetic noncommutative geometry (ANCG). To be continued, so far as now.

2 Noncommutative modular curves

The results of this section are mostly based on Yuri Manin and Matilde Marcolli [112], with additional material from Marcolli [113] and [115] (not at hand). Included are necessary preliminary notions on modular curves based on Manin [103] and those on noncommutative tori based on Connes [22] and Rieffel [140].

2.1 Modular curves

Let G be a finite index subgroup of the (projective) (elliptic) modular group $\Gamma = PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1_2\}$, where 1_2 is the 2×2 identity matrix in $GL_2(\mathbb{C})$ or $SL_2(\mathbb{Z})$, with $[1_2] = 1_\Gamma$ the unit class. Denote by X_G the quotient space $G \backslash \mathbb{H}^2$ as the **modular** curve, where \mathbb{H}^2 is the real 2-dimensional hyperbolic plane, namely the upper half plane $\{z \in \mathbb{C} \mid \text{Im}(z) > 0\} = \mathbb{R} \times \mathbb{R}_+$ with the metric $ds^2 = |dz|^2 (\text{Im}(z))^{-2}$. Equivalently, we may identify \mathbb{H}^2 with the Poincaré disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ with the metric $ds^2 = 4|dz|^2 (1 - |z|^2)^{-2}$.

Proof. (Added). May refer to [151]. Let $\mathbb{C}^+ = \mathbb{C} \cup \{\infty\} \approx S^2$ be the Riemann (or real 2-dimensional) sphere. Any element of $GL_2(\mathbb{C})$ defines the linear **fractional**

transformation of \mathbb{C}^+ as

$$f_g(z) = g(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}, \quad z \in \mathbb{C} \quad \text{and} \quad g(\infty) = \begin{cases} \frac{a}{c} & \text{if } ac \neq 0, \\ 0 & \text{if } a = 0, \\ \infty & \text{if } c = 0. \end{cases}$$

◇ Note that if $c = 0$, then $f_g(z) = \frac{a}{d}z + \frac{b}{d}$, and if $c \neq 0$, then

$$f_g(z) = \frac{a}{c} - \frac{(ad - bc)c^{-2}}{z + \frac{d}{c}}.$$

As well, suppose that $g \in SL_2(\mathbb{Z})$ with $ad - bc = 1$. If $c = 0$, then either $a = d = 1$ or $a = d = -1$. If $c \neq 0$, then

$$\frac{1}{z + \frac{d}{c}} = \frac{1}{\operatorname{Re}(z) + \frac{d}{c} + i\operatorname{Im}(z)} = \frac{\operatorname{Re}(z) + \frac{d}{c} - i\operatorname{Im}(z)}{|z + \frac{d}{c}|^2}.$$

It then follows that if $\operatorname{Im}(z) > 0$, then $\operatorname{Im}(f_g(z)) > 0$, where $g \in SL_2(\mathbb{Z})$. \square

If $g, h \in GL_2(\mathbb{C})$, then $f_{gh} = f_g \circ f_h$ and $(f_g)^{-1} = f_{g^{-1}}$.

◇ If $g \in SL_2(\mathbb{Z})$, then $-g \in SL_2(\mathbb{Z})$ and $f_g = f_{-g}$. Thus, we may define $f_{[g]} = f_g$ for $[g] \in PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1_2\}$. \square

◇ Note as well that for $z = x + iy$,

$$\begin{aligned} ds^2 &= |dz|^2 (\operatorname{Im}(z))^{-2} = (dx + idy)(dx - idy)(\operatorname{Im}(z))^{-2} = (dx^2 + dy^2)(\operatorname{Im}(z))^{-2} \\ &= \langle ((\operatorname{Im}(z))^{-2} \oplus (\operatorname{Im}(z))^{-2})(dx, dy)^t, (dx, dy)^t \rangle \equiv Q_{\mathbb{H}^2}(dz) \end{aligned}$$

on \mathbb{H}^2 . And on the Poincaré disk D ,

$$\begin{aligned} ds^2 &= 4|dz|^2(1 - |z|^2)^{-2} = (dx^2 + dy^2)4(1 - |z|^2)^{-2} \\ &= \langle (4(1 - |z|^2)^{-2} \oplus 4(1 - |z|^2)^{-2})(dx, dy)^t, (dx, dy)^t \rangle \equiv Q_D(dz). \quad \square \end{aligned}$$

The **Cayley** transform is the holomorphic isomorphism from \mathbb{H}^2 to D defined by

$$Cl(z) = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} z = \frac{z - i}{z + i} \in D.$$

If fact, for $z \in \mathbb{C}$, $1 - |Cl(z)|^2 = \frac{4\operatorname{Im}(z)}{|z+i|^2}$. Hence $Cl(\mathbb{H}^2) = D$, $Cl(-\mathbb{H}^2 \setminus \{-i\}) = \overline{D}^c$ the complement of the closure of D , and $Cl(\mathbb{R}) = S^1$ the unit 1-sphere (or circle).

◇ As a note, $(Cl)^*Cl = 2 \oplus 2$ the diagonal sum, and

$$\frac{4}{(1 - |Cl(z)|^2)^2} = \frac{|z + i|^4}{4\operatorname{Im}(z)^2} = \frac{(x^2 + (y + 1)^2)^2}{4\operatorname{Im}(z)^2}. \quad \square$$

(The end of the proof.) \square

Remark. (Added). May refer to [116]. Any **conformal** mapping between the unit open disk D in \mathbb{C} is given as $w = \varepsilon(z - z_0)(1 - \bar{z}_0 z)^{-1}$ with $|\varepsilon| = 1$, $|z_0| < 1$, so that $|dw|(1 - |w|^2)^{-1} = |dz|(1 - |z|^2)^{-1}$ which is called as Poincaré's differential invariant. The disk D with line element $ds = |dz|(1 - |z|^2)^{-1}$ as another Poincaré metric is a Lobachevskii's non-Euclidean space.

◊ In fact, we have $1 - |w|^2 = 1 - |z - z_0|^2 |1 - \bar{z}_0 z|^{-2}$. In particular, if $z_0 = 0$, then $1 - |w|^2 = 1 - |z|^2$. $\square \blacktriangleleft$

Let $\varphi : \mathbb{H}^2 \rightarrow X_G$ be the quotient map. Denote by \mathbb{P} the coset space Γ/G (which may be trivial). Then X_G can be identified with the quotient $\Gamma \backslash (\mathbb{H}^2 \times \mathbb{P})$.

◊ It says that the element $\Gamma(z, hG) \supset (Gz, hG)$ is identified with $\varphi(z) = Gz$. \square

The action of $PSL_2(\mathbb{Z})$ by fractional linear transformations is written in terms of the generators: for $z = x + iy \in \mathbb{H}^2$,

$$S(z) = -\frac{1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z = \frac{-x + iy}{|z|^2} \quad \text{and} \quad T(z) = z + 1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z$$

as inversion (with respect to both y -axis and the unit circle) and translation (along x -axis), with $[S]^2 = [S^2] = [-1_2]$ in $PSL_2(\mathbb{Z})$. In particular, $S(i) = i$, $S(iy) = iy^{-1}$, as well as $S(e^{i\theta}) = e^{i(\pi - \theta)}$ for $\theta \in \mathbb{R}$.

Also, $PSL_2(\mathbb{Z})$ can be identified with the free product group $\mathbb{Z}_2 * \mathbb{Z}_3$, with generators $\sigma = [S]$ and $\tau = [ST]$, where

$$ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad (ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (ST)^3 = -1_2.$$

The tessellation of the hyperbolic plane \mathbb{H}^2 by fundamental domains for the action of $PSL_2(\mathbb{Z})$ (by S and T) can be illustrated but not included.

◊ May let $G = \Gamma = PSL_2(\mathbb{Z})$. For instance, $\Gamma\{i\}$ contains $n + i$ and $\frac{-n+i}{n^2+1}$ for $n \in \mathbb{Z}$, and more. \square

Remark. (Added). May refer to [116]. A group Γ acting on a locally compact Hausdorff space X is said to be a **discontinuous** group if the action of Γ on X is properly discontinuous in the sense that for any compact subset S of X , there are only finitely many elements $\gamma \in \Gamma$ such that $\gamma S \cap S \neq \emptyset$.

A **fundamental** domain for a discontinuous group Γ acting on X is defined to be a complete representative set F for the quotient space $\Gamma \backslash X$ such that $\Gamma F = X$ and $\gamma F \cap F = \emptyset$ for any $\gamma \in \Gamma \setminus \{1_\Gamma\}$.

The decomposition $X = \cup_{\gamma \in \Gamma} \gamma F$ as a union is said to be the **tessellation** by a fundamental domain F .

The case of the Riemann sphere \mathbb{C}^+ . The group $\text{Aut}_a(\mathbb{C}^+)$ of analytic automorphisms on \mathbb{C}^+ consists of complex, special, linear fractional transformations f_g for $g \in SL_2(\mathbb{C})$, and is identified with the semi-simple Lie group $PSL_2(\mathbb{C})$. Since \mathbb{C}^+ is compact, a discontinuous group on \mathbb{C}^+ is a finite group, viewed as a finite subgroup of the motion group $SO(3)$ of the real 2-dimensional sphere, and hence either a cyclic group or a regular polyhedral group.

The case of the disk D identified with \mathbb{H}^2 . The group $\text{Aut}_a(\mathbb{H}^2)$ consists of real, special, linear fractional (Möbius) transformations f_g for $g \in SL_2(\mathbb{R})$, and

is identified with the semi-simple Lie group $PSL_2(\mathbb{R}) = G$. The group G acts on \mathbb{H}^2 transitively and its stabilizer group for $i \in \mathbb{H}^2$ is $PSO(2) = K$, so that $\mathbb{H}_2 \cong G/K$. Since K is compact, a discontinuous group Γ on \mathbb{H}^2 is given by a discrete subgroup of G .

Moreover, the action of Γ on \mathbb{H}^2 extends analytically to $\mathbb{C}^+ \approx S^2$. A **limit** point of Γ is defined to an accumulation point of \mathbb{C}^+ for the orbit Γz for some $z \in \mathbb{C}^+$ and is contained in $\mathbb{R} \cup \{\infty\} = \mathbb{R}^+ \approx S^1$ the unit sphere. The set L of limit points of Γ is either one or two points, or an infinite set, and is either $\mathbb{R} \cup \{\infty\}$ or nowhere dense (in other words, \bar{L} is a border set in the sense that its complement \bar{L}^c is dense in \mathbb{R}^+ or \mathbb{C}^+).

A **fundamental** (Dirichlet) domain F for the action of Γ is defined to be a (curved) polygon with boundary composed by (curved) edges as arcs (geodesics in \mathbb{H}^2) or line segments, orthogonal to the real axis, so that equivalent edges of the polygon F and their relations correspond to generators and relations of Γ . The number of edges of F is finite if and only if Γ is finitely generated. In this case, Γ is said to be a **Fuchsian** group. A Fuchsian group is a finitely generated, discrete subgroup of $PSL_2(\mathbb{R})$. Also, a **Kleinian** group is defined to be a discrete subgroup of $PSL_2(\mathbb{C})$.

A Fuchsian group Γ is said to be of the **first kind** if $L = \mathbb{R} \cup \{\infty\} \approx \mathbb{P}^1(\mathbb{R})$ the real 1-dimensional projective line, and be of the second kind if otherwise. Being of the first kind is equivalent to that $\Gamma \backslash \mathbb{H}^2$ has volume finite. A point $x \in \mathbb{R}^+$ is said to be a (parabolic) **cusps** if the stabilizer group Γx consists of parabolic transformations such as T and its conjugates. On the other hand, a fixed point $z \in \mathbb{H}^2$ under Γ is said to be an **elliptic** point since the stabilizer group Γz is a finite cyclic groups of elliptic transformations with respect to K .

The (elliptic) modular group $\Gamma = SL_2(\mathbb{Z})$ is a Fuchsian group of the first kind. The set of cusps of Γ is $\mathbb{Q} \cup \{\infty\} = \mathbb{Q}^+ \approx \mathbb{P}^1(\mathbb{Q})$ in $L = \mathbb{R}^+$. The point $i = \sqrt{-1}$ is an elliptic point for Γ . ◀

Table 5: The simply connected Riemann surfaces X

X	Elliptic	Parabolic	Hyperbolic
Space name	Riemann sphere	Gauss plane	Open unit disk
Definition formula	$\mathbb{C}^+ = \mathbb{C} \cup \{\infty\}$	Complex \mathbb{C}	$D = \{z \in \mathbb{C} \mid z < 1\}$
Analytic $\text{Aut}_a(X)$	$PSL_2(\mathbb{C})$	SDP $\mathbb{C} \rtimes \mathbb{C}^*$	Real $PSL_2(\mathbb{R})$
Discrete subgroups	Kleinian	Discrete $az + b$	Fuchsian

(SDP = semi-direct product (group)).

Example 2.1. (Edited). An example of finite index subgroups G of Γ is given by the congruence subgroups $\Gamma_0(N)$ of Γ :

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\}.$$

A fundamental domain for $\Gamma_0(N)$ is given by F and $\cup_{k=0}^{N-1} ST^k(F)$, where F is a fundamental domain for Γ .

◊ Note that

$$ST^k z = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix} z = \frac{-1}{z+k}.$$

As well, in $\Gamma_0(N)$,

$$\begin{pmatrix} a & b \\ kN & d \end{pmatrix} \begin{pmatrix} a' & b' \\ k'N & d' \end{pmatrix} = \begin{pmatrix} aa' + bk'N & ab' + bd' \\ (ka' + dk')N & kb'N + dd' \end{pmatrix}$$

and $\begin{pmatrix} a & b \\ kN & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -kN & a \end{pmatrix}$. ◻ ◀

The quotient space X_G has the structure as a non-compact Riemann surface. This has a natural algebro-geometric compactification, by adding the cusp points as points at infinity. The space of **cusps** points is identified with the quotient

$$G \backslash \mathbb{P}^1(\mathbb{Q}) \cong \Gamma \backslash (\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}),$$

where $\mathbb{P}^1(\mathbb{R}) \supset \mathbb{P}^1(\mathbb{Q}) \approx \mathbb{Q}^+$. Thus, define the compactification of X_G as

$$\overline{X_G} = G \backslash (\mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q})) \cong \Gamma \backslash ((\mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q})) \times \mathbb{P}).$$

Remark. (Added). The union $\mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q})$ may be replaced by $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \mathbb{Q} \cup \{i\infty\}$ (cf. [145, 3.2]). ◀

The modular curve X_Γ for $\Gamma = SL_2(\mathbb{Z})$ is the moduli space of elliptic curves, with the point $\tau \in \mathbb{H}^2$ parameterizing the lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ in \mathbb{C} and the corresponding elliptic curve uniformized by $E_\tau = \mathbb{C}/\Lambda$.

The unique cusp point corresponds to the degeneration of the elliptic curve to the cylinder \mathbb{C}^* , when τ goes to ∞ in \mathbb{H}^2 .

The other modular curves, obtained as quotients X_G by congruence subgroups, can also be interpreted as moduli spaces of elliptic curves with level structure. Namely, for **elliptic** curves $E = \mathbb{C}/\Lambda$, this is an additional information on the torsion points $\frac{1}{N}\Lambda/\Lambda \subset \mathbb{Q}\Lambda/\Lambda$ for some level N .

Example 2.2. (Edited). The principal **congruence** subgroups $\Gamma(N)$ of $\Gamma = PSL_2(\mathbb{Z})$ consists of matrices $M \in \Gamma$ such that $M \equiv 1_2 \pmod{N}$. Points in the modular curve $\Gamma(N) \backslash \mathbb{H}^2$ classify elliptic curves $E_\tau = \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, together with a basis $\{\frac{1}{N}, \frac{\tau}{N}\}$ for the torsion subgroup $\frac{1}{N}\Lambda/\Lambda$. The projection from $X_{\Gamma(N)}$ to X_Γ forgets the extra structure.

In the case of the groups $\Gamma_0(N)$, points in the quotient space $\Gamma_0(N) \backslash \mathbb{H}^2$ classify elliptic curves, together with a cyclic subgroup of E_τ of order N . This extra information is equivalent to an isogeny $\varphi : E_\tau \rightarrow E_{\tau'}$, where the cyclic group is the kernel $\ker(\varphi)$. Recall that an isogeny is a morphism $\varphi : E_\tau \rightarrow E_{\tau'}$ such that $\varphi(0) = 0$. These are implemented by the action of $GL_2^+(\mathbb{Q})$ on \mathbb{H}^2 . Namely, E_τ and $E_{\tau'}$ are **isogenous** if and only if τ and τ' in \mathbb{H}^2 are in the same orbit of $GL_2^+(\mathbb{Q})$. ◀

Modular symbols. Given two points or cusps $\alpha, \beta \in \mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q}) \equiv \overline{\mathbb{H}^2}$, the real homology class $[\alpha, \beta]_G \in H_1(\overline{X}_G, \mathbb{R})$ (corrected) as a **modular symbol** is defined as

$$[\alpha, \beta]_G = \int_{\alpha}^{\beta} \varphi^*(\omega), \quad \varphi^*(\omega) = \omega \circ \varphi, \quad \varphi: \mathbb{H}^2 \rightarrow X_G = G \backslash \mathbb{H}^2$$

as a functional with variable ω holomorphic differentials (of the first kind) on X_G , and the integration of the pull back to \mathbb{H}^2 is along the geodesic arc connecting α and β .

The functionals above as modular symbols in $H_1(\overline{X}_G, \mathbb{R})$ (for α, β cusps) are in fact in (rational) $H_1(\overline{X}_G, \mathbb{Q})$ (by Manin [103] and Drinfel'd [72]).

The modular symbols satisfy the following additivity and invariance properties:

$$[\alpha, \beta]_G + [\beta, \gamma]_G = [\alpha, \gamma]_G \quad \text{and} \quad [g\alpha, g\beta]_G = [\alpha, \beta]_G \quad g \in G.$$

◇ Indeed, we have

$$\begin{aligned} \int_{\alpha}^{\beta} \varphi^*(\omega) + \int_{\beta}^{\gamma} \varphi^*(\omega) &= \int_{\alpha}^{\gamma} \varphi^*(\omega), \\ \int_{g\alpha}^{g\beta} \varphi^*(\omega) &= \int_{\alpha}^{\beta} \omega \circ \varphi \circ g = \int_{\alpha}^{\beta} \omega \circ \varphi. \quad \square \end{aligned}$$

By additivity, (for cusps) it is sufficient to consider modular symbols of the form $[0, \alpha]$, with $\alpha \in \mathbb{Q}$.

◇ Because $[\alpha, \beta]_G = [\alpha, 0]_G + [0, \beta]_G = -[0, \alpha]_G + [0, \beta]_G$. □

Moreover, (detailed)

$$\begin{aligned} [0, \alpha]_G &= \sum_{k=1}^n \left[\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k} \right]_G = \sum_{k=1}^n \int_{p_{k-1}q_k^{-1}}^{p_k q_k^{-1}} \varphi^* \omega \\ &= - \sum_{k=1}^n \int_{g_k(0)}^{g_k(i\infty)} \varphi^* \omega = - \sum_{k=1}^n [g_k(0), g_k(i\infty)]_G \end{aligned}$$

where any rational α has the finite, **continued fraction** expansion

$$\alpha = [a_0, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_n}}},$$

and (detailed)

$$\begin{aligned} g_k &= g_k(\alpha) = \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix} \in GL_2(\mathbb{Z}), \\ g_k(z) &= \frac{p_{k-1}z + p_k}{q_{k-1}z + q_k}, \quad (z \in \mathbb{C}^+), \quad g_k(0) = \frac{p_k}{q_k}, \quad g_k(\infty) = g_k(i\infty) = \frac{p_{k-1}}{q_{k-1}} \end{aligned}$$

with

$$\frac{p_k}{q_k} \equiv \frac{p_k(\alpha)}{q_k(\alpha)} = [a_0, a_1, \dots, a_k] \quad (0 \leq k \leq n)$$

the successive, reduced fractional approximations for $\alpha = \frac{p_n}{q_n}$.

Remark. (Added). May recall the following from [116]. For any real number α , let $\alpha = \lfloor \alpha \rfloor + \frac{1}{b_1}$ with $a_0 = \lfloor \alpha \rfloor$ (Gauss symbol) the largest integer no more than α , and $b_1 > 1$ if $\alpha \neq \lfloor \alpha \rfloor$. Define inductively

$$b_j = a_j + \frac{1}{b_{j+1}} \quad \text{and} \quad a_j = \lfloor b_j \rfloor, \quad j \geq 1.$$

Then α has the unique simple **continued fraction**, denoted as $\alpha = [a_0, a_1, \dots]$. If α is a rational number, then $\alpha = [a_0, a_1, \dots, a_n]$ for some n . For $0 \leq k \leq n$, we may set

$$[a_0, a_1, \dots, a_k] = \frac{p_k(\alpha)}{q_k(\alpha)} \equiv \frac{p_k}{q_k}$$

the k -th approximate (reduced) fractional for α .

The following recursive formulae hold:

$$p_k(\alpha) = a_k p_{k-1}(\alpha) + p_{k-2}(\alpha) \quad \text{and} \quad q_k(\alpha) = a_k q_{k-1}(\alpha) + q_{k-2}(\alpha),$$

for $k \geq 0$, where we set $p_{-2}(\alpha) = 0$, $p_{-1}(\alpha) = 1$, $q_{-2}(\alpha) = 1$, and $q_{-1}(\alpha) = 0$.

◇ For instance, $p_0(\alpha) = a_0$, $q_0(\alpha) = 1$, and $\frac{p_1}{q_1} = \frac{a_1 p_0 + 1}{a_1 \cdot 1 + 0}$, and

$$\frac{p_2}{q_2} = \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1} = \frac{a_2 p_1 + a_0}{a_2 q_1 + q_0}. \quad \square$$

It then follows that

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}, \quad k \geq -1.$$

Hence

$$\det \begin{pmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{pmatrix} = (-1)^k.$$

◇ For instance,

$$p_1 q_0 - p_0 q_1 = (a_1 a_0 + 1) - a_0 a_1 = 1,$$

$$p_2 q_1 - p_1 q_2 = (a_2 p_1 + a_0) q_1 - p_1 (a_2 q_1 + q_0) = a_0 a_1 - (a_1 a_0 + 1) = -1. \quad \square \quad \blacktriangleleft$$

In the classical theory of the modular symbols of [103] and [121] (cf. also [80], [120]), the cohomology classes obtained from cusp forms are evaluated against the relative homology classes given by modular symbols. Namely, given a **cusp** form Φ_ω on \mathbb{H}^2 , obtained as a pull back $\Phi_\omega = \varphi^*(\omega) \frac{1}{dz}$ under the quotient map $\varphi: \mathbb{H}^2 \rightarrow X_G$ (corrected), we define the **intersection** numbers

$$\Delta_\omega(s) = \int_{g_s(0)}^{g_s(i\infty)} \Phi_\omega(z) dz, \quad g_s G = s \in \mathbb{P}^1(\mathbb{Q}).$$

These intersection numbers can be interpreted in terms of special values of L -functions associated to the automorphic form which determines the cohomology class.

That is rephrased in cohomological terms as follows, following [121]. Denote by $\langle i \rangle$ and $\langle \rho \rangle$ the elliptic points, defined as the orbits $\langle i \rangle = \Gamma i$ and $\langle \rho \rangle = \Gamma \rho$ respectively, where $\rho = e^{\frac{2\pi i}{3}}$ (with $\rho' = e^{\frac{\pi i}{3}}$, corrected, but which may be exchanged if necessary). Let

$$I = \langle \varphi(i) \rangle = G \backslash \langle i \rangle \quad \text{and} \quad R = \langle \varphi(\rho) \rangle = G \backslash \langle \rho \rangle$$

be the image in X_G of the elliptic points.

Set $H_A^B = H_1(\overline{X_G} \setminus A, B; \mathbb{Z})$ the **relative** (singular) homology group for disjoint A and B in $\overline{X_G}$, where either of which may be empty sets.

There is the (intersection) pairing $\bullet : H_A^B \times H_B^A \rightarrow \mathbb{Z}$.

The modular symbols $[g(0), g(i\infty)]_G$ for $gG \in \mathbb{P} = \Gamma/G$ define classes in H^{cp} and generate it by Manin [103], where cp (as well as $\{\text{cp}\}$) denotes the set of cusps.

For σ and τ the generators of $PSL_2(\mathbb{Z})$ with $\sigma^2 = 1$ and $\tau^3 = 1$, set $\mathbb{P}_I = \langle \sigma \rangle \backslash \mathbb{P}$ and $\mathbb{P}_R = \langle \tau \rangle \backslash \mathbb{P}$.

There is an isomorphism $\mathbb{Z}^{|\mathbb{P}|} \cong H_{\text{cp} \cup I}^R$.

There are exact sequences of groups (cf. Remark below detailed and edited):

$$0 \rightarrow H_{\text{cp}} \longrightarrow H_{\text{cp}}^R \xrightarrow{\pi_R} \mathbb{Z}^{|\mathbb{P}_R|} \rightarrow \mathbb{Z} \rightarrow 0$$

and

$$0 \rightarrow \mathbb{Z}^{|\mathbb{P}_I|} \longrightarrow \mathbb{Z}^{|\mathbb{P}|} \cong H_{\text{cp} \cup I}^R \xrightarrow{\pi_I} H_{\text{cp}}^R \rightarrow 0.$$

The image $\pi_I(x^\sim) \in H_{\text{cp}}^R$ of an element $x^\sim = \sum_{s \in \mathbb{P}} \lambda_s s$ in $\mathbb{Z}^{|\mathbb{P}|} \cong H_{\text{cp} \cup I}^R$ with coefficients $\lambda_s \in \mathbb{Z}$ represents an element $x \in H_{\text{cp}}$ if and only if the image $\pi_R(\pi_I(x^\sim)) = 0$ in $\mathbb{Z}^{|\mathbb{P}_R|}$.

As proved in [121], for $s = gG \in \mathbb{P} = \Gamma/G$, the intersection pairing $\bullet : H^{\text{cp}} \times H_{\text{cp}} \rightarrow \mathbb{Z}$ gives, for $x^\sim = \sum_{s \in \mathbb{P}} \lambda_s s$ in $\mathbb{Z}^{|\mathbb{P}|}$, with $x \in H_{\text{cp}}$,

$$[g(0), g(i\infty)]_G \bullet x = \lambda_s - \lambda_{\sigma s}, \quad \sigma \in PSL_2(\mathbb{Z}), \sigma^2 = 1.$$

Thus, may write the intersection number as a function $\Delta_x : \mathbb{P} \rightarrow \mathbb{R}$ defined by $\Delta_x(s) = \lambda_s - \lambda_{\sigma s}$.

Remark. (Added). As obtained in [121], there are group isomorphisms:

$$\mathbb{Z}^{\Gamma/G} \cong H_R^{\text{cp} \cup I} \quad \text{and} \quad \mathbb{Z}^{\Gamma/G} \cong H_{\text{cp} \cup I}^R,$$

which are induced by sending gG to $\varphi(g\gamma) \equiv [g]^0$, with $[1]^0 = I$, and gG to $\varphi(g\delta) \equiv [g]_0$, with $[1]_0 = R$ respectively, where γ is the geodesic between i and $i\infty$, and δ is between ρ' to a point z_0 in \mathbb{H}^2 with $0 < \text{Re}(z_0) < \frac{1}{2}$ and $|z_0| > 1$, to $-\bar{z}_0$, and to ρ . The intersection pairing

$$\bullet : H_R^{\text{cp} \cup I} \times H_{\text{cp} \cup I}^R \rightarrow \mathbb{Z}$$

is given by

$$[g]^0 \bullet [h]_0 = \begin{cases} 1 & \text{if } gG = hG, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$H_0(I, \mathbb{Z}) \cong \mathbb{Z}^{|I|} \cong \mathbb{Z}^{|\mathbb{P}_I|} \quad \text{and} \quad H_0(R, \mathbb{Z}) \cong \mathbb{Z}^{|R|} \cong \mathbb{Z}^{|\mathbb{P}_R|}.$$

There are **long exact** sequences of relative homology groups (modified):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}^{|\mathbb{P}_I|} & \longrightarrow & H_{\text{cp} \cup I} = H_{\text{cp} \cup I}^\emptyset & \xrightarrow{j_*} & H_{\text{cp}} = H_{\text{cp}}^\emptyset & \longrightarrow & 0 \\ & & \parallel & & \downarrow 1-1 \quad j_* & & j_* \downarrow 1-1 & & \\ 0 & \longrightarrow & \mathbb{Z}^{|\mathbb{P}_I|} & \longrightarrow & H_{\text{cp} \cup I}^R \cong \mathbb{Z}^{|\mathbb{P}|} & \xrightarrow[\text{onto}]{\pi_I = j_*} & H_{\text{cp}}^R & \longrightarrow & 0 \\ & & & & \downarrow \partial_* & & \downarrow \pi_R = \partial_* & & \\ 0 & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z}^{|\mathbb{P}_R|} = H_0(R) & \xlongequal{\quad} & \mathbb{Z}^{|\mathbb{P}_R|} = H_0(R) & & \end{array}$$

as well as

$$\begin{array}{ccccccc} 0 & \longleftarrow & \mathbb{Z}^{|\mathbb{P}_I|} \cong H_0(I) & \xleftarrow{q_*} & H_0(\{\text{cp}\} \cup I) & \xleftarrow{i_*} & H_0(\{\text{cp}\}) & \longleftarrow & 0 \\ & & \parallel & & \uparrow \partial_* & & \uparrow \partial_* & & \\ 0 & \longleftarrow & \mathbb{Z}^{|\mathbb{P}_I|} & \xleftarrow{\pi^I} & H^{\text{cp} \cup I} = H_\emptyset^{\text{cp} \cup I} & \xleftarrow{j_*} & H^{\text{cp}} = H_\emptyset^{\text{cp}} & \longleftarrow & 0 \\ & & \parallel & & \text{onto} \uparrow \pi^R = j_* & & j_* \uparrow \text{onto} & & \\ 0 & \longleftarrow & \mathbb{Z}^{|\mathbb{P}_I|} & \longleftarrow & H_R^{\text{cp} \cup I} \cong \mathbb{Z}^{|\mathbb{P}|} & \xleftarrow{j_*} & H_R^{\text{cp}} & \longleftarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^{|\mathbb{P}_R|} & \xlongequal{\quad} & \mathbb{Z}^{|\mathbb{P}_R|} & & \end{array}$$

◀

Remark. Recall from [116] the following. Let X be a topological space and $q \in \mathbb{Z}$ non negative. Denote by $S_q(X)$ the free abelian group with basis as the set of all singular q -dimensional simplexes of X , where a **singular q -simplex** of X is a continuous map f from the q -dimensional standard simplex Δ^q to X , where

$$\Delta^q = \{(t = (t_j)_{j=0}^q \in \mathbb{R}^{q+1} \mid t_j \geq 0, \sum_{j=0}^q t_j = 1)\}.$$

In particular, $\Delta^0 = \{1\}$, $\Delta^1 \approx [0, 1]$, and $\Delta^2 \approx \overline{D}$. Thus,

$$S_q(X) = S_q(X, \mathbb{Z}) = \{\sigma = \sum_{k=1}^l n_k f_k \mid f_k : \Delta^q \rightarrow X, n_k \in \mathbb{Z}, l \in \mathbb{N}\}.$$

The **boundary** operator $\partial_q : S_q(X) \rightarrow S_{q-1}(X)$ (with $S_{-1}(X) = 0$) is defined by

$$\partial_q \sigma = \sum_{k=0}^q (-1)^k \sigma \circ \varepsilon_k, \quad \text{where } \varepsilon_k : \Delta^{q-1} \rightarrow \Delta^q$$

is defined by $\varepsilon_k((t_j)_{j=0}^{q-1}) = (t_0, \dots, t_{k-1}, 0, t_k, \dots, t_{q-1})$. It then holds that $\partial_{q-1} \circ \partial_q = 0$, so that the singular **chain** complex $S(X) = \{S_q(X), \partial_q\}_q$ is defined. The integral singular **homology** (group) for X is defined to be

$$H_*(X) = \{H_q(X)\}_q = \bigoplus_{q=0}^{\infty} H_q(X),$$

where $H_q(X) = H_q(X, \mathbb{Z}) = Z_q(X)/B_q(X)$, where $Z_q(X) = \{c \in S_q(X) \mid \partial_q(c) = 0\}$ the group of q -singular cycles and $B_q(X) = \{c \in S_q(X) \mid c \in \partial_{q+1}(S_{q+1}(X))\}$ the group of q -boundary cycles. Two elements c and $c' \in S_q(X)$ are said to be **homologous** if $c - c' \in B_q(X)$, and then their homology classes $[c] = [c'] \in H_q(X)$.

◇ In particular, $\partial_1 f(1) = f(0, 1) - f(1, 0)$ for $f \in S_1(X)$. Hence, $f \in Z_1(X)$ if and only if the image of f is a closed curve (or a cycle) in X . For $f \in S_2(X)$,

$$\partial_2 f(t_0, t_1) = f(0, t_0, t_1) - f(t_0, 0, t_1) + f(t_0, t_1, 0) \in B_1(X),$$

where the union of the domains of these terms is the boundary of Δ^2 , and

$$\begin{aligned} (\partial_1 \circ \partial_2) f(1) &= (f(0, 0, 1) - f(0, 1, 0)) - (f(0, 0, 1) - f(1, 0, 0)) \\ &\quad + (f(0, 1, 0) - f(1, 0, 0)) = 0. \quad \square \end{aligned}$$

If $g : X \rightarrow Y$ is a continuous map, then it induces the chain map from $S(X)$ to $S(Y)$ as well as the induced homomorphism $g_* : H_*(X) \rightarrow H_*(Y)$.

The homology (group or theory) for X has homotopy invariance in the sense that if X and Y are homotopy equivalent as a space, then $H_*(X) \cong H_*(Y)$.

If X is contractible, then $H_0(X) = \mathbb{Z}$ and $H_q(X) = 0$ for $q \geq 1$.

If X is path-connected, then $H_0(X) = \mathbb{Z}$. ◀

Remark. (Added). Further recall from [116] the following. Let X be a topological space and A be a subset of X . The singular chain complex $S(A) = \{S_q(A), \partial_q\}$ of A is viewed as a subcomplex of $S(X) = \{S_q(X), \partial_q\}$. The quotient singular chain complex $S(X)/S(A) = \{S_q(X)/S_q(A), \bar{\partial}_q\}$ is then defined, where

$$\bar{\partial}_q : S_q(X)/S_q(A) \rightarrow S_{q-1}(X)/S_{q-1}(A)$$

is induced by the boundary map ∂_q . The **relative** singular homology (group) $H_*(X, A)$ of such a pair (X, A) is defined to be the homology of the quotient complex $S(X)/S(A)$.

If $f : (X, A) \rightarrow (Y, B)$ is a continuous map of topological pairs such that $f : X \rightarrow Y$ is continuous and $f(A) \subset B$, then it induces the homomorphism $f_* : H_*(X, A) \rightarrow H_*(Y, B)$.

◊ In particular, there is the following group homomorphism:

$$\begin{array}{ccc} H_{A_1 \cup A_2}^B & \xrightarrow{j_*} & H_{A_1}^B \\ \parallel & & \parallel \\ H_1(\overline{X_G} \setminus (A_1 \cap A_2), B) & \xrightarrow{j_*} & H_1(\overline{X_G} \setminus A_1, B) \end{array}$$

induced by j the inclusion map for a pair. □

There is the exact sequence of the chain complexes for a pair (X, A) as

$$0 \rightarrow S(A) \xrightarrow{i} S(X) \xrightarrow{q} S(X)/S(A) \rightarrow 0.$$

It induces the following **long exact** sequence of homology for a pair (X, A) :

$$\dots \xrightarrow{\partial_*} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_*} H_{q-1}(A) \xrightarrow{i_*} \dots$$

where $i : A \rightarrow X$ and $j : (X, \emptyset) \rightarrow (X, A)$ are inclusion maps, and $\partial_*[c] = [\partial c]$.

◊ In particular, there is the following exact sequence:

$$\begin{array}{ccccc} H_1(\overline{X_G} \setminus A) & \xrightarrow{j_*} & H_1(\overline{X_G} \setminus A, B) & \xrightarrow{\partial_*} & H_0(B) \\ \parallel & & \parallel & & \uparrow \partial_* \\ H_A = H_A^\emptyset & \longrightarrow & H_A^B & \longrightarrow & H^B = H_\emptyset^B \quad \square \blacktriangleleft \end{array}$$

The modular complex. For $x, y \in \mathbb{H}^2$, denote by $x \cap y$ the oriented geodesic arc connecting x and y . Let σ and τ be the generators of $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ as a free product group.

Definition 2.3. The **modular** complex is defined to be the cell complex $C = \{C_j, \partial_j\}_{j=0}^2$ defined as follows. The 0-dimensional cells of C_0 are given by the cusps (cp) in $G \setminus \mathbb{P}^1(\mathbb{Q})$ and the elliptic points I and R . The 1-dimensional cells of C_1 are given by the oriented geodesic arcs

$$G \setminus (\Gamma \cdot (i\infty \cap i)) \quad \text{and} \quad G \setminus (\Gamma \cdot (i \cap \rho)),$$

where $\Gamma \cdot$ means the orbit under the action of Γ , and $\rho = e^{\frac{\pi i}{3}}$. The 2-dimensional cells of C_2 are given by classes of $G \setminus \Gamma \cdot E$, where E is the polygon with vertices $i, \rho, 1+i$, and $1+i\infty$ and sides (or edges) as the corresponding geodesic arcs. The **boundary** operator $\partial_2 : C_2 \rightarrow C_1$ is defined by sending

$$gE \mapsto g(i \cap \rho) + g(\rho \cap (1+i)) + g((1+i) \cap i\infty) + g(i\infty \cap i), \quad g \in \Gamma,$$

and the **boundary** map $\partial_1 : C_1 \rightarrow C_0$ is defined by sending

$$g(i\infty \cap i) \mapsto (g \cdot i) - (g \cdot i\infty) \quad \text{and} \quad g(i \cap \rho) \mapsto (g \cdot \rho) - (g \cdot i),$$

and in general, $g(x \cap y) = (g \cdot y) - (g \cdot x)$, and $\partial_0 : C_0 \rightarrow C_{-1} = 0$.

◊ It then follows by cancellation that $\partial_{j-1} \circ \partial_j = 0$ for $j = 2, 1$. We may denote the homology of the modular (cell) complex as $H_*^{cl}(\overline{X}_G)$. \square

That gives a cell decomposition of \mathbb{H}^2 adapted to the action of $PSL_2(\mathbb{Z})$ and congruence subgroups.

It is shown by Manin [103] that

Proposition 2.4. *The (first, only non-trivial) homology of the modular (cell) complex is isomorphic to the first homology of \overline{X}_G :*

$$H_1^{cl}(\overline{X}_G) \cong H_1(\overline{X}_G).$$

The (first **complementary**) relative homology such as $H_A^B = H_1(\overline{X}_G \setminus A, B)$ can be derived from the corresponding relative versions of the modular cell complex.

Indeed, we have $\mathbb{Z}[\{\text{cp}\}] = C_0/\mathbb{Z}[R \cup I]$. Hence, the (modified) quotient complex is induced:

$$0 \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1^\sim} \mathbb{Z}[\{\text{cp}\}] = C_0^\sim \rightarrow 0,$$

where $\partial_1^\sim = q \circ \partial_1 : C_1 \rightarrow C_0 \rightarrow \mathbb{Z}[\{\text{cp}\}]$ with $q : C_0 \rightarrow C_0/\mathbb{Z}[R \cup I]$ the quotient map. It then computes the relative homology $H_1(\overline{X}_G, R \cup I)$.

◊ Note that the kernel $\ker(\partial_1)$ is contained in $\ker(\partial_1^\sim)$, and hence $H_1^{cl}(\overline{X}_G)$ is contained in $H_1^{cl}(\overline{X}_G)^\sim$ the homology corresponding to the modified complex as above. There is also the following exact sequence:

$$H_1(R \cup I) \xrightarrow{i_*} H_1(\overline{X}_G) \xrightarrow{j_*} H_1(\overline{X}_G, R \cup I) \xrightarrow{\partial_*} H_0(R \cup I) \quad \square$$

The cycles (in $\ker(\partial_1^\sim)$) are given by $\mathbb{Z}[\mathbb{P}]$, as combinations of elements $g(i \cap \rho)$ with g ranging over representatives of $\mathbb{P} = \Gamma/G$, and by the elements $\oplus a_g(g(i\infty) \cap g(i))$ satisfying $\sum a_g g(i\infty) = 0$. In fact, these can be represented as relative cycles in $(\overline{X}_G, R \cup I)$.

Similarly, there is the subcomplex

$$0 \rightarrow \mathbb{Z}[\mathbb{P}] \cong \mathbb{Z}^{\Gamma/G} \xrightarrow{\partial} \mathbb{Z}[R \cup I] \rightarrow 0,$$

with $\mathbb{Z}[\mathbb{P}]$ generated by the elements $g(i \cap \rho)$. It computes the homology $H_1(\overline{X}_G \setminus \{\text{cp}\})$.

◊ There is the following sequence:

$$H_1(\overline{X}_G \setminus \{\text{cp}\}) = H_{\text{cp}} \xrightarrow[1 \text{ to } 1]{j_*} H_{\text{cp}}^R \xrightarrow{j_*} H_{\text{cp}}^{R \cup I} \cong \mathbb{Z}[\mathbb{P}] \xrightarrow{\partial_*} H_0(R \cup I). \quad \square$$

The relative homology

$$H_1(\overline{X}_G \setminus \{\text{cp}\}, R \cup I) \cong \mathbb{Z}[\mathbb{P}]$$

is generated by the relative cycles $g(i \cap \rho)$, with g ranging over representatives of \mathbb{P} .

There is a long exact sequence of relative homology (edited as)

$$\begin{array}{ccccccc}
0 \rightarrow H_{\text{cp}} & \longrightarrow & H_{\text{cp}}^{R \cup I} \cong \mathbb{Z}[\mathbb{P}] & \xrightarrow{(\beta_R^{\sim}, \beta_I^{\sim})} & H_0(R) \oplus H_0(I) & \rightarrow \mathbb{Z} \rightarrow 0 \\
\parallel & & \parallel & & \parallel & \\
H_1(\overline{X_G} \setminus \{\text{cp}\}) & \xrightarrow{j_*} & H_1(\overline{X_G} \setminus \{\text{cp}\}, R \cup I) & \xrightarrow{\partial_*} & H_0(R \cup I) \cong \mathbb{Z}[\mathbb{P}_I] \oplus \mathbb{Z}[\mathbb{P}_R] &
\end{array}$$

(with that $\mathbb{Z} \cong H_0(\overline{X_G} \setminus \{\text{cp}\}) = H_0(X_G)$ at the right corner in the first line, because X_G is path-connected, since so is \mathbb{H}^2).

The pairing between $H_{\text{cp}}^{R \cup I}$ and $H_{R \cup I}^{\text{cp}}$ gives the identification of $\mathbb{Z}[\mathbb{P}]$ and $\mathbb{Z}^{|\mathbb{P}|}$, obtained by identifying the elements of \mathbb{P} with the corresponding delta functions.

◇ It says that the identification holds:

$$\mathbb{Z}^{|\mathbb{P}|} \ni (\lambda_{g_j G})_{g_j G, 1 \leq j \leq k} \times (0)_{hG} \Leftrightarrow \sum_{j=1}^k \lambda_{g_j G} \chi_{g_j G} \in \mathbb{Z}[\mathbb{P}], \quad \chi_{g_j G} = g_j G \in \mathbb{P},$$

where $\chi_{g_j G}$ is the characteristic function on $\mathbb{P} = \Gamma/G$ at $g_j G$, which may be identified with the left regular representation ($\lambda_{\Gamma/G}$ usually denoted so) of Γ/G evaluated at $g_j G$, which is also identified with the class $g_j G \in \mathbb{P}$. \square

The long exact sequence above can be rewritten as

$$\begin{array}{ccccccc}
0 \rightarrow H^{\text{cp}} & \longrightarrow & \mathbb{Z}^{|\mathbb{P}|} \cong H_{R \cup I}^{\text{cp}} & \xrightarrow{(\beta_R, \beta_I)} & \mathbb{Z}^{|\mathbb{P}_I|} \oplus \mathbb{Z}^{|\mathbb{P}_R|} & \rightarrow \mathbb{Z} \rightarrow 0 \\
\partial_* \downarrow & & \parallel & & \parallel & \\
H_0(\{\text{cp}\}) & \xleftarrow{\partial_*} & H_1(\overline{X_G} \setminus \{R \cup I\}, \text{cp}) & \xleftarrow{j_*} & H_1(\overline{X_G} \setminus \{R \cup I\}) &
\end{array}$$

In order to understand more explicitly the map (β_R, β_I) , we give the equivalent algebraic formulation of the modular complex as follows (cf. [103], [121]).

Again, the homology group $H_{\text{cp}}^{R \cup I} \cong \mathbb{Z}[\mathbb{P}]$ is generated by the images in $X_G = G \setminus \mathbb{H}^2$ of the geodesic segments $g\gamma_0 \equiv g(i \cap \rho)$, with g ranging over a chosen set of representatives of the coset space $\mathbb{P} = \Gamma/G$.

May identify the dual basis δ_s of $H_{R \cup I}^{\text{cp}} \cong \mathbb{Z}^{|\mathbb{P}|}$ with the images in X_G of the paths $g\eta_0$, where the path η_0 is given by the geodesic arcs connecting $i\infty$ to z_0 , z_0 to τz_0 , and τz_0 to 0. It then holds that the intersection pairing (corrected)

$$\bullet : H_{\text{cp}}^{R \cup I} \times H_{R \cup I}^{\text{cp}} \rightarrow \mathbb{Z}, \quad \begin{cases} [g\gamma_0] \bullet [h\eta_0] = 1 & \text{if } gG = hG, \\ [g\gamma_0] \bullet [h\eta_0] = 0 & \text{if } gG \neq hG. \end{cases}$$

By exactness in the long exact sequence above, it holds that $H^{\text{cp}} \cong \ker(\beta_R, \beta_I)$, which is also obtained by the identification of the modular symbol $[g(0), g(i\infty)]_G$ with $[g\eta_0]$. Therefore, the relations imposed on the generators δ_s by vanishing under β_I correspond to the relations $\delta_s \oplus \delta_{\sigma s}$ (or σs if $s = \sigma s$), and the vanishing under β_R gives another set of relations $\delta_s \oplus \delta_{\tau s} \oplus \delta_{\tau^2 s}$ (or δ_s if $s = \tau s$).

Remark. (Added). May recall from [116] the following (modified). There is the non-degenerate bilinear **intersection** form between homology groups:

$$Q : H_q(M, M_1; \Lambda) \times H_{n-q}(M, M_2; \Lambda) \rightarrow \Lambda,$$

where M is assumed to be an oriented connected n -dimensional manifold and Λ is a unital commutative ring, and $H_q(M) = 0$ if q not in $0 \leq q \leq n$. That intersection form is obtained by composing D and \cap^\sim as follows. The **Poincaré-Lefschetz** (PL) **duality** theorem (or **Poincaré** duality theorem when the boundary $\partial M = \emptyset$) says that there is a duality isomorphism D between homology and relative cohomology groups

$$D : H^q(M, \partial M; \Lambda) \cong H_{n-q}(M; \Lambda),$$

which implies that there is an isomorphism D between relative, homology and cohomology groups

$$D : H^q(M, M_1; \Lambda) \cong H_{n-q}(M, M_2; \Lambda),$$

where M_1 and M_2 are $(n-1)$ -dimensional manifolds such that $\partial M = M_1 \cup M_2$ and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. The homology group $H_n(M, \partial M)$ is isomorphic to \mathbb{Z} , and is generated by the fundamental homology class $[M]$ of M , which is mapped to $[\partial M]$ under the boundary map $\partial_n : H_n(M, \partial M) \rightarrow H_{n-1}(\partial M)$. The PL duality D is defined by $D(\xi) = \xi \cap [M]$.

The **cap** (or intersection) product \cap for a topological space X is defined as

$$\begin{aligned} \cap : H^q(X; \Lambda) \times H_{p+q}(X; \Lambda) &\rightarrow H_p(X; \Lambda), \\ v \cap c &= \sum_i (\sigma_i \circ \varepsilon) \otimes v(\sigma_i \circ \varepsilon') \lambda_i \in S_p(X) \otimes \Lambda \end{aligned}$$

(corrected) for $v \in \text{Hom}(S_q(X), \Lambda)$, $c = \sum_i \sigma_i \otimes \lambda_i \in S_{p+q}(X) \otimes \Lambda$, where $\varepsilon : \Delta^p \rightarrow \Delta^{p+q}$ and $\varepsilon' : \Delta^q \rightarrow \Delta^{p+q}$ are defined by $\varepsilon(t_0, \dots, t_p) = (t_0, \dots, t_q, 0, \dots, 0)$ and $\varepsilon'(t_p, \dots, t_{p+q}) = (0, \dots, 0, t_p, \dots, t_{p+q})$.

With $p = n - q$, the duality D is induced by

$$\cap : H^q(M, \partial M; \Lambda) \times H_n(M, \partial M; \Lambda) \rightarrow H_{n-q}(M, \Lambda).$$

In particular, if $q = n$ and $p = n - q = 0$, then

$$\cap : H^n(M, \partial M, \Lambda) \times H_n(M, \partial M, \Lambda) \rightarrow H_0(M, \Lambda) \cong \Lambda,$$

so that $H^n(M, \partial M, \Lambda) \cong \Lambda$. As well, if $p = 0$, then we can define

$$\cap^\sim : H^q(M, \Lambda) \times H^{n-q}(M, \partial M; \Lambda) \rightarrow H_0(M, \Lambda) \cong \Lambda$$

with $H^{n-q}(M, \partial M, \Lambda) \cong H_q(M, \Lambda)$ by the PL duality. Therefore, it is obtained that

$$Q = (\cap^\sim) \circ (D^{-1}, D^{-1}).$$

Also, if $q = 1$ and $p = 0$, then

$$\cap : H^1(X; \Lambda) \times H_1(X; \Lambda) \rightarrow H_0(X; \Lambda) \cong \Lambda$$

when X is connected, with $H^1(X, \partial X; \Lambda) \cong H_1(X; \Lambda)$ by D if X is a 2-dimensional manifold. The composed map $\cap \circ (D^{-1}, \text{id})$ in this case is just equal to \bullet the intersection product. \blacktriangleleft

Remark. (Added as in [116]). The n -dimensional **homotopy** group $\pi_n(X)$, with some $x \in X$ as the base point, is defined to be the set of all homotopy classes of continuous maps from $(I^n, \partial I^n)$ to (X, x) , with $I = [0, 1]$ and ∂I^n the boundary of I^n .

The n -dimensional **relative** homotopy group $\pi_n(X, A)$, with $x \in A \subset X$, is defined to be the set of all homotopy classes of continuous maps from $(I^n, \partial I^n, J^{n-1})$ to (X, A, x) , where J^{n-1} is the closure of $\partial I^n \setminus (I^{n-1} \times \{0\})$ in ∂I^n . If $n \geq 2$, then it is a group, and if $n \geq 3$, then it is abelian.

There is the **Hurewicz** homomorphism τ from the relative homotopy group $\pi_n(X, A)$ to the homology group $H_n(X, A; \mathbb{Z})$, defined as $\tau([f]) = f_*(\varepsilon_n)$, where ε_n is the generator of $H_n(I^n, \partial I^n)$.

If X is a path-connected space, then the fundamental (or Poincaré) group $\pi_1(X)$ of X has the quotient abelian group $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ isomorphic to $H_1(X, \mathbb{Z})$, by Hurewicz.

The **Hurewicz** isomorphism theorem says that

$$\tau : \pi_n(X, A) \cong H_n(X, A) \quad \text{and} \quad H_i(X, A) = 0, \quad i \leq n - 1,$$

for (X, A) $(n - 1)$ -connected and n -simple, in the respective senses that

$$\pi_0(A) = \pi_0(X) = \pi_i(X, A) = 0 \quad 1 \leq i \leq n - 1$$

and that X and A are path-connected and the action of $\pi_1(A)$ on $\pi_n(X, A)$ is trivial. \blacktriangleleft

2.2 The noncommutative boundary of modular curves

The main idea that bridges between the algebro-geometric theory of modular curves and noncommutative geometry consists of replacing $\mathbb{P}^1(\mathbb{Q})$ in the classical compactification, which gives rise to a finite set of cusps, with $\mathbb{P}^1(\mathbb{R})$. This substitution can be done naively, since the quotient $G \backslash \mathbb{P}^1(\mathbb{R})$ is ill behaved as a topological space, as G does not act on $\mathbb{P}^1(\mathbb{R})$ discretely.

When we regard the quotient $\Gamma \backslash \mathbb{P}^1(\mathbb{R})$, or more generally, $\Gamma \backslash (\mathbb{P}^1(\mathbb{R}) \times \mathbb{P})$, as a noncommutative space, it becomes a geometric object that is rich enough to recover several aspects of the classical theory of modular curves. In particular, it makes sense to study in terms of the geometry of such spaces the limiting behavior for certain arithmetic invariants defined on modular curves when $\tau \rightarrow \theta \in \mathbb{R} \setminus \mathbb{Q}$

Moduli of noncommutative elliptic curves. The boundary $\Gamma \backslash \mathbb{P}^1(\mathbb{R})$ of the modular curve $\Gamma \backslash \mathbb{H}^2$, viewed as a noncommutative space, has a modular

interpretation, as observed originally Connes-Douglas-Schwarz [34]. In fact, we can think of the quotient spaces of the circle S^1 by the actions of irrational rotations $2\pi n\theta$ for $n \in \mathbb{Z}$ by irrational numbers θ , as the noncommutative 2-tori and as particular degenerations of the classical elliptic curves, which are invisible to ordinary algebraic geometry. The quotient space $\Gamma \backslash \mathbb{P}^1(\mathbb{R})$ classifies the noncommutative tori up to Morita equivalence ([22] and [140]) and completes the moduli space $\Gamma \backslash \mathbb{H}^2$ of the classical elliptic curves. Thus, from a conceptual point of view, it is reasonable to think of $\Gamma \backslash \mathbb{P}^1(\mathbb{R})$ as the boundary of $\Gamma \backslash \mathbb{H}^2$, when we allow points as elliptic curves in the classical moduli space to have non-classical degenerations to noncommutative tori.

Noncommutative tori are viewed in a sense as a prototype example of noncommutative spaces, so as to need the full range of techniques of noncommutative geometry (cf. [22] and [24]). Noncommutative tori are irrational rotation algebras as C^* -algebras. Recall some basic properties of noncommutative tori, which justify the claim that the C^* -algebras behave like a noncommutative version of elliptic curves. For this, may follow [22], [27], and [140].

Irrational rotation and Kronecker foliation

Definition 2.5. The (irrational) **rotation** C^* -algebra \mathfrak{A}_θ for a given $\theta \in \mathbb{R}$ (irrational) is defined to be the universal C^* -algebra $C^*(u, v)$, generated by two unitary operators u and v , subject to the commutation relation $vu = e^{2\pi i\theta}uv$ (corrected as exchanged).

The C^* -algebra \mathfrak{A}_θ can be realized as a C^* -subalgebra of bounded operators on the Hilbert space $H = L^2(S^1)$, with the circle S^1 identified with \mathbb{R}/\mathbb{Z} . For a given $\theta \in \mathbb{R}$, consider two operators u and v , which act on a complete orthonormal basis $\{e_n (= e^{2\pi i n t})\}_{n \in \mathbb{Z}}$ ($t \in [0, 1]$) of H as

$$ue_n = e_{n+1} (= e^{2\pi i t} e_n) \quad \text{and} \quad ve_n = e^{2\pi i n \theta} e_n, \quad n \in \mathbb{Z}.$$

◇ Note that

$$\begin{aligned} \langle ue_n, ue_m \rangle &= \langle e_{n+1}, e_{m+1} \rangle = \delta_{n+1, m+1} = \delta_{n, m} = \langle e_n, e_m \rangle, \\ \langle ve_n, ve_m \rangle &= \langle e^{2\pi i n \theta} e_n, e^{2\pi i m \theta} e_m \rangle = e^{2\pi i(n-m)\theta} \delta_{n, m} = \langle e_n, e_m \rangle. \end{aligned}$$

It then follows that $u^*u = 1$ and $v^*v = 1$. Also, for any $\xi = \sum_j \xi_j e_j \in H$,

$$\begin{aligned} \langle u^*e_n, \xi \rangle &= \langle e_n, u\xi \rangle = \overline{\xi_{n-1}} = \langle e_{n-1}, \xi \rangle, \\ \langle v^*e_n, \xi \rangle &= \langle e_n, v\xi \rangle = e^{-2\pi i n \theta} \overline{\xi_n} = \langle e^{-2\pi i n \theta} e_n, \xi \rangle, \end{aligned}$$

and hence $u^*e_n = e_{n-1}$ and $v^*e_n = e^{-2\pi i n \theta} e_n$. Similarly, it follows that $uu^* = 1$ and $vv^* = 1$. Moreover, for any $\xi, \eta \in H$, check that

$$\begin{aligned} \langle e^{2\pi i \theta} uve_n, \eta \rangle &= e^{2\pi i \theta} \langle ue^{2\pi i n \theta} e_n, \eta \rangle = e^{2\pi i(n+1)\theta} \overline{\eta_{n+1}}, \\ \langle vue_n, \eta \rangle &= \langle ve_{n+1}, \eta \rangle = e^{2\pi i(n+1)\theta} \overline{\eta_{n+1}}. \quad \square \end{aligned}$$

The irrational rotation C^* -algebra can be described in more geometric terms by the foliation on the usual commutative 2-torus \mathbb{T}^2 lines with an irrational slope. On $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, consider the Kronecker **foliation** given by $\theta dx = dy$ (corrected), for $x, y \in \mathbb{R}/\mathbb{Z}$.

◇ Indeed, integrating the differential equation implies that $L_c : y = \theta x + c$ with $c \in \mathbb{R}/\mathbb{Z}$ as constants. Therefore, $\mathbb{T}^2 = \sqcup_{[c] \in (\mathbb{R}/\mathbb{Z})/\theta\mathbb{Z}} L_c$. \square

The space X of leaves L_c is described as $X = \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}) \cong S^1/\theta\mathbb{Z}$. This quotient is ill behaved as a topological space, hence it can not be well described by ordinary geometry.

A **transversal** to the foliation is given for instance by the choice $T = \mathbb{T} \times \{0\}$, with $T \cong S^1 \cong \mathbb{R}/\mathbb{Z}$. Then the noncommutative 2-torus \mathbb{T}_θ^2 is obtained ([22], [27]) as

$$\mathbb{T}_\theta^2 = \{(f_{ab}) \text{ (or just } f_{ab}) \mid a, b \in T \text{ in the same leaf}\} \cong \mathfrak{A}_\theta$$

(in a suitable sense), where (f_{ab}) (or just f_{ab}) may be identified with a (finite) power series $\sum_{n \in \mathbb{Z}} b_n v^n$ (in the crossed product C^* -algebra below, under an interpretation as given in [150] or below), where each $b_n \in C(S^1)$ the C^* -algebra of all complex-valued, continuous functions on S^1 .

◇ The noncommutative 2-torus is just defined as the **foliation** C^* -algebra corresponding to the Kronecker foliation on the 2-torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. In fact, let L be the leaf (orbit, or line with slope θ) passing the origin $(0, 0)$, which is dense in \mathbb{T}^2 . Consider the set of (weighted) finite paths in L , with vertexes in $T \cap L$ and edges as line segments in L between vertexes. May assign each finite path between $a = e^{2\pi i \theta a'}$ and $b = e^{2\pi i \theta b'}$ in $L \cap T$ to a finite sum $f_{a'b'} = \sum_{j=a'}^{b'} b_j v^j$, where each $b_j \in C(S^1)$ may be assumed to be a weight to an edge and each v^j may correspond to an vertex. Then for instance, may identify

$$\begin{aligned} f_{m,m+1} &= (\alpha_{m,n} e_n + \alpha_{m,n+1} e_{n+1}) v^m + (\alpha_{m+1,n} e_n + \alpha_{m+1,n+1} e_{n+1}) v^{m+1} \\ &= \begin{pmatrix} \alpha_{m,n} & \alpha_{m,n+1} \\ \alpha_{m+1,n} & \alpha_{m+1,n+1} \end{pmatrix} \end{aligned}$$

but the matrix multiplication is not the same as the usual one, and does contain the convolution with the action α_θ below involved. For example, $e_n v e_m v = e_n \alpha_\theta(e_m) v^2$. \square

The action of \mathbb{Z} on $C(S^1)$ is given by the automorphism $\alpha_\theta = \theta$ (in short) by a unitary v such that

$$\alpha_\theta(h) = v h v^{-1} = h \circ r_\theta^{-1}, \quad h \in C(S^1),$$

with $r_\theta(x) = x + \theta \in \mathbb{R} \pmod{1}$. The C^* -algebra $C(S^1)$ is generated by the function $u(t) = e^{2\pi i t}$ for $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Then these generators u and v (not the same as those u and v above, or exchanged) satisfy the relation $v u = e^{-2\pi i \theta} u v$.

◇ Because $v u v^{-1} = u \circ r_\theta^{-1} = e^{2\pi i(t-\theta)} = e^{-2\pi i \theta} u$. \square

It is then obtained through this description that there is an identification of \mathfrak{A}_θ as well as \mathbb{T}_θ^2 with the **crossed product** C^* -algebra $C(S^1) \rtimes_{r_\theta} \mathbb{Z}$, generated

by unitaries u and v , subject to the relation above. It represents the quotient space $S^1/\theta\mathbb{Z}$ as a noncommutative space.

Degenerations of elliptic curves. An **elliptic** curve E_τ over \mathbb{C} can be described as the quotient $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ of the complex plane \mathbb{C} by a 2-dimensional lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, where we can take $\text{Im}(\tau) > 0$. It is also possible to describe the elliptic curve E_q for $q \in \mathbb{C}^*$ with $q = e^{2\pi i\tau}$ and $|q| = e^{-2\pi\text{Im}(\tau)} < 1$, in terms of its Jacobi uniformization, namely as the quotient of \mathbb{C}^* by the action of the group generated by a single hyperbolic element q of D (not of $PSL_2(\mathbb{C})$, corrected), so that $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

The fundamental domain for the action of $q^{\mathbb{Z}}$ is an annulus

$$\{z \in \mathbb{C} \mid |q| < |z| \leq 1\}$$

of radii 1 and $|q|$. The identification of the two boundary circles is obtained via the combination of scaling and rotation given by multiplication by q .

Now let us consider a **degeneration** such that q goes to $e^{2\pi i\theta} \in S^1$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. We can say heuristically that in this degeneration the elliptic curve becomes a noncommutative 2-torus as $E_q \Rightarrow \mathbb{T}_\theta^2 \cong \mathfrak{A}_\theta$ in the sense that the annulus as the fundamental domains shrinks to the circle S^1 , and left is the quotient $S^1/e^{2\pi i\theta\mathbb{Z}}$ of S^1 by $\mathbb{Z} \cong e^{2\pi i\theta\mathbb{Z}}$ by the irrational rotation by $e^{2\pi i\theta}$.

Since the quotient space is ill behaved as a space, such degenerations do not admit a description within the context of classical geometry. However, we may replace the quotient by the corresponding crossed product C^* -algebra $C(S^1) \rtimes_\theta \mathbb{Z}$, and can consider such C^* -algebras as noncommutative (degenerate) elliptic curves.

More precisely, when we consider the degeneration of elliptic curves $E_\tau = \mathbb{C}^*/q^{\mathbb{Z}}$ for $q = e^{2\pi i\tau}$, what is obtained in the limit is the suspension $\mathfrak{A}_\theta \otimes C_0(\mathbb{R}) \equiv S\mathfrak{A}_\theta = \mathfrak{A}_\theta S$ of \mathfrak{A}_θ . In fact, as the parameter q degenerates to $e^{2\pi i\theta} \in S^1$, the nice quotient $E_\tau = \mathbb{C}^*/q^{\mathbb{Z}}$ degenerates to the bad quotient $E_\theta = \mathbb{C}^*/e^{2\pi i\theta\mathbb{Z}}$, whose noncommutative algebra of coordinates is Morita equivalent to the suspension C^* -algebra of \mathfrak{A}_θ , where $z \in \mathbb{C}^*$ has the polar decomposition as $z = |z|e^{2\pi i\text{arg}(z)}$, with $|z| = e^\rho$ for $\rho \in \mathbb{R}$ as the radial coordinate.

The Connes' Thom isomorphism [23] or the Bott periodicity in K-theory for C^* -algebras, and the Pimsner-Voiculescu six-term exact sequence [135] imply that the K-theory groups of

$$\mathfrak{A}_\theta \otimes C_0(\mathbb{R}) \cong C_0(S^1 \times \mathbb{R}) \rtimes_{\theta, \text{id}} \mathbb{Z} \cong_M C_0(\mathbb{R}^2) \rtimes_{t, \text{id}} \mathbb{Z} \rtimes_{\theta, \text{id}} \mathbb{Z}$$

(corrected) as the noncommutative space, identified with E_θ , are obtained as

$$\begin{aligned} K_0(E_\theta) &\cong K_1(\mathfrak{A}_\theta) \cong \mathbb{Z}^2 \cong \mathbb{Z}[u] \oplus \mathbb{Z}[v], \\ K_1(E_\theta) &\cong K_0(\mathfrak{A}_\theta) \cong \mathbb{Z}^2 \cong \mathbb{Z}[1] \oplus \mathbb{Z}[\theta], \end{aligned}$$

where id means the trivial action, t means the translation action of \mathbb{Z} on \mathbb{R} , so that $C_0(\mathbb{R}) \rtimes_t \mathbb{Z} \cong C(S^1) \otimes \mathbb{K}$, where \mathbb{K} is the C^* -algebra of all compact operators on a Hilbert space, and \cong_M means the stably (or Morita) isomorphism, and $[x]$ means the K-theory class corresponding to an element x of \mathfrak{A}_θ (in this case).

This is again compatible with the identification of $E_\theta = \mathfrak{A}_\theta S$ (rather than \mathfrak{A}_θ) as degenerations of elliptic curves.

Moreover, the Hodge filtration on the H^1 of an elliptic curve and the equivalence between the elliptic curve and its Jacobian have analogs for the noncommutative 2-torus \mathfrak{A}_θ in terms of the filtration on HC^{ev} (corrected) induced by the pairing (not inclusion of) with K_0 (cf. [24, p. 132-139 (up) and p. 348-355 (down)] and [30, XIII] missing). By the Bott periodicity or the Connes' Thom isomorphism, these appear again on the $HC^{\text{od}} = HC^1$ in the case of the noncommutative elliptic curve $E_\theta = S\mathfrak{A}_\theta$.

The point of view as degenerations is sometimes a useful guideline. For instance, we can study the limiting behavior of arithmetic invariants defined on the parameter space of elliptic curves as well as on modular curves, in the limit when τ goes to $\theta \in \mathbb{R} \setminus \mathbb{Q}$. An instance of this type of result is obtained as theory of limiting modular symbols of [112], which is reviewed in this chapter.

Remark. (Added). Let \mathcal{A}_θ denote the dense (smooth) subalgebra of \mathfrak{A}_θ generated by u and v . As in [24, Theorem 53] (or [27, III, 2.β]), for any $\theta \notin \mathbb{Q}$ that satisfies a Diophantine condition,

$$\begin{aligned} H^{\text{ev}}(\mathcal{A}_\theta) &\equiv \varinjlim H_\lambda^{2n}(\mathcal{A}_\theta) \cong H^2(\mathcal{A}_\theta) \cong HC^2(\mathcal{A}_\theta) \equiv H_\lambda^2(\mathcal{A}_\theta) \cong \mathbb{C}[S(\tau)] \oplus \mathbb{C}[\varphi], \\ H^{\text{od}}(\mathcal{A}_\theta) &\equiv \varinjlim H_\lambda^{2n-1}(\mathcal{A}_\theta) \cong HC^3(\mathcal{A}_\theta) \cong H_\lambda^1(\mathcal{A}_\theta) \cong H^1(\mathcal{A}_\theta, \mathcal{A}_\theta^*) \cong \mathbb{C}[\varphi_1] \oplus \mathbb{C}[\varphi_2], \end{aligned}$$

where in general, for \mathcal{A} a locally convex topological algebra, $H^*(\mathcal{A})$ is defined to be $HC^*(\mathcal{A}) \otimes_{HC^*(\mathbb{C})} \mathbb{C}$ as the inductive limit of the groups $H_\lambda^n(\mathcal{A})$, with $HC^*(\mathbb{C})$ identified with a polynomial ring $\mathbb{C}[\sigma]$ with the generator σ of degree two, with $HC^{2n}(\mathbb{C}) = \mathbb{C}$ and $H^{2n-1}(\mathbb{C}) = 0$, and there is the long exact sequence of the Connes' cyclic cohomology $HC = H_\lambda$ and the Hochschild cohomology H :

$$\begin{array}{ccccccc} H^1(\mathcal{A}_\theta, \mathcal{A}_\theta^*) & \xleftarrow[\cong]{I=i^*} & H_\lambda^1(\mathcal{A}_\theta) & \xleftarrow[0]{B} & H^2(\mathcal{A}_\theta, \mathcal{A}_\theta^*) & & \\ & & & & & \parallel & \\ & & & & & & \\ B \downarrow 0 & & & & & & \\ H_\lambda^0(\mathcal{A}_\theta) & \xrightarrow[A_\theta \circ (\sigma \#)]{S} & H_\lambda^2(\mathcal{A}_\theta) & \xrightarrow[\text{inclusion}^*]{I} & H^2(\mathcal{A}_\theta, \mathcal{A}_\theta^*) & \xrightarrow[0]{B=A \circ B_0} & H_\lambda^1(\mathcal{A}_\theta) \\ & & \parallel & & \parallel & & S \downarrow \cong \\ \mathbb{C}[\text{tr}] & \xrightarrow[A_\theta \circ (\sigma \cup)]{S} & \mathbb{C}[S(\text{tr})] \oplus \mathbb{C}[\varphi] & \xrightarrow[i^*]{I} & \mathbb{C} & & H_\lambda^3(\mathcal{A}_\theta) \end{array}$$

where it holds that

$$HC^1(\mathcal{A}_\theta) \cong H^1(\mathcal{A}_\theta, \mathcal{A}_\theta^*) \cong H^1(\mathcal{A}_\theta, \mathcal{A}_\theta^*)/\text{im}(I \circ B)$$

and $HC^0(\mathcal{A}_\theta) = \mathbb{C}$ for any $\theta \notin \mathbb{Q}$, and τ is the canonical trace on \mathcal{A}_θ defined as sending $\tau(\alpha 1) = \alpha$ and otherwise zero, and $\varphi_j(x_0, x_1) = \tau(x_0 \delta_j(x_1))$, and

$$\varphi(x_0, x_1, x_2) = \frac{1}{2\pi i} \tau(x_0(\delta_1(x_1)\delta_2(x_2) - \delta_2(x_1)\delta_1(x_2)))$$

for $x_j \in \mathcal{A}_\theta$, where δ_j are the basis derivations of \mathcal{A}_θ .

If $\theta \notin \mathbb{Q}$ does not satisfy any Diophantine condition, then $H^j(\mathcal{A}_\theta, \mathcal{A}_\theta^*)$ for $j = 1, 2$ are infinite dimensional. If $\theta \in \mathbb{Q}$, then $H^j(\mathcal{A}_\theta, \mathcal{A}_\theta^*)$ for $0 \leq j \leq 2$ are infinite dimensional. If $\theta \notin \mathbb{Q}$, then $H^0(\mathcal{A}_\theta, \mathcal{A}_\theta^*) \cong \mathbb{C}$.

On the other hand, $H^j(\mathcal{A}_\theta, \mathcal{A}_\theta^*) = 0$ for $j \geq 3$ and $\theta \in \mathbb{R}$. Hence,

$$\begin{array}{ccccccc}
H^2(\mathcal{A}_\theta, \mathcal{A}_\theta^*) & \xleftarrow{I=i^*} & H_\lambda^2(\mathcal{A}_\theta) & & & & \\
B \downarrow 0 & & & & & & \\
H_\lambda^1(\mathcal{A}_\theta) & \xrightarrow[A \circ (\sigma\#)]{S} & H_\lambda^3(\mathcal{A}_\theta) & \xrightarrow[\text{inclusion}^*]{I} & H^3(\mathcal{A}_\theta, \mathcal{A}_\theta^*) & \xrightarrow[0]{B=A \circ B_0} & H_\lambda^2(\mathcal{A}_\theta) \\
\parallel & & \parallel & & \parallel & & S \downarrow \cong \\
\mathbb{C}^2 & \xrightarrow[A \circ (\sigma\cup)]{S} & \mathbb{C}^2 & \xrightarrow[i^*]{I} & 0 & & H_\lambda^4(\mathcal{A}_\theta)
\end{array}$$

Note that it holds that

$$HC^{2n}(\mathfrak{A}) \cong HC^0(\mathfrak{A}) \quad \text{and} \quad HC^{2n+1}(\mathfrak{A}) \cong 0$$

for any nuclear C^* -algebra \mathfrak{A} . If such a C^* -algebra \mathfrak{A} has a trace, then $H^{\text{ev}}(\mathfrak{A}) \cong \mathbb{C}$ and $H^{\text{od}}(\mathfrak{A}) \cong 0$. In particular, $H^{\text{ev}}(\mathfrak{A}_\theta) \cong \mathbb{C}$ and $H^{\text{od}}(\mathfrak{A}_\theta) \cong 0$.

As well, recall from [27, III, 1.β] that for a Hochschild cochain $\varphi \in C^{n+1}(\mathfrak{A}, \mathfrak{A}^*)$,

$$(B_0\varphi)(a_0, \dots, a_n) = \varphi(1, a_0, \dots, a_n) - (-1)^{n+1}\varphi(a_0, \dots, a_n, 1),$$

with $B_0\varphi \in C^n(\mathfrak{A}, \mathfrak{A}^*)$, and $A : C^n(\mathfrak{A}, \mathfrak{A}^*) \rightarrow C^n(\mathfrak{A}, \mathfrak{A}^*)$ is the linear map defined as $A\varphi = \sum_{\gamma \in \Gamma} \varepsilon(\gamma)\varphi^\gamma$, where Γ is the group of cyclic permutations of the set $\{0, 1, \dots, n\}$, and $\varphi^\gamma(a_0, \dots, a_n) = \varphi(a_{\gamma(0)}, \dots, a_{\gamma(n)})$.

The subspace $C_\lambda^n(\mathfrak{A})$ of $C^n(\mathfrak{A}, \mathfrak{A}^*)$ consists of the **cyclic** Hochschild cochains φ such that $\varphi^\gamma = \varepsilon(\gamma)\varepsilon$ for any $\gamma \in \Gamma$.

The subcomplex $(C_\lambda^n(\mathfrak{A}), b)$ of the Hochschild complex $(C^n(\mathfrak{A}, \mathfrak{A}^*), b)$ is defined to define the (Hochschild-Connes) **cyclic** cohomology groups $H_\lambda^n(\mathfrak{A}) = HC^n(\mathfrak{A})$, where $C^n(\mathfrak{A}, \mathfrak{A}^*)$ is the space of linear maps from \mathfrak{A}^n to \mathfrak{A}^* the dual space of all linear functionals on \mathfrak{A} , and any element $\varphi \in C^n(\mathfrak{A}, \mathfrak{A}^*)$ can be identified with a linear functional $\varphi^\sim : \mathfrak{A}^{n+1} \rightarrow \mathbb{C}$ defined by

$$\varphi^\sim(a_0, a_1, \dots, a_n) = \varphi(a_1, \dots, a_n)(a_0).$$

As well, the Hochschild **coboundary** map b with $b\varphi = b\varphi^\sim \in C^{n+1}(\mathfrak{A}, \mathfrak{A}^*)$ is defined by

$$\begin{aligned}
(b\varphi^\sim)(a_0, \dots, a_{n+1}) &= \varphi^\sim(a_0 a_1, a_2, \dots, a_{n+1}) \\
&+ \sum_{j=1}^n (-1)^j \varphi^\sim(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) + (-1)^{n+1} \varphi^\sim(a_{n+1} a_0, \dots, a_n).
\end{aligned}$$

Moreover, the **cup** (or **cocap**) product

$$\# = \cup : HC^m(\mathfrak{A}) \otimes HC^m(\mathfrak{B}) \rightarrow HC^{m+m}(\mathfrak{A} \otimes \mathfrak{B}), \quad [\varphi] \otimes [\psi] \mapsto [\varphi\#\psi]$$

is induced from defining the cup product $\varphi\#\psi$ by $(\varphi\otimes\psi)\circ\pi$ for $\varphi\in C^n(\mathfrak{A},\mathfrak{A}^*)$, $\psi\in C^m(\mathfrak{B},\mathfrak{B}^*)$, where $\pi:\Omega^*(\mathfrak{A}\otimes\mathfrak{B})\rightarrow\Omega^*(\mathfrak{A})\otimes\Omega^*(\mathfrak{B})$ is the natural homomorphism by the universal property of the universal graded differential algebra $\Omega^*(\cdot)$, where $\Omega^n(\mathfrak{A})=\Omega^1(\mathfrak{A})\otimes_{\mathfrak{A}}\cdots\otimes_{\mathfrak{A}}\Omega^1(\mathfrak{A})$, with $\Omega^1(\mathfrak{A})=(\mathfrak{A}\oplus\mathbb{C}1)\otimes_{\mathbb{C}}\mathfrak{A}$ the \mathfrak{A} -bimodule, with the derivation $d:\mathfrak{A}\rightarrow\Omega^1(\mathfrak{A})$ defined by $da=1\otimes a$ for $a\in\mathfrak{A}$. The generator for $HC^2(\mathbb{C})$ is given by the 2-cocycle $\sigma(1,1,1)=1$ for $1\in\mathbb{C}$. ◀

Morita equivalence between NC 2-tori. To extend the modular interpretation of the quotient $\Gamma\backslash\mathbb{H}^2$ as moduli of elliptic curves to the noncommutative boundary $\Gamma\backslash\mathbb{P}^1(\mathbb{R})$, it is necessary to check that points in the same orbit under the action of the modular group $PSL_2(\mathbb{Z})$ by fractional linear transformations on $\mathbb{P}^1(\mathbb{R})$ define Morita (or stably) equivalent noncommutative 2-tori.

It is shown by Connes [22] (cf. also [140]) that the noncommutative 2-tori \mathfrak{A}_θ and $\mathfrak{A}_{-\frac{1}{\theta}}$ are Morita equivalent. Geometrically, in terms of the Kronecker foliation, the Morita equivalence of \mathbb{T}_θ^2 and $\mathbb{T}_{-\frac{1}{\theta}}^2$ corresponds to changing the choice of the transversal from $T=\mathbb{T}\times\{0\}$ to $T'=\{0\}\times\mathbb{T}$ in $\mathbb{T}^2=(\mathbb{R}/\mathbb{Z})^2$, or to from $\theta dx=dy$ to $-\theta^{-1}dx=dy$.

In fact, all Morita equivalence among noncommutative 2-tori arise in that way. Then \mathfrak{A}_θ and $\mathfrak{A}_{\theta'}$ are Morita equivalent if and only if θ is equivalent to θ' under the action of $PSL_2(\mathbb{Z})$.

Constructed explicitly by Connes [22] the bimodules $M_{\theta,\theta'}$ realizing the Morita equivalence between \mathbb{T}_θ^2 and $\mathbb{T}_{\theta'}^2$, with

$$\theta'=g\theta=\frac{a\theta+b}{c\theta+d}, \quad g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z})=\Gamma,$$

defined to be the **Schwartz** space $\mathcal{S}(\mathbb{R}\times\mathbb{Z}/c)$ (?), with the right and left actions of \mathfrak{A}_θ and $\mathfrak{A}_{\theta'}$ respectively, defined by

$$\begin{aligned} (fu)(x,n) &= f\left(x-\frac{c\theta+d}{c},n-1\right), \\ (fv)(x,n) &= e^{2\pi i\left(x-\frac{nd}{c}\right)}f(x,n), \quad \text{and} \\ (uf)(x,n) &= f\left(x-\frac{1}{c},n-a\right), \\ (vf)(x,n) &= e^{2\pi i\left(\frac{x}{c\theta+d}-\frac{n}{c}\right)}f(x,n). \end{aligned}$$

◊ In particular, $S(\theta)=-\frac{1}{\theta}$, with $S=1\otimes-1$ (transposed-diagonal sum). ◻

Remark. (Added as in [140]). Note that $\mathfrak{A}_\theta\cong\mathfrak{A}_{\theta'}$ as a C^* -algebra if and only if $\theta'=\theta$ or $1-\theta\pmod{1}$. Also, \mathfrak{A}_θ is strongly Morita (sM) equivalent to $\mathfrak{A}_{\theta'}$ if and only if θ is equivalent to θ' under the action of $GL_2(\mathbb{Z})$, which is generated by $S'=1\otimes 1$ and T . In particular, $S'(\theta)=\frac{1}{\theta}$. Hence $\mathfrak{A}_\theta\cong_M\mathfrak{A}_{\frac{1}{\theta}}$.

◊ As a note, \mathfrak{A}_θ is stably equivalent to both of $\mathfrak{A}_{-\frac{1}{\theta}}$ and $\mathfrak{A}_{\frac{1}{\theta}}$ since $SL_2(\mathbb{Z})\subset GL_2(\mathbb{Z})$. There are infinitely many sM equivalent NC tori. ◻

By definition, two C^* -algebras \mathfrak{A} and \mathfrak{B} are **strongly** Morita equivalent if there is an \mathfrak{A} - \mathfrak{B} equivalence (or imprimitivity) bimodule. The sM equivalent,

unital C^* -algebras are stably isomorphic, and as well each is a (full) corner $pM_n p$ of the matrix algebra M_n over the other, of a suitable size n and a projection p (not contained in any two-sided ideal). ◀

Other properties as NC elliptic curves. There are other ways by which the irrational rotation algebra behaves like an NC elliptic curve, most notably as in the relation between elliptic curves and their Jacobians and some aspects of the theory of theta functions, which recalled briefly (cf. [24] and [30] missing).

The (commutative) 2-torus $\mathbb{T}^2 = S^1 \times S^1$ is connected, so that the C^* -algebra $C(\mathbb{T}^2)$ does not have non-trivial projections. On the contrary, the noncommutative 2-tori \mathfrak{A}_θ contain non-trivial (**Rieffel**) projections. Indeed, it is shown by Rieffel [140] that for a given irrational number θ and any $\alpha \in (\mathbb{Z} + \mathbb{Z}\theta)$ in $[0, 1]$, there exists a projection $p_\alpha \in \mathfrak{A}_\theta$ such that the trace $\tau(p_\alpha) = \alpha$. Also, a different construction of projections of \mathfrak{A}_θ is given by Boca [12], with arithmetic relevance, inasmuch as those projections that correspond to the theta functions for noncommutative 2-tori, defined by Manin [108].

The method of constructing projections of C^* -algebras is based on the following two steps (cf. [140] and [110] missing):

(I) Suppose given a $\mathfrak{A}\text{-}\mathfrak{B}$ bimodule \mathcal{M} . If an element $\xi \in M$ admits an invertible $*$ -invariant square root $\langle \xi, \xi \rangle_{\mathfrak{B}}^{\frac{1}{2}} (= \sqrt{\xi^* \xi} = |\xi|)$, then the element $\nu = \xi \langle \xi, \xi \rangle_{\mathfrak{B}}^{-\frac{1}{2}} (= \xi |\xi|^{-1})$ satisfies $\nu \langle \nu, \nu \rangle_{\mathfrak{B}} (= \xi |\xi|^{-1} |\xi|^{-1} \xi^* \xi |\xi|^{-1}) = \nu$.

(II) Let $\nu \in \mathcal{M}$ be a non-trivial element such that $\nu \langle \nu, \nu \rangle_{\mathfrak{B}} = \nu$. Then the element $p = \mathfrak{A} \langle \nu, \nu \rangle_{\mathfrak{B}} (= \xi |\xi|^{-1} |\xi|^{-1} \xi^*)$ is a projection if $p = p^* = p^2$.

In the Boca construction, the elements ξ are obtained from Gaussian elements in some Heisenberg modules, in such a way that the corresponding $\langle \xi, \xi \rangle_{\mathfrak{B}}$ is a quantum theta function in the sense of Manin [110](missing). An introduction to the relation between the Heisenberg groups and the theory of theta functions is given in the third volume of the Mumford Tata Lectures on theta [125].

Remark. (Added). As in [140], the (**Rieffel**) projection p_α constructed has the form $h\nu^* + f + g\nu$ for some suitably chosen $h, f, g \in C(\mathbb{T})$, so that $f = f^*$, $h = \nu^* g^*$, and that (1) $g(t)g(t - \theta) = 0$, (2) $g(t)(1 - f(t) - f(t - \theta)) = 0$, and (3) $f(t)(1 - f(t)) = |g(t)|^2 + |g(t + \theta)|^2$, for $t \in \mathbb{R}$, if p_α is an idempotent. The condition (1) says that $g(t) = 0$ or $g(t - \alpha) = 0$, (2) if $g(t) \neq 0$, then $f(t) + f(t - \theta) = 1$, (3) if $f(t) \neq 0, 1$, then $g(t) \neq 0$ or $g(t + \alpha) \neq 0$. Indeed, may assume that $\theta \in [0, \frac{1}{2}]$. For any $0 < \varepsilon < \theta$, with $\theta + \varepsilon < \frac{1}{2}$, define

$$f(t) = \begin{cases} \frac{1}{\varepsilon}t & \text{if } t \in [0, \varepsilon], \\ 1 & \text{if } t \in [\varepsilon, \theta] \\ 1 - \frac{1}{\varepsilon}(t - \theta) & \text{if } t \in [\theta, \theta + \varepsilon], \\ 0 & \text{if } t \in [\theta + \varepsilon, 1], \end{cases} \quad \text{with} \quad \int_0^1 f(t)dt = \theta,$$

and $g(t) = \begin{cases} \sqrt{f(t)(1 - f(t))}, & \text{if } t \in [\theta, \theta + \varepsilon], \\ 0 & \text{otherwise.} \end{cases} \quad \blacktriangleleft$

Remark. As well, recall from [141] the following. Assume that \mathfrak{A} and \mathfrak{B} are full hereditary C^* -subalgebras of a C^* -algebra \mathfrak{C} . Define an \mathfrak{A} - \mathfrak{B} **bimodule** \mathcal{M} as the closed linear space $\overline{\mathfrak{A}\mathfrak{C}\mathfrak{B}}$. Define an \mathfrak{A} - \mathfrak{B} **equivalence** (or **imprimitivity**) bimodule to be such \mathcal{M} , equipped with an \mathfrak{A} -valued inner product and a \mathfrak{B} -valued inner product on \mathcal{M} as a left \mathfrak{A} -module and a right \mathfrak{B} -module respectively, defined (for instance as)

$$\mathfrak{A}\langle x, y \rangle (= xy^*) \quad \text{and} \quad \langle x, y \rangle_{\mathfrak{B}} (= x^*y), \quad x, y \in \mathcal{M}$$

such that (1) positivity, (2) symmetry, and (3) linearity hold:

- (1) $0 \leq \mathfrak{A}\langle x, x \rangle (= xx^*) \in \mathfrak{A}_+$ and $0 \leq \langle x, x \rangle_{\mathfrak{B}} (= x^*x) \in \mathfrak{B}_+$, $x \in \mathcal{M}$,
- (2) $\mathfrak{A}\langle x, y \rangle^* = \mathfrak{A}\langle y, x \rangle$ and $\langle x, y \rangle_{\mathfrak{B}}^* = \langle y, x \rangle_{\mathfrak{B}}$, $x, y \in \mathcal{M}$,
- (3) $\mathfrak{A}\langle ax, y \rangle = a\mathfrak{A}\langle x, y \rangle$, $x, y \in \mathcal{M}, a \in \mathfrak{A}$ and $\langle x, yb \rangle_{\mathfrak{B}} = \langle x, y \rangle_{\mathfrak{B}}b$, $x, y \in \mathcal{M}, b \in \mathfrak{B}$,

and moreover, (4) compatibility, (5) boundedness of representations of \mathfrak{A} and \mathfrak{B} on \mathcal{M} , and (6) density hold;

- (4) $\mathfrak{A}\langle x, y \rangle z = x\langle y, z \rangle_{\mathfrak{B}}$, $x, y, z \in \mathcal{M}$,
- (5) $\langle ax, ax \rangle_{\mathfrak{B}} \leq \|a\|^2 \langle x, x \rangle_{\mathfrak{B}}$ in \mathfrak{B}_+ , $a \in \mathfrak{A}, x \in \mathcal{M}$, and $\langle xb, xb \rangle_{\mathfrak{A}} \leq \|b\|^2 \langle x, x \rangle_{\mathfrak{A}}$ in \mathfrak{A}_+ , $b \in \mathfrak{B}, x \in \mathcal{M}$,
- (6) $\overline{\mathfrak{A}\langle \mathcal{M}, \mathcal{M} \rangle} = \mathfrak{A}$ and $\overline{\langle \mathcal{M}, \mathcal{M} \rangle_{\mathfrak{B}}} = \mathfrak{B}$, as a C^* -algebra.

As well, define a norm on \mathcal{M} by

$$\|x\| (= \|x\|_{\mathfrak{C}}) = \sqrt{\|\mathfrak{A}\langle x, x \rangle\|} (= \mathfrak{A}\|xx^*\|) = \sqrt{\|\langle x, x \rangle_{\mathfrak{B}}\|} (= \|x^*x\|_{\mathfrak{B}}). \quad \blacktriangleleft$$

2.3 Limiting modular symbols

Consider the action of the group $\Gamma = PGL_2(\mathbb{Z})$ on the upper and lower half planes \mathbb{H}^{\pm} in the complex plane \mathbb{C} and the modular curves defined to be the quotient spaces $X_G^{\pm} = G \backslash \mathbb{H}^{\pm}$, for G a finite index subgroup of Γ . Then the noncommutative **compactification** of the modular curves is obtained by extending the action of Γ on \mathbb{H}^{\pm} to the action on the full $\mathbb{P}^1(\mathbb{C}) = \mathbb{H}^{\pm} \cup \mathbb{P}^1(\mathbb{R})$, so that we define

$$\overline{X_G^{\pm}} = G \backslash \mathbb{P}^1(\mathbb{C}) = \Gamma \backslash (\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}), \quad \text{with } \mathbb{P} = \Gamma/G.$$

Due to the fact that Γ does not act discretely on $\mathbb{P}^1(\mathbb{R})$, the quotient only makes sense as a noncommutative space as the crossed product $C(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}) \rtimes \Gamma$. Note that $\mathbb{P}^1(\mathbb{R})$ in $\mathbb{P}^1(\mathbb{C})$ is the limit set of the group Γ , in the sense that it is the set of accumulation points of orbits of elements of Γ on $\mathbb{P}^1(\mathbb{C})$. Let us see another instance of noncommutative geometry arising from the action of a group of Möbius transformations of $\mathbb{P}^1(\mathbb{C})$ on its limit set, in the context of the geometry at the archimedean primes of arithmetic varieties.

Generalized Gauss shift and dynamics. The crossed product C^* -algebra $C(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}) \rtimes \Gamma$ is described as the noncommutative **boundary** of modular curves. It is also able to describe the quotient space $\Gamma \backslash (\mathbb{P}^1(\mathbb{R}) \times \mathbb{P})$ in the following equivariant way. If $\Gamma = PGL_2(\mathbb{Z})$, then Γ -orbits in $\mathbb{P}^1(\mathbb{R})$ are the same as equivalence classes of points of $[0, 1]$ under the equivalence relation defined as that for $x, y \in [0, 1]$, $x \sim_T y$ if there exist $n, m \in \mathbb{Z}$ (positive) such that $T^n x = T^m y$, where the classical **Gauss** shift T of the continued fraction expansion is defined as

$$Tx = x^{-1} - \lfloor x^{-1} \rfloor \in [0, 1], \quad x \in (0, 1] \quad \text{and} \quad T0 = 0 \quad (\text{or } 1).$$

Namely, the equivalence relation is that of having the same tail of the continued fraction expansion as shift-tail equivalence.

◇ Note that $x = (x^{-1})^{-1} = (\lfloor x^{-1} \rfloor + Tx)^{-1}$ for $0 < x \leq 1$. □

As a simple generalization of that classical result above,

Lemma 2.6. *The $PGL_2(\mathbb{Z}) = \Gamma$ -orbits in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$ are the same as equivalence classes of points in $[0, 1] \times \mathbb{P}$ under the equivalence relation defined as that for $(x, s), (y, t)$, $(x, s) \sim_T (y, t)$ if there exist $n, m \in \mathbb{Z}$ (positive) such that $T^n(x, s) = T^m(y, t)$, where the shift T (by the same symbol) on $[0, 1] \times \mathbb{P}$ generalizing the classical shift T of the continued fraction expansion is defined by*

$$T(x, s) = (Tx, Ts) = \left(\frac{1}{x} - \lfloor x^{-1} \rfloor, \begin{pmatrix} -\lfloor x^{-1} \rfloor & 1 \\ 1 & 0 \end{pmatrix} s \right).$$

◇ Note that

$$\begin{aligned} \begin{pmatrix} -\lfloor x^{-1} \rfloor & 1 \\ 1 & 0 \end{pmatrix} s &\equiv \begin{pmatrix} -\lfloor x^{-1} \rfloor & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} / G \\ &= \begin{pmatrix} -\lfloor x^{-1} \rfloor a + c & -\lfloor x^{-1} \rfloor b + d \\ a & b \end{pmatrix} / G. \quad \square \end{aligned}$$

The quotient by the equivalence relation by T (extended) is described as a noncommutative space by the C^* -algebra of the **groupoid** of the equivalence relation

$$\mathcal{G}([0, 1] \times \mathbb{P}, T) = \{((x, s), m - n, (y, t)) \mid T^m(x, s) = T^n(y, t)\}$$

with objects $\mathcal{G}^0 = \{((x, s), 0, (x, s))\}$.

In fact, for any T -invariant subset E of $[0, 1] \times \mathbb{P}$, we can consider the equivalence relation by T . The corresponding **groupoid** C^* -algebra denoted as $C^*(\mathcal{G}(E, T))$ encodes the dynamical properties of the map T on E .

Geometrically, the equivalence relation on $[0, 1] \times \mathbb{P}$ is related to the action of the geodesic flow on the horocycle foliation on the modular curves.

Arithmetic of modular curves and noncommutative boundary. The result as Lemma 2.6 shows that the properties of the dynamical system in $[0, 1] \times \mathbb{P}$ by T can be used to describe the geometry of the noncommutative boundary

of modular curves. There are various types of results that can be obtained by this method ([112], [113]), which are discussed soon in the rest of this subsection.

(1) Using the properties of this dynamical system, it is possible to recover and enrich the theory of modular curves on X_G , by extending the notion of modular symbols from geodesics connecting cusps to images (in $\overline{X_G}$) of geodesics in \mathbb{H}^2 connecting irrational points on the boundary $\mathbb{P}^1(\mathbb{R})$. In fact, the irrational points of $\mathbb{P}^1(\mathbb{R})$ define limiting modular symbols. In the case of quadratic irrationalities, these can be expressed in terms of the classical modular symbols and recover the generators of the homology of the classical compactification $\overline{X_G}$ by cusps. In the remaining cases, the limiting modular symbol vanishes almost everywhere.

(2) It is possible to reinterpret Dirichlet series related to modular forms of weight two in terms of integrals on $[0, 1]$ of certain intersection numbers obtained from homology classes defined in terms of the dynamical system. In fact, even when the limiting modular symbol vanishes, it is possible to associate a non-trivial (co)homology class in $H_1(\overline{X_G})$ (corrected) to irrational points on the boundary, in such a way that an average of the corresponding intersection numbers give Mellin transforms of modular forms of weight two on X_G .

(3) The Selberg zeta function of the modular curve can be expressed as a Fredholm determinant of the Perron-Frobenius operator associated to the dynamical system on the boundary.

(4) Using the formulation of the boundary as the noncommutative space as the crossed product C^* -algebra, we can obtain a canonical identification of the modular complex with a sequence of K-theory groups of the C^* -algebra. The resulting exact sequence of K-groups can be interpreted, using the orbit description of the quotient space, in terms of the Baum-Connes assembly map and the Connes' Thom isomorphism.

All these show that the noncommutative space as $C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma$ considered as a boundary stratum of $C(\overline{\mathbb{H}^2} \times \mathbb{P}) \rtimes \Gamma$ (corrected) contains a part of the arithmetic information on the classical modular curve. The fact that information on the bulk space is stored in its boundary at infinity can be seen as an instance of the physical principle of holography as bulk and boundary correspondence in string theory (cf. [111]). Discussed is the holography principle in more details in relation to the geometry of the archimedean fibers of arithmetic varieties.

Limiting modular symbols. Let γ_β denote an infinite geodesic in the hyperbolic half plane \mathbb{H} with one end at $i\infty$ and the other end at $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Let $x \in \gamma_\beta$ be a fixed base point, and $y(\tau)$ be the point along γ_β at a distance τ as the geodesic arc length, from x towards the end β . Let $[x, y(\tau)]_G$ denote the homology class in $H_1(\overline{X_G})$ (corrected) determined by the image of the geodesic arc $x \cap y(\tau)$ in \mathbb{H} .

Definition 2.7. The limiting **modular** symbol is defined to be

$$[[*, \beta]]_G \equiv \lim_{y(\tau) \rightarrow \beta} \frac{1}{\tau} [x, y(\tau)]_G \in H_1(\overline{X_G}, \mathbb{R}),$$

whenever such limit exists.

The limit is independent of the choice of the initial point x as well as of the choice of the geodesic in \mathbb{H} ending at β , as discussed in [112]. The (modified) notation is as introduced in [112], where $*$ as the first variable indicates the independence of the choice of the initial point x , the double brackets indicate the fact that the homology class is computed as a limiting cycle.

Dynamics of continued fractions. The classical (Gauss) **shift** map T of the continued fraction is defined by $T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor \in [0, 1]$ for $x \in [0, 1]$ with $x \neq 0$ and $T(0) = 0$ (or 1). The generalized shift on $[0, 1] \times \mathbb{P}$ is defined by

$$T(x, s) = (Tx, Ts) = \left(\frac{1}{x} - \lfloor x^{-1} \rfloor, \begin{pmatrix} -\lfloor x^{-1} \rfloor & 1 \\ 1 & 0 \end{pmatrix} s \right).$$

Recall the following basic notation regarding **continued fraction** expansion. Let k_1, \dots, k_n be independent variables for $n \geq 1$, and let

$$[k_1, \dots, k_n] \equiv \frac{1}{k_1 + \frac{1}{k_2 + \dots + \frac{1}{k_n}}} \equiv \frac{p_n(k_1, \dots, k_n)}{q_n(k_1, \dots, k_n)},$$

where p_n and q_n are polynomials with integral coefficients, which can be calculated inductively from the relations

$$\begin{aligned} q_{n+1}(k_1, \dots, k_n, k_{n+1}) &= k_{n+1}q_n(k_1, \dots, k_n) + q_{n-1}(k_1, \dots, k_{n-1}), \\ p_n(k_1, \dots, k_n) &= q_{n-1}(k_2, \dots, k_n), \end{aligned}$$

with $p_0 = q_{-1} = 0$ and $q_0 = 1$.

◇ For instance, $[k_1] = \frac{1}{k_1}$, so that $q_1 = k_1 = k_1q_0 + q_{-1}$ and $p_1 = 1 = q_0$. Also, $]k_1, k_2] = \frac{k_2}{k_2k_1+1}$, so that $q_2 = k_2q_1 + q_0$ and $p_2 = k_2 = q_1(k_2)$. \square
It is obtained that

$$[k_1, \dots, k_{n-1}, k_n + x_n] = \begin{pmatrix} p_{n-1}(k_1, \dots, k_{n-1}) & p_n(k_1, \dots, k_n) \\ q_{n-1}(k_1, \dots, k_{n-1}) & q_n(k_1, \dots, k_n) \end{pmatrix} x_n$$

as the standard matrix notation for fractional linear transformation.

◇ For instance,

$$[k_1 + x_1] = \frac{1}{x_1 + k_1} = \frac{p_0x_1 + p_1}{q_0x_1 + q_1} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} x_1.$$

As well,

$$\begin{aligned} [k_1, k_2 + x_2] &= \frac{x_2 + k_2}{k_1x_2 + k_2k_1 + 1} = \frac{p_1x_2 + p_2}{q_1x_2 + q_2} \\ &= \begin{pmatrix} 1 & k_2 \\ k_1 & k_1k_2 + 1 \end{pmatrix} x_2 = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & k_2 \end{pmatrix} x_2. \quad \square \end{aligned}$$

If $\alpha \in (0, 1)$ is an irrational number, then there is a unique sequence of integers $k_n(\alpha) \geq 1$ such that $\alpha = \lim_{n \rightarrow \infty} [k_1(\alpha), \dots, k_n(\alpha)]$.

◊ Indeed,

$$\begin{aligned}\alpha &= \frac{1}{\alpha} = \frac{1}{\lfloor \frac{1}{\alpha} \rfloor + T(\alpha)} = \frac{1}{\lfloor \frac{1}{\alpha} \rfloor + \frac{1}{\lfloor T(\alpha)^{-1} \rfloor + T(T(\alpha))}} \\ &= [\lfloor \frac{1}{\alpha} \rfloor, \lfloor T(\alpha)^{-1} \rfloor, \lfloor T^2(\alpha)^{-1} \rfloor, \dots]. \quad \square\end{aligned}$$

Moreover, there is a unique sequence $x_n(\alpha) \in (0, 1)$ such that

$$\alpha = [k_1(\alpha), \dots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)]$$

for each $n \geq 1$. It is obtained (by induction) that

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & k_1(\alpha) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & k_2(\alpha) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n(\alpha) \end{pmatrix} x_n(\alpha).$$

Set

$$[k_1(\alpha), \dots, k_n(\alpha)] = \frac{p_n(k_1(\alpha), \dots, k_n(\alpha))}{q_n(k_1(\alpha), \dots, k_n(\alpha))} \equiv \frac{p_n(\alpha)}{q_n(\alpha)}.$$

Also set

$$\begin{aligned}\alpha &= [k_1(\alpha), \dots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)] \\ &= \begin{pmatrix} p_{n-1}(k_1(\alpha), \dots, k_{n-1}(\alpha)) & p_n(k_1(\alpha), \dots, k_n(\alpha)) \\ q_{n-1}(k_1(\alpha), \dots, k_{n-1}(\alpha)) & q_n(k_1(\alpha), \dots, k_n(\alpha)) \end{pmatrix} x_n(\alpha) \equiv g_n(\alpha)x_n(\alpha)\end{aligned}$$

with $g_n(\alpha) \in GL_2(\mathbb{Z})$ with $\det g_n(\alpha) = (-1)^n$.

The Gauss shift T is then given by

$$\begin{aligned}T(\alpha) &= T([k_1(\alpha), k_2(\alpha), \dots]) = T([\lfloor \frac{1}{\alpha} \rfloor, \lfloor T(\alpha)^{-1} \rfloor, \lfloor T^2(\alpha)^{-1} \rfloor, \dots]) \\ &= [\lfloor T(\alpha)^{-1} \rfloor, \lfloor T^2(\alpha)^{-1} \rfloor, \dots] = [k_2(\alpha), k_3(\alpha), \dots]\end{aligned}$$

in terms of the continued fraction expansion.

The properties of the generalized shift on $[0, 1] \times \mathbb{P}$ can be used to extend the notion of modular symbols to geodesics with irrational ends ([112]). Such geodesics correspond to infinite geodesics on the modular curve X_G which exhibit a variety of interesting possible behaviors, from closed geodesics to geodesics that approximate some limiting cycle, to geodesics that wind around different homology class exhibiting a typically chaotic behavior.

Lyapunov spectrum. A measure of how chaotic a dynamical system is, or better of how fast nearby orbits tend to diverge, is given by the Lyapunov exponent.

Definition 2.8. The **Lyapunov** exponent of a (differentiable) map T from $[0, 1]$ to $[0, 1]$ is defined as

$$\begin{aligned}\lambda(\beta) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log |(T^n)'(\beta)| = \lim_{n \rightarrow \infty} \frac{1}{n} \log |T'(T^{n-1}(\beta))| |(T^{n-1})'(\beta)| \\ &= \dots = \lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |T'(T^k(\beta))|, \quad \beta \in [0, 1].\end{aligned}$$

The function $\lambda(\beta)$ is T -invariant.

◊ Indeed,

$$\begin{aligned} \frac{1}{n} \log |(T^n)'(T\beta)| &= \frac{1}{n} \log \prod_{k=1}^n |T'(T^k(\beta))| \\ &= \frac{n+1}{n} \frac{1}{n+1} \log \prod_{k=0}^n |T'(T^k(\beta))| - \frac{1}{n} \log |T'(\beta)| \\ &\rightarrow \lambda(T\beta) = \lambda(\beta) \quad (n \rightarrow \infty). \quad \square \end{aligned}$$

Moreover, in the case of the classical continued fraction shift T on $[0, 1]$, the Lyapunov exponent is given by

$$\lambda(\beta) = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n(\beta),$$

with $q_n(\beta)$ the successive denominators of the continued fraction expansion for β .

◊ Note that

$$\begin{aligned} q_2(\beta) &= 1 + k_1(\beta)k_2(\beta) = q_0(\beta) + q_1(\beta)k_2(\beta), \\ q_3(\beta) &= k_1(\beta) + (1 + k_1(\beta)k_2(\beta))k_3(\beta) = q_1(\beta) + q_2(\beta)k_3(\beta), \quad \text{and} \\ q_n(\beta) &= q_{n-2}(\beta) + q_{n-1}(\beta)k_n(\beta) \quad (n \geq 2). \quad \square \end{aligned}$$

◊ Note that if $\frac{1}{2} < x \leq 1$, then $1 \leq \frac{1}{x} < 2$. Hence, $T(x) = \frac{1}{x} - 1$. Moreover, if $\frac{1}{n+1} < x \leq \frac{1}{n}$, then $n \leq \frac{1}{x} < n+1$. Thus, $T(x) = \frac{1}{x} - n$. Therefore, T is discontinuous at $\frac{1}{n}$ for $n \geq 2$ and at 0, but continuous and even differentiable otherwise, with $T'(x) = -\frac{1}{x^2}$. \square

In particular, it is shown as the Khintchine-Lévy theorem that for almost all β with respect to the Lebesgue measure on $[0, 1]$, the above limit is equal to $\lambda_0 \equiv (6 \log 2)^{-1} \pi^2$. However, there is an exceptional set in $[0, 1]$ of Hausdorff dimension $\dim_H = 1$ but with Lebesgue measure $m_L = 0$, for elements of which, the limit defining the Lyapunov exponent does not exist.

As seen later below, in some cases, the value $\lambda(\beta)$ can be computed from the spectrum of the Perron-Frobenius operator of the shift T .

The Lyapunov spectrum is introduced (cf. [137]) by decomposing the unit interval into level sets of the Lyapunov exponent $\lambda(\beta)$. Let

$$L_c = \{\beta \in [0, 1] \mid \lambda(\beta) = c \in \mathbb{R}\}.$$

These (level) sets provide a T -invariant decomposition of the unit interval as

$$[0, 1] = (\sqcup_{c \in \mathbb{R}} L_c) \sqcup \{\beta \in [0, 1] \mid \lambda(\beta) \text{ does not exist}\}.$$

These level sets are uncountable dense T -invariant subsets of $[0, 1]$, of varying Hausdorff dimension ([137]). The Lyapunov spectrum measures how the Hausdorff dimension varies, as a function $h(c) = \dim_H L_c$.

Limiting modular symbols and iterated shifts. Define a function $\varphi : \mathbb{P} = \Gamma/G \rightarrow H^{cp} = H_1(\overline{X_G}, \{cp\})$ by $\varphi(s) = [g(0), g(i\infty)]_G$, where $g \in \Gamma = PGL_2(\mathbb{Z})$ (or $PSL_2(\mathbb{Z})$) is a representative of the coset $s = gG \in \mathbb{P}$.

Then the limiting modular symbol $[[*, \beta]]_G$ as a limit is computed in the following way:

Theorem 2.9. *For all $\beta \in L_c$ the level set of the Lyapunov exponent $\lambda(\beta)$ for a fixed $c \in \mathbb{R}$, the limiting modular symbol is computed as*

$$[[*, \beta]]_G = \lim_{n \rightarrow \infty} \frac{1}{cn} \sum_{k=1}^n \varphi \circ T^k(t_0),$$

where $T = T(\beta, t) = (T(\beta), T(\beta, t))$ and $t_0 \in \mathbb{P}$.

Without loss of generality, we may consider the geodesic γ_β in $\mathbb{H} \times \mathbb{P}$ with one end at $(i\infty, t_0)$ and the other at (β, t_0) , for $\varphi(t_0) = [0, i\infty]_G$.

The argument given in [112, 2.3] is based on the fact that we may replace the homology class defined by the vertical geodesic with the class obtained by connecting the successive rational approximations $\frac{p_n(\beta)}{q_n(\beta)}$ to β in the continued fraction expansion. Namely, we may replace the path $x_0 \cap y_n$ with the union of arcs

$$(x_0 \cap y_0) \cup (y_0 \cap \frac{p_0}{q_0}) \cup \dots \cup (\frac{p_{k-1}}{q_{k-1}} \cap \frac{p_k}{q_k}) \cup \dots \cup (\frac{p_{n-1}}{q_{n-1}} \cap y_n)$$

(corrected) representing the same homology class in $H_1(\overline{X_G}, \mathbb{Z})$, where $y_n \in \mathbb{H}^2$ is the intersection of γ_β and the geodesic with ends at $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$.

◇ For instance, $\frac{p_0}{q_0} = 0$ and $\frac{p_1}{q_1} = \frac{1}{k_1} = ([\beta^{-1}])^{-1}$. □

The result above then follows by estimating the geodesic distance, so that

$$\tau \sim -\log \operatorname{Im}(y) + O(1) \quad (y(\tau) \rightarrow \beta)$$

and $(2q_n q_{n+1})^{-1} < \operatorname{Im}(y_n) < (2q_n q_{n-1})^{-1}$.

◇ Note that $y_n \in \frac{p_{n-1}}{q_{n-1}} \cap \frac{p_n}{q_n}$ with $\operatorname{Im}(y_{n+1}) < \operatorname{Im}(y_n)$, and

$$\frac{1}{2q_n q_{n+1}} = \frac{1}{2} \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \quad \text{and} \quad \frac{p_{n+1}}{q_{n+1}} < \beta < \frac{p_n}{q_n} \quad \text{for any } n.$$

By taking logarithm,

$$\log(2q_n q_{n-1}) < -\log(\operatorname{Im}(y_n)) < \log(2q_n q_{n+1}). \quad \square$$

The inverse matrix $g_k^{-1}(\beta)$ with $g_k(\beta) \in GL_2(\mathbb{Z})$ as

$$g_k^{-1}(\beta) = \begin{pmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{pmatrix}^{-1} = (-1)^k \begin{pmatrix} q_k & -p_k \\ -q_{k-1} & p_{k-1} \end{pmatrix}$$

acts on points $(\beta, t) \in [0, 1] \times \mathbb{P}$ as the k -th power of the shift map T , so that

$$\varphi(T^k t_0) = [g_k^{-1}(\beta)(0), g_k^{-1}(\beta)(i\infty)]_G = \left[\frac{(-1)^{k+1} p_k(\beta)}{p_{k-1}(\beta)}, \frac{(-1)^{k+1} q_k(\beta)}{q_{k-1}(\beta)} \right]_G$$

(corrected).

Ruelle and Perron-Frobenius operators. A general principle in the theory of dynamical systems is that it is often possible to study the dynamical properties of such a map T (like ergodicity) via the spectral theory of an associated operator. This allows us to employ techniques of functional analysis and to derive conclusions on dynamics.

In the case of the (generalized) **shift** map T on $[0, 1] \times \mathbb{P}$, with $\mathbb{P} = \Gamma/G$, associated to the map is the operator defined as

$$(L_\sigma f)(x, t) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^{2\sigma}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} t\right)$$

depending on a complex parameter σ .

More generally, the **Ruelle** transfer operator of a (differentiable) map T is defined as

$$(L_\sigma f)(x, t) = \sum_{(y,s) \in T^{-1}(x,t)} \exp(h(y, s)) f(y, s),$$

where we take $h(x, t) = -2\sigma \log |T'(x, t)|$.

◇ Check that for any (or positive) integer k and $x \in [0, 1)$ with $[x+k] = k$,

$$T\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} t\right) = \left(x+k - [x+k], \begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} t\right) = (x, t),$$

$$\exp(-2\sigma \log |T'(y, s)|) = \frac{1}{|T'(y, s)|^{2\sigma}} = \frac{1}{|-\frac{1}{y^2}|^{2\sigma}} = \frac{1}{(x+k)^{4\sigma}},$$

with $y = \frac{1}{x+k}$ (so that we may remove 2 in 2σ in $\exp(\dots)$, or replace σ with $\frac{\sigma}{2}$ in either $\exp(\dots)$ or L_σ , and so on). \square

Clearly, that operator L_σ is well suited for capturing the dynamical properties of the map T as it is defined as a weighted sum over the inverse image of each (x, t) by T .

On the other hand, there is another operator that can be associated to a dynamical system and which typically has better spectral properties, but is less clearly related to the dynamics. The best circumstances arise when these two (operators) agree (or coincide) for a particular value of the parameter σ . The other operator is called the **Perron-Frobenius** operator PF and is defined by the relation

$$\int_{[0,1] \times \mathbb{P}} f(g \circ T) d\mu = \int_{[0,1] \times \mathbb{P}} (PF(f)) g d\mu,$$

the integration with respect to the Lebesgue measure μ . In the case of the shift T on $[0, 1] \times \mathbb{P}$ we in fact have that $PF = L_\sigma$ with $\sigma = 1$.

Spectral theory of the PF operator $PF = L_1$. In the case of the modular group $G = \Gamma$, the spectral theory of the Perron-Frobenius operator of the Gauss shift is studied by D. Mayer [119] (unchecked). More recently, Chang and Mayer [18] (unchecked) extend the results to the case of congruence subgroups. A similar approach is used in [112] to study the properties of the shift on $[0, 1] \times \mathbb{P}$.

Theorem 2.10. *Defined on the Banach space of holomorphic functions on $D \times \mathbb{P}$ continuous up to the boundary, with $D = \{z \in \mathbb{Z} \mid |z - 1| < \frac{3}{2}\}$ (analytically extended from $[0, 1] \times \mathbb{P}$), under the condition as irreducibility that*

$$\mathbb{P} = \bigcup_{n=0}^{\infty} \left\{ \prod_{j=1}^n \begin{pmatrix} 0 & 1 \\ 1 & k_j \end{pmatrix} (t_0) \mid k_1, \dots, k_n \geq 1 \right\},$$

(for some base point $t_0 \in \mathbb{P}$), the Perron-Frobenius operator $PF = L_1$ (the generalized Gauss-Kuzmin operator) defined as

$$(L_1 f)(x, s) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} s\right)$$

for the shift on $[0, 1] \times \mathbb{P}$ has the following properties:

- (1) L_1 is a nuclear operator, of trace class.
- (2) L_1 has top eigenvalue 1, which is simple.
- (3) The eigenfunction corresponding to 1 is $\frac{1}{1+x}$ up to normalization.
- (4) The spectrum of L_1 except 1 is contained in the ball with radius < 1 .
- (5) There is a complete set of eigenfunctions for L_1 .

The irreducibility condition for \mathbb{P} above is satisfied by congruence subgroups.

The function $\frac{1}{1+x}$ is called Gauss density on $[0, 1]$.

For other T -invariant subsets $E \subset [0, 1] \times \mathbb{P}$, may also consider the operators $L_{E, \sigma}$ and PF_E as the restrictions of L_{σ} and PF to E respectively. When the set E has the property that $PF_E = L_{E, h_E}$ with $h_E = \dim_H E$ the Hausdorff dimension, may use the spectral theory of the operator PF_E to study the dynamical properties of T .

In particular, the Lyapunov exponent can be derived from the spectrum of the family of operators $L_{E, \sigma}$ as follows. Let λ_{σ} denote the top eigenvalue of $L_{E, \sigma}$.

Lemma 2.11. *The Lyapunov exponent is given as*

$$\lambda(\beta) = \frac{d}{d\sigma} \lambda_{\sigma}$$

at $\sigma = h_E = \dim_H E$, where μ_H a.e. in E .

The Gauss problem. Now let us define a function for $n \in \mathbb{N}$ on $[0, 1]$ as

$$m_n(x) = \mu\{\alpha \in (0, 1) \mid x_n(\alpha) \leq x\}$$

valued as measure of the set, with $\alpha = [k_1(\alpha), \dots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)]$ with $x_n(\alpha) \in (0, 1)$.

The asymptotic behavior of the measures m_n is known to be a famous problem on the distribution of continued fractions, formulated by Gauss. It is conjectured by Gauss that

$$m(x) \equiv \lim_{n \rightarrow \infty} m_n(x) = \frac{1}{\log 2} \log(1+x).$$

The convergence as so above is only proved by R. Kuzmin in 1928. Other proofs are then given by P. Lévy (1929), K. Babenko (1978), and D. Mayer (1991). In the arguments used by Babenko and Mayer, used is the spectral theory on the Perron-Frobenius operator. Of these different arguments, only the latter extends to the case of the generalized Gauss shift on $[0, 1] \times \mathbb{P}$.

◊ It is evident that $m_n(1) = 1$ and $m_n(0) = 0$. Hence $m(1) = 1 = \frac{\log 2}{\log 2}$ and $m(0) = \frac{\log 1}{\log 2}$. Note also that

$$\frac{\log(1+x)}{\log 2} = \left(\int_0^1 \frac{1}{1+t} dt \right)^{-1} \left(\int_0^x \frac{1}{1+t} dx \right). \quad \square$$

The Gauss problem can be formulated in terms of a recursive relation as

$$m'_{n+1}(x) = (L_1 m'_n)(x) \equiv \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} m'_n \left(\frac{1}{x+k} \right),$$

where the right hand side is the image of m'_n under the Gauss-Kuzmin (or PF) operator L_1 (in the case of the group $\Gamma = PGL_2(\mathbb{Z})$).

Let us define functions m_n on $[0, 1] \times \mathbb{P}$, valued as measure of the set:

$$m_n(x, t) \equiv \mu\{(y, s) \in [0, 1] \times \mathbb{P} \mid x_n(y) \leq x, g_n(y)^{-1}(s) = t\}.$$

Or as $\mu\{\alpha \in (0, 1) \mid x_n(\alpha) \leq x, g_n(\alpha)^{-1}(t_0) = t\}$ (cf. [112]).

As a consequence of the theorem above, it is obtained that

Theorem 2.12. *It follows that $m'_n(x, t) = L_1^n(1)(x, t)$, and the following limit exists and equals to*

$$m(x, t) \equiv \lim_{n \rightarrow \infty} m_n(x, t) = \frac{1}{|\mathbb{P}| \log 2} \log(1+x).$$

◊ Note again that $m'_n = L_1 m'_{n-1} = \dots = L_1^n m'_0$ at x as well as (x, t) , with $m'_0 = 1$, where $x_0(\alpha) = 0$, so that $m_0(x) = x$. Hence $m'_0(x) = 1$. Indeed,

$$m'_{n+1}(x, t) = (L_1 m'_n)(x, t) \equiv \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} m'_n \left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} (t_0) \right). \quad \square$$

This shows that there exists a unique T -invariant measure on $[0, 1] \times \mathbb{P}$. This is uniform in the discrete set \mathbb{P} with the counting measure, and it is the Gauss measure of the shift of the continued fraction expansion on $[0, 1]$.

Two theorems on limiting modular symbols. The result of the T -invariant measure allows to study the general behavior of limiting modular symbols.

A special role is played by limiting modular symbols $[[*, \beta]]_G$, where β is a quadratic irrationality in $\mathbb{R} \setminus \mathbb{Q}$ (such as \sqrt{p} for p a prime).

Theorem 2.13. *Let $g \in G$ be hyperbolic, with eigenvalue $0 < \Lambda_g < 1$ corresponding to the attracting fixed point α_g^- . Let $\Lambda(g) \equiv |\log \Lambda_g|$ (geodesic distance) and let l be the period of the continued fraction expansion of $\beta = \alpha_g^-$. Then*

$$[[*, \beta]]_G = \frac{[0, g(0)]_G}{\Lambda(g)} = \frac{1}{\lambda(\beta)l} \sum_{k=1}^l [g_k^{-1}(\beta)0, g_k^{-1}(\beta)i\infty]_G.$$

(Added). Recall from [112] (or [113]) that any hyperbolic $g \in G$ has two fixed points α^\pm , repelling and attracting, on \mathbb{R} . Let Λ_g^\pm be the respective eigenvalues, with $0 < \Lambda_g^+ < 1 < \Lambda_g^-$. The oriented geodesic in \mathbb{H} connecting α_g^- to α_g^+ is g -invariant, and the action of g induces on it the shift by the geodesic distance $\log \Lambda_g^-$. And then \dots \blacktriangleleft

This shows that, in this case, the limiting modular symbols are linear combinations of classical modular symbols, with coefficients in the field generated over \mathbb{Q} by the Lyapunov exponents $\lambda(\beta)$ of the quadratic irrationalities.

In terms of geodesics on modular curve, this is the case, where the (winding) geodesic has a limiting cycle, given by the closed geodesic $[0, g(0)]_G$ (no figure).

On the other hand, there is the generic case, where, contrary to the previous example, the geodesics wind around many different cycles in such a way that the resulting homology class averages out to zero over long distances, as limiting modular symbols, as chaotic tangling and untangling (no figure).

Theorem 2.14. *For a T -invariant $E \subset [0, 1] \times \mathbb{P}$, under the irreducibility condition for E , the function $R_\tau(\beta, s) \equiv \frac{1}{\tau}[x, y(\tau)]_G$ converges weakly to zero. Namely, for all integrable $f \in L^1(E, d\mu_H)$,*

$$\lim_{\tau \rightarrow \infty} \int_E R_\tau(\beta, s) f(\beta, s) d\mu_H(\beta, s) = 0.$$

This weak convergence can be improved to strong convergence (without f) $\mu_H(E)$ -almost everywhere. Thus, the limiting modular symbol satisfies that $[[, \beta]]_G \equiv 0$ a.e. on E .*

This result depends upon the properties of the PF operator L_1 and the result on the T -invariant measure. In fact, to get the result on the weak convergence ([112]), notice that the limit computing limiting modular symbols as $\lim_{n \rightarrow \infty} \frac{1}{cn} \sum_{k=1}^n \varphi \circ T^k(t_0)$ can be evaluated in terms of a limit of iterates of the PF operator by

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_0 n} \sum_{k=1}^n \int_{[0,1] \times \mathbb{P}} f(\varphi \circ T^k) = \lim_{n \rightarrow \infty} \frac{1}{\lambda_0 n} \sum_{k=1}^n \int_{[0,1] \times \mathbb{P}} (L_1^k f) \varphi,$$

where $\lambda(\beta) = \lambda_0$ a.e. in $[0, 1]$.

By the convergence of $L_1^k 1$ to the density function h of the T -invariant mea-

sure, so that $L_1^k f$ converges to $(\int f d\mu)h$, it yields that

$$\begin{aligned} & \int_{[0,1] \times \mathbb{P}} [[*, \beta]]_G f(\beta, t) d\mu(\beta, t) \\ &= \left(\int_{[0,1] \times \mathbb{P}} f d\mu \right) \left(\int_{[0,1] \times \mathbb{P}} \varphi h d\mu \right) = \left(\int_{[0,1] \times \mathbb{P}} f d\mu \right) \frac{1}{2|\mathbb{P}|} \sum_{s \in \mathbb{P}} \varphi(s). \end{aligned}$$

It is then checked that the sum $\sum_{s \in \mathbb{P}} \varphi(s) = 0$ since each term in the sum changes sign under the action of the inversion $\sigma \in PGL_2(\mathbb{Z})$ with $\sigma^2 = \text{id}$, but the sum is globally invariant under σ .

The argument above can be extended to the case of other T -invariant subsets E of $[0, 1] \times \mathbb{P}$, for which the corresponding PF operator $L_{E.h_E}$ has analogous properties (cf. [113]). The weak convergence, improved to strong convergence, can be obtained by applying the strong law of large numbers to the random variables (or measurable, real-valued functions) $\varphi_k = \varphi \circ T^k$ (cf. [113] for more details). The result plays the role of an ergodic theorem for the shift T on E , effectively.

(Added). Recall from [116] that that the strong law of large numbers holds for random variables φ_j (on $[0, 1]$) means that there is a sequence $\{a_n\}$ such that $\frac{1}{n} \sum_{j=1}^n (\varphi_j - a_j)$ converges to zero almost surely, in the sense that

$$\mu\{x \in [0, 1] \mid \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (\varphi_j(x) - a_j) = 0\} = 1.$$

In particular, it holds when

$$\sum_{j=1}^{\infty} \frac{1}{j^2} E[(\varphi_j - E[\varphi_j])^2] < \infty,$$

with mean (or expectation) $E[\varphi_j] = \int_0^1 \varphi_j(x) dx$ and variance $E[(\varphi_j - E[\varphi_j])^2]$.

◀

2.4 Hecke eigenforms

An important question may be that what happens to modular forms at the noncommutative boundary of the modular curves. There is a variety of phenomena in the theory of modular forms that hint to the fact that a suitable class of modular forms survives on the noncommutative boundary. Introduced by Zagier is the term as quantum modular forms, to denote this but not yet sufficiently understood class of examples. Some aspects of such modular forms, pushed to the noncommutative boundary are analyzed by [112], in the form of certain averages involving modular symbols and Dirichlet series related to modular forms of weight two. Recall the main results in this case, as in the following.

Now consider the case of congruence subgroups $G = \Gamma_0(p)$ for p a prime.

Let ω be a holomorphic differential (of the first kind) on the modular curve $X_G = G \backslash \mathbb{H}$, with $G = \Gamma_0(p)$. Let $\Phi_\omega = \varphi^*(\omega) \frac{1}{dz}$ (a cusp form) defined as the pullback $\varphi^*(\omega) = \omega \circ \varphi$ under the projection $\varphi : \mathbb{H} \rightarrow X_G$.

Let Ψ be an eigenfunction for all the **Hecke** operators

$$H_n = \sum_{d|n} \sum_{b=0}^{d-1} \begin{pmatrix} \frac{n}{d} & b \\ 0 & d \end{pmatrix}$$

with $(p, n) = 1$, such that $H_n \Psi = c_n \Psi$ for some c_n . Then the **L**-function of ω is given by

$$L_\omega(s) = -\frac{(2\pi)^s}{2\pi i \Gamma(s)} \int_0^\infty \Psi(iy) y^{s-1} dy = \sum_{n=1}^\infty \frac{c_n}{n^s}$$

(where Ψ is identified with Φ_ω ?). Such a series on the right hand is said to be a **Dirichlet** series, and if c_n is assumed to be a Dirichlet character, then it is said to be the *L*-function associated to the character (cf. [20, 3.3]).

(Added). Recall from [20, 3.3] that the **Gamma** function is defined to be

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}, \quad \operatorname{Re}(s) > 0.$$

Note that $\frac{d(t_0 t)}{t_0 t} = \frac{dt}{t}$ for any $t_0 > 0$, which says that $\frac{dt}{t}$ is a Haar measure on the multiplicative group $\mathbb{R}_+^* = \{t \in \mathbb{R} \mid t > 0\}$. For any $n > 0$, compute that

$$\int_0^\infty t^s e^{-nt} \frac{dt}{t} = \int_0^\infty \left(\frac{x}{n}\right)^s e^{-x} \frac{dx}{x} = \frac{\Gamma(s)}{n^s}, \quad (x = nt). \quad \blacktriangleleft$$

There are known many interesting arithmetic properties of the integrals of such Hecke eigenforms on modular symbols (Manin [103] uncheked).

In particular, there is the following relation between L_ω and modular symbols (cf. [112, 2.2]):

$$\begin{cases} (\sigma(n) - c_n) L_\omega(1) \\ (\sigma(n) - c_n) \int_0^{i\infty} \varphi^*(\omega) \end{cases} \quad \text{or} \quad \text{(corrected)} = \sum_{d|n} \sum_{b \bmod d} \int_{[0, \frac{b}{d}]_G} \omega,$$

($b = 1, \dots, d$), where $\sigma(n) = \sum_{d|n} d$, from which it is obtained that

$$\sum_{q:(q,p)=1} \frac{1}{q^{2+t}} \sum_{q' \leq q, (q,q')=1} \int_{[0, \frac{q'}{q}]_G} \omega = \left[\frac{\zeta^{(p)}(1+t)}{\zeta^{(p)}(2+t)} - \frac{L_\omega^{(p)}(2+t)}{\zeta^{(p)}(2+t)^2} \right] \int_0^{i\infty} \varphi^* \omega,$$

where $L_\omega^{(p)}$ is the Mellin transform of Φ_ω with omitted Euler p -factor, and $\zeta^{(p)}$ is the corresponding Riemann zeta. Namely,

$$L_\omega^{(p)}(s) = \sum_{n:(n,p)=1} \frac{c_n}{n^s} \quad \text{and} \quad \zeta^{(p)}(s) = \sum_{n:(n,p)=1} \frac{1}{n^s}.$$

◇ Also, $\int_{[x,y]_G} \omega = \int_x^y \omega \circ \varphi$. Note that

$$L_\omega(1) = i \int_0^\infty \Psi(iy) dy = \sum_{n=1}^\infty \frac{c_n}{n}. \quad \square$$

Therefore, it seems to be interesting to study the properties of integrals on limiting modular symbols (cf. [112]) and their extensions. For this purpose, a suitable class of functions on the boundary $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$ (or just $[0, 1] \times \mathbb{P}$) is introduced in [112]. Such functions have the form

$$l(f, \beta) = \sum_{k=1}^\infty f(g_k(\beta), q_{k-1}(\beta)),$$

where f is a complex-valued (**Lévy**) function defined on pairs of coprime integers (q, q') with $q \geq q' \geq 1$ and with $f(q, q') = O(q^{-\varepsilon})$ ($q \rightarrow \infty$) for some $\varepsilon > 0$, and $q_k(\beta)$ are the successive denominators of the continued fraction expansion of $\beta \in (0, 1]$, so that $\frac{p_k(\beta)}{q_k(\beta)} \rightarrow \beta$ ($k \rightarrow \infty$).

The reason for that choice of functions is just the following classical result by Lévy (1929):

Proposition 2.15. *For such a function f of Lévy as above,*

$$\int_0^1 l(f, \alpha) d\alpha = \sum_{q \geq q' \geq 1, (q, q')=1} \frac{f(q, q')}{q(q+q')},$$

where both the integral and the sum converge absolutely and uniformly.

May interpret the summing over pairs of successive denominators as a property that replaces modularity, at when pushed to the boundary.

Through this class of Lévy functions, it is possible to recast certain averages related to modular symbols on X_G , completely in terms of function theory on the boundary space $\Gamma \backslash (\mathbb{P}^1(\mathbb{R}) \times \mathbb{P})$. A typical result of this sort is given as the following ([112]):

Proposition 2.16. (Edited). *Upon choosing*

$$f(q, q') = \frac{q+q'}{q^{1+t}} \int_{[0, \frac{q'}{q}]_G} \omega,$$

it is obtained that

$$\int_0^1 l(f, \beta) d\mu(\beta) = \left[\frac{\zeta^{(p)}(1+t)}{\zeta^{(p)}(2+t)} - \frac{L_\omega^{(p)}(2+t)}{\zeta^{(p)}(2+t)^2} \right] \int_0^{i\infty} \varphi^* \omega,$$

which expresses (or contains) the Mellin transform $L_\omega^{(p)}$ of a Hecke eigenform Ψ (or Φ_ω) in terms of a boundary average.

Analogous results exist for different choices of $f(q, q')$, for Eisenstein series twisted by modular symbols, double logarithms at roots of unity, and so on.

In fact that result also shows that, even when the limiting modular symbol vanishes, non-trivial homology classes still can be associated to the geodesics with irrational ends.

In fact, in the case of $G = \Gamma_0(p)$, for f the function chosen as in the proposition above, let us consider

$$C(f, \beta) \equiv \sum_{n=1}^{\infty} \frac{q_{n+1}(\beta) + q_n(\beta)}{q_{n+1}(\beta)^{1+t}} \left[0, \frac{q_n(\beta)}{q_{n+1}(\beta)} \right]_G,$$

which defines a homology class in $H_1(X_G, \{\text{cp}\}, \mathbb{R})$ for $\text{Re}(t) > 0$ and for almost all β , such that the integral average $l(f, \beta) = \int_{C(f, \beta)} \omega$ on $[0, 1]$ recovers Mellin transforms of cusp forms by

$$\begin{aligned} \int_0^1 \left(\int_{C(f, \beta)} \omega \right) d\mu(\beta) &= \int_0^1 l(f, \beta) d\mu(\beta) \\ &= \left[\frac{\zeta^{(p)}(1+t)}{\zeta^{(p)}(2+t)} - \frac{L_{\omega}^{(p)}(2+t)}{\zeta^{(p)}(2+t)^2} \right] \int_0^{i\infty} \varphi^* \omega. \end{aligned}$$

The series involved are absolutely convergent by an estimate:

$$\frac{q_{n+1}(\beta) + q_n(\beta)}{q_{n+1}(\beta)^{1+t}} \left[0, \frac{q_n(\beta)}{q_{n+1}(\beta)} \right]_G \sim e^{-(5+2t)n\lambda(\beta)} \sum_{k=1}^n \varphi \circ T^k(s).$$

2.5 Selberg zeta function

Infinite geodesics on $\overline{X_G}$ are the images of infinite geodesics on $\mathbb{H}^2 \times \mathbb{P}$ with ends in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$. Then they may be coded by (w^-, w^+, s) with end points $(w^{\pm}, s) \in \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$ with $w^- \in (-\infty, -1]$ and $w^+ \in [0, 1]$.

If w^{\pm} are not cusp points, then consider infinite continued fraction expansions of these end points:

$$w^+ = [k_0, k_1, \dots, k_r, \dots] \quad \text{and} \quad w^- = [k_{-1}, k_{-2}, \dots, k_{-n}, \dots],$$

so that the corresponding geodesics are coded by (w, s) with $s \in \mathbb{P}$, where w is a doubly infinite sequence

$$w = (\dots k_{-n} \dots k_{-1} k_0 k_1 \dots k_r \dots) \equiv (w^-)^t w_+.$$

The equivalence relation in passing to the quotient by the group action is implemented by the invertible double sided shift:

$$T(w^+, w^-, s) = \left(\frac{1}{w^+} - \lfloor \frac{1}{w^+} \rfloor, \frac{1}{w^- \pm \lfloor \frac{1}{w^+} \rfloor}, \begin{pmatrix} -\lfloor \frac{1}{w^+} \rfloor & 1 \\ 1 & 0 \end{pmatrix} s \right),$$

where both the sign \pm may be allowed.

In particular, the closed geodesics in $\overline{X_G}$ correspond to the case where the end points w^\pm are the attractive and repelling fixed points of a hyperbolic element in the group.

The **Selberg** zeta function may be a suitable generating function for the closed geodesics in $\overline{X_G}$, and is given by

$$Z_G(s) = \prod_{m=0}^{\infty} \prod_{\gamma} (1 - e^{-(s+m)l(\gamma)}),$$

where γ runs over primitive closed geodesics of length $l(\gamma)$.

That can be expressed in terms of the generalized Gauss shift as follows:

Theorem 2.17. ([18], [112]). *For $G \subset PSL_2(\mathbb{Z})$ of finite index, the Selberg zeta function is computed as the Fredholm determinant of the Ruelle transfer operator:*

$$Z_G(s) = \det(1 - L_s).$$

There is an analogous result that it holds for finite index subgroups of $SL_2(\mathbb{Z})$, with $\det(1 - L_s^2)$ involved. Also, there are generalizations to other T -invariant subsets E , where $\det(1 - L_{E,s})$ corresponds to a suitable Selberg type zeta function, that only counts certain classes of closed geodesics.

(Added). Recall from [112] that the **Selberg** zeta function is defined as

$$Z_G(s) = \prod_{m=0}^{\infty} \prod_g \det \left[1 - \frac{\det(g)^m}{N(g)^{s+m}} \rho_{\mathbb{P}}(g) \right],$$

where g runs over a set of representatives of $GL_2(\mathbb{Z})$ -conjugacy classes of primitive hyperbolic elements of $GL_2(\mathbb{Z})$, and $\mathbb{P} = GL_2(\mathbb{Z})/G$, and $\rho_{\mathbb{P}}$ is the natural representation of $GL_2(\mathbb{Z})$ on the space of functions on \mathbb{P} , and for $g \in GL_2(\mathbb{Z})$,

$$N(g) = \left(\frac{\operatorname{tr}(g) + \sqrt{d(g)}}{2} \right)^2 \quad \text{and} \quad d(g) = \operatorname{tr}(g)^2 - 4 \det(g),$$

and where an element $g \in GL_2(\mathbb{Z})$ is said to be hyperbolic if both $\operatorname{tr}(g)$ and $d(g)$ are positive, and such a hyperbolic matrix is primitive if it is not a non-trivial power of an element of $GL_2(\mathbb{Z})$. And then \dots . \blacktriangleleft

(Added). May also recall from [116] the following. Let Γ be a Fuchsian group of first kind in $SL_2(\mathbb{R})$. An element $\gamma \in \Gamma$ is said to be **hyperbolic** if it has two different real eigenvalues a and a^{-1} with $|a| > 1$. Set $N(\gamma) = a^2$ as the norm of γ . If γ is hyperbolic, then so is γ^n for any $n \in \mathbb{N}$. A hyperbolic element of Γ is said to be **primitive** if it is not a power of other hyperbolic elements. Let $\{\gamma_j\}$ be a set of representatives of conjugacy classes of a primitive hyperbolic element. For any character χ of Γ , the **Selberg** zeta function is defined to be an analytic function as an infinite product

$$Z_{\Gamma}(s, \chi) = \prod_j \prod_{n=0}^{\infty} \left(1 - \frac{\chi(\gamma_j)}{N(\gamma_j)^{s+n}} \right).$$

If any non-trivial element of Γ is hyperbolic, then \mathbb{H}^2/Γ is compact and $\mu(\mathbb{H}^2/\Gamma) = 4\pi(g-1)$, with g the genus of \mathbb{H}^2/Γ . In this case, the Selberg zeta function is analytically extended to \mathbb{C} , with genus at most 2, and with zero at $s = -n$ of order $(2n+1)(2g-2)$ for $n \in \mathbb{Z}$ non-negative, and with other zeros on the line (or fiber) $\operatorname{Re}(s) = \frac{1}{2}$, except possible finite points in $(0, 1)$. Moreover, \dots (May further read). \blacktriangleleft

Remark. ([116]). For real $s > 1$, the following infinite (Riemann) sum and the infinite (Euler) product are equal and converge absolutely:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p: \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}.$$

In particular, for m positive integers,

$$\zeta(2m) = 2^{2m-1} \pi^{2m} \frac{B_{2m}}{2m!},$$

For $s \in \mathbb{C}$, the Riemann zeta function $\zeta(s)$ is a holomorphic function for $\operatorname{Re}(s) > 1$, and is analytically extended to \mathbb{C} , with only pole at $s = 1$ of order 1. It follows from the integral expression as

$$\xi(s) \equiv \frac{1}{\pi^{\frac{s}{2}}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s} + \frac{1}{1-s} + \int_0^{\infty} (x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}) \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x} \right) dx.$$

The π -gamma-zeta function satisfies the functional equation as $\xi(s) = \xi(1-s)$. The Laurent expansion at $s = 1$ for the zeta function is given by

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} C_k (s-1)^k,$$

where γ is the Euler constant and C_k are called the generalized Euler constants.

By the Euler product, the zeta function has no zeros for $\operatorname{Re}(s) > 1$, and as well $\operatorname{Re}(s) = 1$. For $\operatorname{Re}(s) \leq 0$, the trivial zeros are given by $-2m$ of order 1 for m positive integers. It is conjectured as the Riemann hypothesis that the non-trivial zeros on the critical strip $0 < \operatorname{Re}(s) < 1$ are all on the critical line that $\operatorname{Re}(s) = \frac{1}{2}$.

◇ For instance,

$$\begin{aligned} \zeta(2) &= \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 2\pi^2 \frac{B_2}{2!}, \quad B_2 = \frac{1}{6}, \\ \xi(2) &= \frac{1}{\pi} \Gamma(1) \zeta(2) = \frac{\pi}{6} = \frac{4}{6} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \right). \end{aligned}$$

Is there the uniqueness theorem for the zeta function? □ \blacktriangleleft

Remark. ([116]). The **Bernoulli** polynomials are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n(x) t^n.$$

Then it holds that

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k(0) x^{n-k}.$$

The **Bernoulli** numbers B_n are defined to be $|B_n(0)|$, where $B_{2n+1} = 0$ for $n \geq 1$. The Bernoulli polynomials satisfy the following properties:

$$B_n(x+1) - B_n(x) = nx^{n-1} \quad \text{and} \quad \frac{d}{dx} B_n(x) = nB_{n-1}(x).$$

◇ For instance, consider the equation at $x = 0$

$$\begin{aligned} t &= (e^t - 1) \left(\sum_{n=0}^{\infty} \frac{1}{n!} B_n(0) t^n \right) \\ &= \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) \left(B_0(0) + B_1(0)t + \frac{1}{2!} B_2(0)t^2 + \cdots \right). \end{aligned}$$

Because of convergence and uniqueness, comparing coefficients determines that $B_0(0) = 1$, and $0 = \frac{1}{2}B_0(0) + B_1(0)$, so that $B_1(0) = -\frac{1}{2}$, and that $0 = \frac{1}{6}B_0(0) + \frac{1}{2}B_2(0) + \frac{1}{2}B_1(0)$, so that $B_2(0) = \frac{1}{6}$. Moreover, \cdots . $\square \blacktriangleleft$

Remark. ([116]). The gamma function is defined to be the **infinite** product:

$$\Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)^x \frac{1}{1 + \frac{x}{n}},$$

and is also defined to be the **Euler** integral of the second kind:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad x > 0.$$

The gamma function is determined uniquely as the solution for the difference equation $\Gamma(x+1) = x\Gamma(x)$, with $\Gamma(1) = 1$ and $\frac{d^2}{dx^2} \log \Gamma(x) \geq 0$ for $x > 0$. There is also the infinite (**Weierstrass**) product expression:

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n} \right) \frac{1}{e^{\frac{x}{n}}},$$

where $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n \frac{1}{k} - \log n)$ the Euler constant.

For $s \in \mathbb{C}$, $\Gamma(s)$ has poles of order 1 at $s = -n$ for integers $n \geq 0$.

◇ For instance,

$$\begin{aligned} 1 &= \Gamma(1) = \prod_{n=1}^{\infty} 1 = \int_0^{\infty} e^{-t} dt \\ &= \frac{1}{\Gamma(1)} = e^{\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n} \right) \frac{1}{e^{\frac{1}{n}}} = \lim_{N \rightarrow \infty} \frac{1}{N} \prod_{n=1}^N e^{\frac{1}{n}} \prod_{n=1}^N \left(1 + \frac{1}{n} \right) e^{-\frac{1}{n}} \end{aligned}$$

with $\frac{1}{N} \prod_{n=1}^N \left(1 + \frac{1}{n} \right) = \frac{N+1}{N} \rightarrow 1$ as $N \rightarrow \infty$. Moreover, the other conditions for the uniqueness of the gamma function may be checked. $\square \blacktriangleleft$

2.6 The modular complex and K-theory of C^* -algebras

The boundary $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$ with the action of Γ may be interpreted as the crossed product C^* -algebra $C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma$ as a noncommutative space. It is then possible to interpret some of the arithmetic properties of modular curves in terms of such operator algebras. For instance, the modular symbols determine certain elements of the K-theory of that C^* -algebra, and as well, the modular complex and the exact sequence of relative homology can be identified with the Pimsner six-term exact sequence for the K-theory of that C^* -algebra.

Now let $\Gamma = PSL_2(\mathbb{Z})$ and G be a finite index subgroup of Γ . Let $X = \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$ and $\Gamma_\sigma = \langle \sigma \rangle = \mathbb{Z}_2$ and $\Gamma_\tau = \langle \tau \rangle = \mathbb{Z}_3$.

The group Γ acts on the tree, with the set of edges, identified with Γ , and the set of vertices, identified with $\Gamma/\Gamma_\sigma \cup \Gamma/\Gamma_\tau$. This tree is embedded into \mathbb{H}^2 with vertices as elliptic points Γi and $\Gamma \rho$ and edges as geodesics segments $\Gamma(i \cap \rho)$ as the dual graph of the triangulation (or compactification) of the modular complex.

For such a group acting on a tree, there is the **Pimsner** [134] (or Natsume (after Lance)) six-term exact sequence of the K-theory groups of the corresponding crossed product C^* -algebra as

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \xrightarrow{\alpha} & K_0(\mathfrak{A} \rtimes \Gamma_\sigma) \oplus K_0(\mathfrak{A} \rtimes \Gamma_\tau) & \xrightarrow{\alpha^\sim} & K_0(\mathfrak{A} \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} \rtimes \Gamma) & \xleftarrow{\beta^\sim} & K_1(\mathfrak{A} \rtimes \Gamma_\sigma) \oplus K_1(\mathfrak{A} \rtimes \Gamma_\tau) & \xleftarrow{\beta} & K_1(\mathfrak{A}) \end{array}$$

with $\mathfrak{A} = C(X)$ (cf. [130]).

It follows from a direct inspection of the maps in the exact diagram above that it contains a subsequence canonically identified with the algebraic presentation of the modular complex, compatible with the covering maps between modular curves for different congruence subgroups. Namely,

Theorem 2.18. (Edited). *The algebraic presentation of the modular complex:*

$$\begin{array}{ccccccc} 0 \rightarrow \mathbb{H}^{\text{cp}} & \longrightarrow & \mathbb{Z}[\mathbb{P}] & \xrightarrow{(\beta_I, \beta_R)} & \mathbb{Z}[\mathbb{P}_I] \oplus \mathbb{Z}[\mathbb{P}_R] & \longrightarrow & \mathbb{Z} \rightarrow 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \parallel \\ 0 \rightarrow \mathbb{H}_{\text{cp}} & \longrightarrow & \mathbb{Z}[\mathbb{P}] \cong \mathbb{Z}[I \cup R] & \xrightarrow{(\beta_I^\sim, \beta_R^\sim)} & \mathbb{Z}[\mathbb{P}_I] \oplus \mathbb{Z}[\mathbb{P}_R] & \longrightarrow & \mathbb{Z} \rightarrow 0 \\ \parallel & & \downarrow \cong & & \downarrow \cong & & \parallel \\ H_1(\overline{X}_G \setminus \{\text{cp}\}) & \xrightarrow{j_*} & H_{\text{cp}}^{I \cup R} (\cong H_{R \cup I}^{\text{cp}}) & \xrightarrow{\partial_*} & H_0(I) \oplus H_0(R) & \longrightarrow & H_0(X_G) \end{array}$$

with $\mathbb{P}_I = \langle \sigma \rangle \backslash \mathbb{P}$ and $\mathbb{P}_R = \langle \tau \rangle \backslash \mathbb{P}$, is canonically isomorphic to the following exact sequence:

$$0 \rightarrow \ker(\beta) \rightarrow K_1(\mathfrak{A}) \xrightarrow{\beta} K_1(\mathfrak{A} \rtimes \Gamma_\sigma) \oplus K_1(\mathfrak{A} \rtimes \Gamma_\tau) \xrightarrow{\beta^\sim} \text{im}(\beta^\sim) \rightarrow 0$$

with $\mathfrak{A} = C(X) = C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P})$.

A more geometric explanation for that exact sequence is given as in the following. The connected Lie group $PSL_2(\mathbb{R})$ can be identified with the circle (or unit sphere) bundle over the hyperbolic plane $\mathbb{H}^2 = PSL_2(\mathbb{R})/PSO(2)$. Thus, the unit tangent bundle $TX_G^1 = \cup_{p \in X_G} T_p X_G^1$ to the modular curve $X_G = G \backslash \mathbb{H}^2$ is given by the quotient $G \backslash PSL_2(\mathbb{R})$, where G is a finite index subgroup of the modular group $PSL_2(\mathbb{Z})$, and $T_p X_G^1$ means the unit sphere in the tangent space $T_p X_G$.

On the other hand, we may identify $\mathbb{P}^1(\mathbb{R})$ with $B \backslash PSL_2(\mathbb{R})$ the quotient by the group B of upper triangular 2×2 matrices in $PSL_2(\mathbb{R})$. Thus, there is a strong Morita equivalence between the C^* -algebra crossed products:

$$\begin{array}{ccc} C(\mathbb{P}^1(\mathbb{R})) \rtimes G & \xrightarrow[\text{sM}]{\cong} & C(TX_G^1) \rtimes B \\ \parallel & & \parallel \\ C(B \backslash PSL_2(\mathbb{R})) \rtimes G & \xrightarrow[\text{sM}]{\cong} & C(PSL_2(\mathbb{R})/G) \rtimes B. \end{array}$$

The C^* -algebra crossed (or semi-direct) product by B is decomposed into the successive crossed products by two actions by \mathbb{R} . Therefore, the Connes' Thom isomorphism ([23]) implies the K-theory group isomorphism:

$$K_*(C(TX_G^1) \rtimes B) \cong K_*(C(TX_G^1)), \quad * = 0, 1.$$

As well, the topological K-theory group $K^*(TX_G^1)$ is related to $K^*(X_G)$ by the Gysin exact sequence.

◇ Recall from [158] that $SL_2(\mathbb{R})$ is homeomorphic to the direct product $SO(2) \times \mathbb{R}^2$ as a topological space (Cartan-Mal'cev-Iwasawa decomposition), with $SO(2) \approx S^1$. Note that $PSL_2(\mathbb{R}) \cong PSO(2) \times \mathbb{R}^2$, with $PSO(2) \cong P^1(\mathbb{R}) \cong S^1$, and

$$PSL_2(\mathbb{R})/PSO(2) \cong \mathbb{R}^2 \cong B \cong \mathbb{R} \rtimes \mathbb{R}_+^* \cong \mathbb{H}^2,$$

as a topological space, where $B \cong \mathbb{R} \rtimes \mathbb{R}_+^*$ as a Lie group, which is a connected solvable Lie group defined as the semi-direct product:

$$\left\{ (a, e^t) \text{ (or } (a, t)) \equiv \begin{pmatrix} e^t & a \\ 0 & e^{-t} \end{pmatrix} \in B \mid a \in \mathbb{R}, t \in \mathbb{R} \text{ (or } e^t \in \mathbb{R}_+^*) \right\}.$$

Note as well that

$$\begin{array}{ccc} PSO(2) \cong \mathbb{P}^1(\mathbb{R}) \cong S^1 & \longrightarrow & G \backslash \mathbb{P}^1(\mathbb{R}) \\ \downarrow & & \downarrow \\ PSL_2(\mathbb{R}) \cong PSO(2) \times \mathbb{R}^2 & \longrightarrow & TX_G^1 = G \backslash PSL_2(\mathbb{R}) \\ \downarrow & & \downarrow \\ PSL_2(\mathbb{R})/PSO(2) \approx \mathbb{H}^2 & \longrightarrow & X_G = G \backslash \mathbb{H}^2. \quad \square \end{array}$$

Remark. (Added). Recall from [142] the following. Let G be a locally compact (second countable) topological group and H and K be two closed subgroups. Assume that K acts on G/H by left translation and H acts on $K\backslash G$ by right translation. Then the C^* -algebra crossed products are strongly Morita equivalent as

$$C_0(G/H) \rtimes K \xrightarrow[\text{sM}]{\cong} C_0(K\backslash G) \rtimes H (\cong C_0(G/K) \rtimes H),$$

where $C_0(X)$ is the C^* -algebra of all continuous \mathbb{C} -valued functions on X vanishing at infinity. \blacktriangleleft

Remark. (Added). Recall from [81] that for a fiber space X over a base space B with F a fiber which has the same homology theory of the p -sphere S^p with $p \geq 1$, so that $H_q(F, \mathbb{Z}) = \mathbb{Z}$ if $q = 0, p$ and $= 0$ otherwise, then there are Gysin long exact sequences in homology and cohomology theories as

$$\begin{aligned} \cdots \rightarrow H_n(B, G) \rightarrow H_{n-p-1}(B, \mathfrak{S}) \rightarrow H_{n-1}(X, G) \xrightarrow{\pi_*} H_{n-1}(B, G) \rightarrow \cdots, \\ \cdots \leftarrow H^n(B, G) \leftarrow H^{n-p-1}(B, \mathfrak{S}) \leftarrow H^{n-1}(X, G) \xleftarrow{\pi^*} H^{n-1}(B, G) \leftarrow \cdots, \end{aligned}$$

where $\pi : X \rightarrow B$ is the projection, and $\mathfrak{S} = \mathfrak{S}_0 \otimes G$ with \mathfrak{S}_0 the Serre local system for the p -dimensional homology in \mathbb{Z} for the fiber space $X \rightarrow B$, where if \mathfrak{S}_0 is simple, then $\mathfrak{S}_0 \cong \mathbb{Z}$ and $\mathfrak{S} \cong G$. \blacktriangleleft

Of crucial importance in noncommutative geometry is a construction of the classifying space BG for proper actions by Baum-Connes [9], the homotopy quotient $X \times_G EG$, and the assembly map:

$$\mu : K^*(X \times_G EG) \rightarrow K_*(C(X) \rtimes G), \quad * = 0, 1$$

relating the topological K-theory of spaces and the analytic K-theory of C^* -algebras. In a sense, for noncommutative spaces as C^* -algebras that are obtained as quotients by bad equivalence relations, the homotopy quotient is viewed as a commutative shadow, from which much crucial information on the topology can be read (cf. [9], [25]).

In the case of G (or Γ) of X_G , we have $EG = \mathbb{H}^2$ and the map μ is an isomorphism, so that the Baum-Connes conjecture holds. By retracting $EG = \mathbb{H}^2$ to the tree of $PSL_2(\mathbb{Z})$ and applying the Mayer-Vietoris sequence to vertices and edges, obtained is the Pimsner (or Natsume) six-term exact sequence in K-theory (for crossed product C^* -algebras by Γ and subgroups).

2.7 Chaotic cosmology as Intermezzo

As a momentary digression, a topic from general relativity turns out to be closely related to the noncommutative compactification of the modular curve X_G for G the congruence subgroup $\Gamma_0(2)$ ([112]).

An important problem in cosmology seems to be understanding how anisotropy in the early universe affects the long time evolution of the space time. This problem is relevant to the study of the beginning of galaxy formation and in relating

the anisotropy of the background radiation to the appearance of the universe today.

May follow [8] (unchecked) for a brief summary of anisotropic and chaotic cosmology (cf. [7]). The simplest significant cosmological model that presents strong anisotropic properties is given by the **Kasner** metric:

$$\begin{aligned} ds^2 &= -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \\ &= \langle (-1 \oplus t^{2p_1} \oplus t^{2p_2} \oplus t^{2p_3})(dt, dx, dy, dz)^t, (dt, dx, dy, dz)^t \rangle, \end{aligned}$$

with a Riemannian metric on \mathbb{R}^4 $g = (g_{ij})_{i,j=0}^3 = \langle \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle \rangle$ equal to the 4×4 diagonal matrix above (at $(t, x, y, z) \in \mathbb{R}^4$), where the exponents p_j are constants satisfying

$$p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2.$$

◇ Note that $(g_{11}, g_{22}, g_{33}) = (t^{2p_1}, t^{2p_2}, t^{2p_3}) = (e^{2\alpha_1}, e^{2\alpha_2}, e^{2\alpha_3})$ (cf. [19]). □

Note that for $p_j = \frac{d \log g_{jj}}{d \log g}$, the first constraint $\sum_{j=1}^3 p_j = 1$ is just the condition that $\log g_{ij} = 2\alpha \delta_{ij} + \beta_{ij}$ (?) for a traceless $\beta = (\beta_{ij})$ (which may be removed, and the equations may hold by definition in that case), while the second constraint $\sum_{j=1}^3 p_j^2 = 1$ amounts to the condition that, in the Einstein equations written in terms of α and β_{ij} ,

$$\begin{aligned} \left(\frac{d\alpha}{dt} \right)^2 &= \frac{8\pi}{3} \left(T^{00} + \frac{1}{16\pi} \left(\frac{d\beta_{ij}}{dt} \right)^2 \right), \\ e^{-3\alpha} \frac{d}{dt} \left(e^{3\alpha} \frac{d\beta_{ij}}{dt} \right) &= 8\pi \left(T_{ij} - \frac{1}{3} \delta_{ij} T_{kk} \right), \end{aligned}$$

where the term T^{00} is negligible with respect to the term $\frac{1}{16\pi} \left(\frac{d\beta_{ij}}{dt} \right)^2$, which is the effective energy density of the anisotropic motion of empty space, contributing together with a matter term to the Hubble constant.

◇ In fact, if $g_{jj} = t^{2p_j}$ for $1 \leq j \leq 3$, then we have $p_j = \frac{\log g_{jj}}{2 \log t}$. The first constraint implies that $\log(g_{11}g_{22}g_{33}) = 2 \log t$, so that or as well $g_{11}g_{22}g_{33} = t^2$. The second constraint implies that $\sum_{j=1}^3 (\log g_{jj})^2 = 4(\log t)^2$. If $t^{2p_j} = e^{2\alpha_j}$, then $\alpha_j = p_j \log t$. □

Around 1970, a cosmological model as mixmaster universe is introduced by Belinskii, Khalatnikov, and Lifshitz (BKL) ([10] not checked) (cf. [19]), where the exponents p_j of the Kasner metric are allowed to depend on a parameter u (> 1), such that

$$p_1 = \frac{-u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}.$$

◇ Then $\sum_{j=1}^3 p_j = 1$ and as well,

$$\sum_{j=1}^3 p_j^2 = \frac{u^2 + (1+u)^2 + u^2(1+u)^2}{(1+u+u^2)^2} = \frac{1+2u+3u^2+2u^3+u^4}{1+2u+3u^2+2u^3+u^4} = 1! \quad \square$$

Since for fixed u , the model is given by a Kasner space time, the behavior of this universe can be approximated by a Kasner metric for certain large intervals of time. In fact, the evolution is divided into Kasner eras and each era into cycles. During each era, the mixmaster universe goes through a volume compression. Instead of resulting in a collapse, as with the Kasner metric, high negative curvature resulting in a bounce as transition to a new era, which starts again with a behavior approximated by a Kasner metric, but with a different value of the parameter u . Within each era, most of the volume compression is due to the scale factors along one the space axes, while the other scale factors alternate between phases of contraction and expansion. These alternating phases separate cycles within each era.

More precisely, we may consider metrics generalizing the Kasner metric, which still admit $SO(3)$ symmetry on the space like hypersurfaces and present a singularity at $t \rightarrow 0$. In terms of logarithmic time

$$d\Omega = -\frac{dt}{abc} = -\frac{dt}{t^{p_1+p_2+p_3}} = -\frac{dt}{t},$$

so that $\Omega = -\log t$ and $\Omega \rightarrow +\infty$ as $t \rightarrow +0$, the mixmaster universe model of BKL admits a discretization with the following properties.

◇ It may hold that $((\log a)^2)'' = (b^2 - c^2)^2 - a^4$ (cf. [7]). □

Remark. (Added). ([112]). The **mixmaster** universe is defined as the space of solutions of the vacuum Einstein equations admitting $SO(3)$ symmetry of the space like hypersurfaces, whose metric acquires a singularity as $t \rightarrow +0$, given as

$$ds^2 = dt^2 - a(t)^2 dx^2 - b(t)^2 dy^2 - c(t)^2 dz^2$$

with $a(t), b(t), c(t)$ as scale factors. A family of such metrics satisfying Einstein equations is given by Kasner solutions, so that $a(t) = t^{p_1}$, $b(t) = t^{p_2}$, $c(t) = t^{p_3}$ such that $\sum_{j=1}^3 p_j = 1 = \sum_{j=1}^3 p_j^2$. It is discovered that most of the trajectories in the mixmaster universe exhibit a chaotic behavior as $t \rightarrow +0$ backwards in time to the Big Bang.

(1). The time evolution is divided into Kasner eras $[\Omega_n, \Omega_{n+1}]$ for $n \in \mathbb{Z}$. At the beginning of each era, we have a corresponding discrete value of the parameter $u_n > 1$ such as the parameter u of p_j above.

(2). Each era, where the parameter u decreases with the (time) Ω grows, can be subdivided into cycles corresponding to the discrete steps such as $u_n - k$ for k positive integers. A change by -1 corresponds, after acting as a permutation $(12)(3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ on the space coordinate, to changing sign of u , and hence replacing contraction with expansion and vice versa. Within each cycle, the space time metric is approximated by the Kasner metric with the exponents p_j with a fixed value as $u = u_n - k > 1$.

(3). An era ends when, after a number of such cycles, the parameter u_n falls in the range $0 < u_n < 1$. Then the bouncing is given by the transition of u to $\frac{1}{u}$, which gives rise to start a new series of cycles with new Kasner parameters and

a permutation $(1)(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ of the space axis, in order to have again $p_1 < p_2 < p_3$ (with $u > 1$).

The transition formula relating the values u_n and u_{n+1} of two successive Kasner eras $[\Omega_n, \Omega_{n+1}]$ and $[\Omega_{n+1}, \Omega_{n+2}]$ is given by $u_{n+1} = (u_n - \lfloor u_n \rfloor)^{-1}$, which is exactly the shift of the continued fraction expansion as

$$x_{n+1} = \frac{1}{u_{n+1}} = T \left(x_n = \frac{1}{u_n} \right) = u_n - \lfloor u_n \rfloor.$$

The observation done previously becomes the key to a geometric description of solutions of the mixmaster universe in terms of geodesics on a modular curve (cf. [112])

Theorem 2.19. *Each infinite geodesic on the modular curve $X_{\Gamma_0(2)}$ not ending at cusps determines a mixmaster universe.*

In fact, an infinite geodesic on $X_{\Gamma_0(2)}$ is the image of an infinite geodesic on $\mathbb{H}^2 \times \mathbb{P}$ with ends in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$, under the quotient map

$$\pi_\Gamma : \mathbb{H}^2 \times \mathbb{P} \rightarrow \Gamma \backslash (\mathbb{H}^2 \times \mathbb{P}) \cong X_G,$$

where $\Gamma = PGL_2(\mathbb{Z})$, $G = \Gamma_0(2)$, and $\mathbb{P} = \Gamma/G \cong \mathbb{P}^1(\mathbb{F}_2) = \{0, 1, \infty\}$. We consider the elements of $\mathbb{P}^1(\mathbb{F}_2)$ as labels assigned to the three space axes, according to the identification

$$0 = [0 : 1] \mapsto z, \quad \infty = [1 : 0] \mapsto y, \quad 1 = [1 : 1] \mapsto x.$$

As seen, geodesics can be coded in terms of the data (w^-, w^+, s) with the action of the shift T , where $(w^\pm, s) \in \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$, with $-\infty < w^- \leq -1$ and $0 \leq w^+ \leq 1$. The data (w, s) with $w = (w^-)^t w^+$ determines a mixmaster universe, with $k_n = \lfloor u_n \rfloor = \lfloor \frac{1}{x_n} \rfloor$ in the Kasner eras, and with the transition between subsequent Kasner eras given by $x_{n+1} = T x_n \in [0, 1]$ and by the permutation of axes induced by the transformation (corrected)

$$\begin{pmatrix} k_n & 1 \\ 1 & 0 \end{pmatrix} \equiv A_n \quad \text{acting on } \mathbb{P}^1(\mathbb{F}_2), \quad \left(\text{with } A_n z = \frac{k_n z + 1}{z} \pmod{2} \right).$$

It is verified that this acts as the permutation $\begin{pmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{pmatrix}$ if k_n is even (so $k_n = 0 \pmod{2}$), and $\begin{pmatrix} 0 & 1 & \infty \\ \infty & 0 & 1 \end{pmatrix}$ if k_n is odd (so $k_n = 1 \pmod{2}$), that is, the permutation $\begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}$ if k_n is even, and the product $\begin{pmatrix} x & y & z \\ z & x & y \end{pmatrix} = (12)(3) \circ (1)(23)$ of the permutations if k_n is odd. This is precisely what is obtained in the mixmaster universe model by the repeated series of cycles with a Kasner era followed by the transition to the next era.

Data (w, s) and $T^m(w, s)$ for $m \in \mathbb{Z}$ determine the same solution up to a different choice of the initial time.

There is an additional time symmetry in this model of the evolution of mixmaster universes (cf. [8]). In fact, there is an additional parameter δ_n in the system, which measures the initial amplitude of each cycle. It is shown in [8] that this is governed by the evolution of the parameter $v_n = \frac{\delta_{n+1}(1+u_n)}{1-\delta_{n+1}}$ which is subject to the transformation across cycles $v_{n+1} = \lfloor u_n \rfloor + \frac{1}{v_n}$. By setting $y_n = \frac{1}{v_n}$, we obtain $y_{n+1} = (y_n + \lfloor \frac{1}{x_n} \rfloor)^{-1}$, and hence it follows that we can interpret the evolution determined by the data (w^\pm, s) with the shift T either as giving the complete evolution of the u -parameter towards and away from the cosmological singularity, or as giving the simultaneous evolution of the two parameters (u, v) while approaching the cosmological singularity.

This in turn determines the complete evolution of the parameters (u, δ, Ω) , where Ω_n is the starting time of each era. For the explicit recursion as $\Omega_{n+1} = \Omega_{n+1}(\Omega_n, u_n, \delta_n)$ (corrected), may see [8].

(Added). As in [112], with u_n as the initial value in that era, and $\delta_n > 0$ characterizing the relative length of the era,

$$\Omega_{n+1} = (1 + \delta_n k_n (u_n + \frac{1}{x_n})) \Omega_n,$$

where $k_n = \lfloor u_n \rfloor$ as the number of oscillations and $x_n = u_n - k_n \in (0, 1)$. If we put $\rho_n = (1 - \delta_n) \delta_n^{-1}$, then there is the recursion relation as $\rho_n x_n = (k_n + \rho_n x_{n-1})^{-1}$. \blacktriangleleft

The mentioned result on the unique T -invariant measure on $[0, 1] \times \mathbb{P}$ given as

$$d\mu(x, s) = \frac{\delta(s) dx}{(3 \log 2)(1+x)},$$

implies that the alternation of the space axes is uniform over the time evolution, namely the three axes provide the scale factor responsible for volume compression with equal frequencies.

The Perron-Frobenius operator $PF = L_1$ for the shift T on $[0, 1] \times \mathbb{P}$ yields the density of the T -invariant measure $d\mu$ above satisfying $L_1 f = f$. The top eigenvalue η_σ of L_σ is related to the topological pressure P by $\eta_\sigma = \exp(P(\sigma))$. This can be estimated numerically, using the technique of [6] and the integral kernel operator representation of [112, 1.3].

The interpretation of solutions in terms of geodesics provides a natural way to single out and study certain special classes of solutions on the basis of their geometric properties. Physically, such special classes of solutions exhibit different behaviors approaching the cosmological singularity.

For instance, the data (w^+, s) corresponding to an eventually periodic sequence $(k_j)_{j=0}^\infty$ of some period $l+1$ correspond to those geodesics on $X_{\Gamma_0(2)}$ that asymptotically wind around the closed geodesic identified with the doubly infinite sequence $(a_j)_{-\infty < i < \infty}$ of period $l+1$, so that $a_{j+k(l+1)} = a_j$ for $0 \leq j \leq l$ and $k \in \mathbb{Z}$.

Physically, these universes exhibit a pattern of cycles that recurs periodically after a finite number of Kasner eras.

Another special class of solutions is given by the Hensley Cantor sets (cf. [115]). These are the mixmaster universes for which there is a fixed upper bound N to the number of cycles in each Kasner era, called as the **controlled pulse universes**.

In terms of the continued fraction description, those solutions correspond to data (w^+, s) with w^+ in the Hensley Cantor set $E_N \subset [0, 1]$. The set E_N is given by all the points in $[0, 1]$ with add digits in the continued fraction expansion bounded by N (cf. [82]). In more geometric terms, these correspond to geodesics on the modular curve $X_{\Gamma_0(2)}$ that wander only a finite distance into a cusp.

On the set E_N , the Ruelle and Perron-Frobenius operators are given by

$$(L_{\sigma, N} f)(x, s) = \sum_{k=1}^N \frac{1}{(x+k)^{2\sigma}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} s\right).$$

This operator still has a unique invariant measure μ_N , whose density satisfies $L_{\sigma, N} f = f$, with

$$\frac{\sigma}{2} = \dim_H(E_N) = 1 - \frac{6}{\pi^2 N} - \frac{72 \log N}{\pi^4 N^2} + O\left(\frac{1}{N^2}\right)$$

the Hausdorff dimension of the Cantor set E_N . Moreover, the top eigenvalue η_σ of $L_{\sigma, N}$ is related to the Lyapunov exponent by

$$\lambda(\beta) = \frac{d}{d\sigma} \eta_\sigma \Big|_{\sigma = \dim_H(E_N)},$$

for μ_N -almost all $\beta \in E_N$.

A consequence of the characterization of the time evolution in terms of the dynamical system on $[0, 1] \times \mathbb{P}$ is that we can study global properties of suitable moduli spaces of mixmaster universes. For instance, the moduli space for time evolutions of the u -parameter approaching the cosmological singularity as $\Omega \rightarrow \infty$ is given by the quotient of $[0, 1] \times \mathbb{P}$ by the action of the shift T .

Similarly, when we restrict to special classes of solutions, such as the Hensley Cantor set E_N , we can consider the moduli space $(E_N \times \mathbb{P})/T$ modulo the action of T . In this example, the dynamical system by the shift T acting on $E_N \times \mathbb{P}$ is a subshift of finite type, and the resulting noncommutative space is a Cuntz-Krieger C^* -algebra [66], generated by partial isometries with some relations, in an interesting class of C^* -algebras. Another such C^* -algebra plays a fundamental role in the geometry at arithmetic infinity, as the topic of the next section.

In the example of the mixmaster universe dynamics on the Hensley Cantor sets, the shift T is described by the **Markov** partition

$$\mathcal{A}_N = \{(k, t), (l, s) \in \{1, \dots, N\} \times \mathbb{P} \mid U_{k,t} = U_k \times \{t\} \subset T(U_{l,s})\},$$

where each $U_k = [\frac{1}{k+1}, \frac{1}{k}] \cap E_N$ are the clopen subsets of E_N where the local inverses of T are defined. This Markov partition determines the matrix $A_N = (A_{(k,t), (l,s)})$ with entries 1 if $U_{k,t} \subset T(U_{l,s})$ and zero otherwise.

Proposition 2.20. *The 3×3 sub-matrices $A_{k,l} = (A_{(k,t),(l,s)})_{s,t \in \mathbb{P}}$ of the matrix A_N are of the form either*

$$A_{k,2m} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad A_{k,2m+1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This is because the condition $U_{k,t} \subset T(U_{l,s})$ can be written as $\begin{pmatrix} 0 & 1 \\ 1 & l \end{pmatrix} s = t$, together with the fact that the transformation by the same 2×2 matrix acts on $\mathbb{P}^1(\mathbb{F}_2)$ as the permutation $\begin{pmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{pmatrix}$ when l is even, and $\begin{pmatrix} 0 & 1 & \infty \\ 1 & \infty & 0 \end{pmatrix}$ if l is odd.

As a noncommutative space associated to the Markov partition as well as the matrix A_N we may consider the **Cuntz-Krieger** C^* -algebra \mathcal{O}_{A_N} , which is defined to be the universal C^* -algebra generated by partial isometries $S_{k,t}$ such that

$$\sum_{(k,t) \in \{1, \dots, N\} \times \mathbb{P}} S_{k,t} S_{k,t}^* = 1 \quad \text{and} \quad S_{l,s}^* S_{l,s} = \sum_{(k,t)} A_{(k,t),(l,s)} S_{k,t} S_{k,t}^*.$$

Topological invariants of this C^* -algebra reflect dynamical properties of the shift T . (No figure provided).

(Added). As mentioned in [115], the irreducibility for the matrix A_N corresponds to that the associated directed graph is strongly connected in the sense that any two vertices are connected by an oriented path of edges. Since the matrix A_N has the form

$$A_N = \begin{pmatrix} M & M & \dots & M \\ L & L & \dots & L \\ M & M & \dots & M \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

the irreducibility for A_N follows from that for A_2 with $N = 2$. it then follows that the matrix A_N is aperiodic in the sense that one is the period, defined to be the greatest common divisor of the lengths of the closed directed paths. \blacktriangleleft

3 Quantum statistical mechanics and Galois theory

This section is a brief review of Alain Connes and Marcolli ([41], [44]) on quantum statistical mechanics of \mathbb{Q} -lattices, and of CM with Ramachandran ([48] as well as [49]) on the construction of a quantum statistical mechanical system that fully recovers the explicit class field theory for imaginary quadratic fields in such a way similar to what the Bost-Connes system does for the explicit class field theory of \mathbb{Q} of rational numbers. Also included at the end of this chapter is

a brief review of Manin [110] on real quadratic fields and noncommutative tori with real multiplication.

As the main notion in this section, we consider commensurability classes of \mathbb{K} -lattices, where \mathbb{K} denotes either the rational field \mathbb{Q} or an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ for d some positive integer. Begin with the case of \mathbb{Q} .

Definition 3.1. An (n -dimensional) \mathbb{Q} -**lattice** in \mathbb{R}^n the real n -dimensional Euclidean space is defined to be a pair (Λ, φ) of a lattice Λ in \mathbb{R}^n , that is, a cocompact free abelian subgroup of \mathbb{R}^n of rank n , together with a system of labels of its torsion points given by a homomorphism of abelian groups: $\varphi : \mathbb{Q}^n / \mathbb{Z}^n \rightarrow \mathbb{Q}\Lambda / \Lambda$.

A \mathbb{Q} -lattice is said to be **invertible** if φ is an isomorphism.

Two \mathbb{Q} -lattices (Λ_j, φ_j) for $j = 1, 2$ are said to be **commensurable**, denoted as $(\Lambda_1, \varphi_1) \sim (\Lambda_2, \varphi_2)$ if $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\varphi_1 = \varphi_2 \bmod \Lambda_1 + \Lambda_2$.

May check that commensurability indeed defines an equivalence relation among \mathbb{Q} -lattices.

Proof. (Added). If $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\mathbb{Q}\Lambda_2 = \mathbb{Q}\Lambda_3$, then $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_3$. If $\varphi_1 = \varphi_2 \bmod \Lambda_1 + \Lambda_2$ and $\varphi_2 = \varphi_3 \bmod \Lambda_2 + \Lambda_3$, then $\varphi_1 = \varphi_3 \bmod \Lambda_1 + \Lambda_3$. Because the mod $\Lambda_i + \Lambda_j$ may be replaced with the mod Λ_i or Λ_j . (It makes sense.) \square

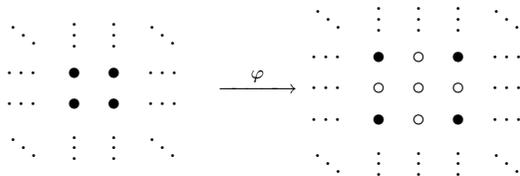
The interesting aspect of this equivalence relation is that the quotient provides another typical case that is described through noncommutative geometry. For this it is crucial to consider non-invertible \mathbb{Q} -lattices. In fact, most (?) \mathbb{Q} -lattices are not commensurable to an invertible one, while two invertible \mathbb{Q} -lattices are commensurable if and only if they are equal.

Example 3.2. (Added). For instance, let $n = 1$. Then $\Lambda = k\mathbb{Z}$ for some positive integer k , with some $\varphi = k\rho : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}\Lambda/\Lambda$ for some $\rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, identified with

$$\text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{T}) = (\mathbb{Q}/\mathbb{Z})^\wedge = \varprojlim (\mathbb{Z}/n\mathbb{Z}),$$

(which is the profinite completion of \mathbb{Z} , denoted as $\overline{\mathbb{Z}}^p$, and in fact is isomorphic to $\prod_{p,k} \mathbb{Z}_{p^k}$ the direct product of indecomposable finite cyclic groups $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$ over prime numbers p and positive integers k (cf. [139, 2.9])), where we may identify \mathbb{Q}/\mathbb{Z} with $\mathbb{Q} \cap [0, 1) \bmod 1$ and $\mathbb{Q}\Lambda/\Lambda = \mathbb{Q} \cap [0, k) \bmod k$. Note that $\mathbb{Q}(k\mathbb{Z}) = \mathbb{Q}(l\mathbb{Z})$ if and only if $k = l$. If so, $k\mathbb{Z} + l\mathbb{Z} = k\mathbb{Z} \subset \mathbb{Z}$. A \mathbb{Q} -lattice in \mathbb{R} is invertible if and only if $\rho \in \text{Aut}(\mathbb{Q}/\mathbb{Z})$. (It seems that most (or non-degenerate) \mathbb{Q} -lattices are invertible.) \blacktriangleleft

◇ As a figure, as a 2-dimensional \mathbb{Q} -lattice $(2\mathbb{Z} \times 2\mathbb{Z}, \varphi = 2(\text{id}_{\mathbb{Z}} \times \text{id}_{\mathbb{Z}}))$ as



(This seems to be invertible in the sense of that definition, but should be non-invertible as an element of $\text{End}(\mathbb{Q}^2/\mathbb{Z}^2)$.) \square

In the following, we denote by Lt_n the set of **commensurability** classes of n -dimensional \mathbb{Q} -lattices in \mathbb{R}^n . The topology on this set is encoded in a noncommutative algebra of coordinates, as a C^* -algebra denoted by $C^*(\text{Lt}_n)$. We discuss in some detail the case of $n = 1$ corresponding to the Bost-Connes system and the case of $n = 2$, considered in ([41], [44]). In these cases, we further consider the spaces of commensurability classes of 1 and 2-dimensional \mathbb{Q} -lattices up to scaling, namely the respective quotients $\text{Lt}_1/\mathbb{R}_+^*$ and Lt_2/\mathbb{C}^* . The corresponding C^* -algebras denoted by $C^*(\text{Lt}_1/\mathbb{R}_+^*)$ and $C^*(\text{Lt}_2/\mathbb{C}^*)$ are endowed with a natural time evolution as a one-parameter family of automorphisms. Thus, may consider the C^* -algebras as quantum statistical mechanical systems, and look for equilibrium states, depending on a thermo-dynamic parameter β as inverse temperature. The interesting connection to arithmetics arises from the fact that the action of symmetries of the system on equilibrium states at zero temperature can be described in terms of Galois theory. In the 1-dimensional case as the Bost-Connes system, this corresponds to the class field theory of \mathbb{Q} , namely the Galois theory of the cyclotomic field \mathbb{Q}^{cyc} . In the two dimensional case of the Connes-Marcolli system, that picture is more elaborate and involves the automorphisms of the field of modular functions.

3.1 Quantum statistical mechanics

In classical statistical mechanics, a state (of the system) is a probability measure μ on the phase (metric) space X (such as a point measure or a point in X), that assigns to each observable (function) f as a μ -measurable function on X , its expectation value in the form of an average as the integration $\int_X f d\mu$.

In particular, for a Hamiltonian system, the macroscopic thermo-dynamic behavior is described via the Gibbs canonical ensemble. This is the normalized (**Gibbs**) measure defined as

$$d\mu_G = \frac{1}{Z} e^{-\beta H} d\mu_L, \quad \text{by} \quad Z = \int_X e^{-\beta H} d\mu_L$$

with a thermo-dynamic parameter $\beta = \frac{1}{kT}$, for T temperature and k the Boltzmann constant (which we can set equal to 1 suitably), where μ_L is the Liouville measure (or a certain Lebesgue measure locally) on a manifold X .

◇ For instance, a **Hamiltonian** system is given as

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial y}(x, y) \\ -\frac{\partial H}{\partial x}(x, y) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_x \\ H_y \end{pmatrix},$$

where H is a real-valued, C^2 -class function on \mathbb{R}^2 (or \mathbb{R}^{2n}) (cf. [152]). \square

A (continuous) quantum statistical mechanical system (QSMS) consists of the data of an algebra of observables as a C^* -algebra \mathfrak{A} , together with a time evolution given as a 1-parameter family of $*$ -automorphisms $\sigma_t \in \text{Aut}(\mathfrak{A})$ for $t \in \mathbb{R}$. We may refer to the triple $(\mathfrak{A}, \sigma, \mathbb{R})$ as a C^* -dynamical system, which defines

the crossed product C^* -algebra $\mathfrak{A} \rtimes_{\sigma} \mathbb{R}$. These data describe the microscopic quantum mechanical evolution of the system. (The discrete version of QSMS may be defined by replacing \mathbb{R} with \mathbb{Z} .)

The macroscopic thermo-dynamical properties are encoded in the equilibrium states of the system, depending on the inverse temperature $\beta = \frac{1}{T}$. While the Gibbs measure in the classical case is defined in terms of the Hamiltonian and the symplectic structure on the phase space, the notion of equilibrium state in the quantum statistical mechanical setting only depends on the algebra of observable (operators) and its time evolution, and does not involve any additional structure such as the symplectic structure or the approximation by regions of finite volume.

Recall the notion of states on C^* -algebras, which can be viewed as probability measures on noncommutative spaces.

Definition 3.3. A **state** on a unital C^* -algebra \mathfrak{A} is defined to be a linear functional $\varphi : \mathfrak{A} \rightarrow \mathbb{C}$ such that normalization $\varphi(1) = 1$ and positivity $\varphi(a^*a) \geq 0$ for any $a \in \mathfrak{A}$ are satisfied.

For a non-unital (or unital) C^* -algebra \mathfrak{A} , the normalization condition may be replaced by the uniform norm $\|\varphi\| = 1$, where

$$\|\varphi\| = \sup_{x \in \mathfrak{A}, \|x\| \leq 1} |\varphi(x)|.$$

Remark. (Added). As obtained in [127], a positive linear functional on a C^* -algebra is bounded. A bounded linear functional on a C^* -algebra is positive if and only if its uniform norm is equal to the limit (or the sup above) with respect to an (or any) approximate unit. ◀

Before giving the general definition of equilibrium states via the KMS condition, we consider a simple case of a system with finitely many quantum degrees of freedom, to see what equilibrium states are. In such a case, the algebra of observable operators is given by a C^* -algebra \mathfrak{A} on a Hilbert space H (or \mathbb{C}^n) (such as $\mathbb{K}(H)$ the C^* -algebra of compact operators, or $M_n(\mathbb{C})$), and the time evolution as an \mathbb{R} -action by $*$ -automorphisms on \mathfrak{A} is defined as $\sigma_t(a) = e^{itK} a e^{-itK}$ for $a \in \mathfrak{A}$ and $t \in \mathbb{R}$, where K is a (positive) self-adjoint operator on H such that $\exp(-\beta K)$ is of trace class for any $\beta > 0$, which may have its inverse unbounded, and does converge to either zero or the projection to its kernel as $\beta \rightarrow \infty$ if K is positive, by functional calculus.

◊ May check that for any $a, b \in \mathfrak{A}$ and $t, s \in \mathbb{R}$,

$$\begin{aligned} \sigma_t(ab) &= e^{itK} a b e^{-itK} = e^{itK} a e^{-itK} e^{itK} b e^{-itK} = \sigma_t(a) \sigma_t(b), \\ \sigma_t(a^*) &= e^{itK} a^* e^{-itK} = (e^{itK} a e^{-itK})^* = (\sigma_t(a))^*, \\ \sigma_{t+s}(a) &= e^{itK} e^{isK} a e^{-isK} e^{-itK} = (\sigma_t \circ \sigma_s)(a). \end{aligned}$$

These conditions are the reason why it has such a form and K should be self-adjoint. The set $\mathbb{T}(H) = \mathfrak{L}^1(H)$ of all trace class operators on H is a self-adjoint ideal in $\mathbb{B}(H)$ the vN C^* -algebra of all bounded operators on H and is contained

in $\mathbb{K}(H)$. Indeed, for any $a \in \mathbb{B}(H)$ and $b \in \mathbb{T}(H)$, $|\operatorname{tr}(ab)| \leq \|a\| \operatorname{tr}(|b|)$ and $\|b\| \leq \operatorname{tr}((|b|^{\frac{1}{2}})^2) = \operatorname{tr}(|b|)$. Note that $\mathbb{B}(H) \cong \mathbb{T}(H)^*$ by this trace duality. \square

For such a system, the analog of the (Gibbs) measure above, as the **Gibbs** equilibrium state is defined as

$$\varphi_K(a) = \frac{1}{Z} \operatorname{tr}(ae^{-\beta K}), \quad Z = \operatorname{tr}(\exp(-\beta K)), \quad a \in \mathfrak{A},$$

with $\operatorname{tr}(\cdot)$ the canonical trace and Z as the normalization factor.

\diamond If \mathfrak{A} is unital, $\varphi_K(1) = 1$. Also, for any $a \in \mathfrak{A}$,

$$\varphi_K(a^*a) = \frac{1}{Z} \operatorname{tr}(a^*ae^{-\frac{\beta}{2}K}e^{-\frac{\beta}{2}K}) = \frac{1}{Z} \operatorname{tr}(e^{-\frac{\beta}{2}K}a^*ae^{-\frac{\beta}{2}K}) \geq 0.$$

Moreover,

$$\varphi_K(a\sigma_{i\beta}(b)) = \frac{1}{Z} \operatorname{tr}(ae^{-\beta K}be^{\beta K}e^{-\beta K}) = \frac{1}{Z} \operatorname{tr}(bae^{-\beta K}) = \varphi_K(ba)$$

(cf. [16, II, 5.3]). Note also that if $\mathfrak{A} = M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, then for $a, b \in \mathfrak{A}$,

$$\operatorname{tr}(ab^*) = \sum_{i=1}^n \sum_{k=1}^n a_{ik} \overline{b_{ik}} = \langle (a_{ij}), (b_{ij}) \rangle_{\mathbb{C}^{n^2}}. \quad \square$$

The Kubo-Martin-Schwinger (KMS) condition describing equilibrium states of more general quantum statistical mechanical systems generalizes such a state φ_K beyond the range of temperatures where $\exp(-\beta K)$ is of trace class (cf. [16, II, 5.3], [77], [78]).

Definition 3.4. Let $(\mathfrak{A}, \sigma, \mathbb{R})$ be a continuous C^* -dynamical system. A state φ on \mathfrak{A} is said to satisfy the **KMS** condition at inverse temperature $0 < \beta < \infty$, called a KMS_β state, if for any $a, b \in \mathfrak{A}$, there exists a holomorphic function $f_{a,b}(z)$ on the strip $0 < \operatorname{Im}(z) < \beta$, which is continuous and bounded on the closure of the strip, such that for any $t \in \mathbb{R}$,

$$f_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad f_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a).$$

(In particular, $\varphi(a\sigma_{i\beta}(b)) = \varphi(ba)$.)

The KMS_∞ states on \mathfrak{A} are defined as weak limits of KMS_β states as

$$\varphi_\infty(a) = \lim_{\beta \rightarrow \infty} \varphi_\beta(a), \quad a \in \mathfrak{A}.$$

The definition of KMS_∞ states is stronger than the one often adopted in the literature, which simply uses the existence of such bounded holomorphic functions $g_{a,b}(z)$ on the upper half plane such that $g_{a,b}(t) = \varphi(a\sigma_t(b))$ only. This notion, which may be called the ground states of the system, is weaker than the notion of KMS_∞ states given above.

\diamond Also, equivalently, a state φ on \mathfrak{A} is said to be a σ **ground** state if $-i\varphi(a^*\delta(a)) \geq 0$ for any a in the domain of the infinitesimal generator δ of σ , defined by $\delta(a) = [iK, a]$ (cf. [16, II, 5.3]). \square

For example, in the simplest case where the time evolution is trivial by the identity map, all states are ground states by trivial functions, while only tracial states are KMS_∞ states. As another advantage, for any $0 < \beta \leq \infty$, the KMS_β states form a weakly compact, convex set \mathfrak{C}_β (with the weak* topology), hence we can consider the set \mathcal{E}_β of all extremal points of \mathfrak{C}_β as equilibrium states of \mathfrak{A} .

◊ If $\varphi, \psi \in \mathfrak{C}_\beta$ with holomorphic $f_{a,b}$ and $g_{a,b}$ respectively and $s \in [0, 1]$, then $t\varphi + (1-t)\psi \in \mathfrak{C}_\beta$ with $tf_{a,b} + (1-t)g_{a,b}$ holomorphic. If a net $\{\varphi_\lambda\} \subset \mathfrak{C}_\beta$ with $f_{a,b,\lambda}$ holomorphic converges weakly to ψ , namely, $\varphi_\lambda(a) \rightarrow \psi(a)$ for any $a \in \mathfrak{A}$, then $\psi \in \mathfrak{C}_\beta$ as well with holomorphic limit. \square

Remark. (Added). It is known (cf. [153, 5.1]) that for $\varphi \in \mathfrak{C}_\beta$, it holds that $\varphi \in \mathcal{E}_\beta$ if and only if the associated GNS representation π_φ of \mathfrak{A} is a factor representation in the sense that the von Neumann algebra generated by $\pi_\varphi(\mathfrak{A})$ is a factor, namely has its center trivial. \blacktriangleleft

These provide a notion of points for the underlying noncommutative space. This becomes especially useful in connection to an arithmetic structure specified by an arithmetic subalgebra of \mathfrak{A} , on which the KMS_∞ states are evaluated. This plays a key role in the relation between the symmetries of the system and the action of the Galois group on states of \mathcal{E}_∞ on the subalgebra.

Symmetries. An important role in quantum statistical mechanics is played by symmetries. Typically, symmetries of such a C^* -algebra \mathfrak{A} (or its subalgebra) compatible with the time evolution induce symmetries the set \mathcal{E}_β of the equilibrium states of \mathfrak{A} at different temperatures. Especially important are the phenomena of symmetry breaking. In such cases, there is a global underlying group G of symmetries of the algebra \mathfrak{A} , but in certain ranges of temperature, the choice of an equilibrium state φ breaks the symmetry to a smaller stabilizer subgroup $G_\varphi = \{g \in G \mid g^*\varphi = \varphi\}$ of G and φ under the induced action of G on states. Various systems are known to exhibit either no, one, or more phase transitions. As a typical situation in physical systems, there is the unique KMS state for all values of the temperature T above a certain critical temperature T_c (so $\beta = \frac{1}{T} < \frac{1}{T_c} = \beta_c$). This corresponds to a chaotic phase such as randomly distributed spins in a ferromagnet. When the system cools down and reaches the critical temperature, the unique equilibrium state branches off into a larger KMS_β set ($\beta > \beta_c$) and the symmetry is broken by the choice of an extremal state in \mathcal{E}_β .

The case of $\text{Lt}_1/\mathbb{R}_+^*$ gives rise to a system with a single phase transition ([13]), while in the case of Lt_2/\mathbb{C}^* the system has multiple phase transitions ([41]).

An important point is that we need to consider both symmetries by automorphisms and by endomorphisms as well.

- [Automorphisms]. A subgroup G of the automorphism group $\text{Aut}(\mathfrak{A})$ of a C^* -algebra \mathfrak{A} (or its subalgebra) is said to be **compatible** with the \mathbb{R} -action $\sigma_t \in \text{Aut}(\mathfrak{A})$ (or σ -compatible, for short) if $g \circ \sigma_t = g\sigma_t = \sigma_t g$ for any $g \in G$ and $t \in \mathbb{R}$. Then there is an induced action of G on the sets \mathfrak{C}_β of KMS states and in particular on the sets \mathcal{E}_β .

◇ Check that for any $g, h \in G$ and $a, b \in \mathfrak{A}$ and $\varphi \in \mathfrak{E}_\beta$,

$$\begin{aligned} (gh)^*\varphi(a) &= \varphi((gh)^{-1}(a)) = \varphi(h^{-1}g^{-1}(a)) = g^*(h^*\varphi(a)), \\ g^*\varphi(a\sigma_t(b)) &= \varphi(g^{-1}(a)g^{-1}(\sigma_t(b))) = \varphi(g^{-1}(a)\sigma_t(g^{-1}(b))) = f_{g^{-1}(a),g^{-1}(b)}(t), \\ g^*\varphi(\sigma_t(b)a) &= \varphi(g^{-1}(\sigma_t(b))g^{-1}(a)) = \varphi(\sigma_t(g^{-1}(b))g^{-1}(a)) = f_{g^{-1}(a),g^{-1}(b)}(t+i\beta). \end{aligned}$$

Also, if $\varphi \in \mathfrak{E}_\beta$, then assume that $g^*\varphi = t\psi + (1-t)\rho$ for some $\psi, \rho \in \mathfrak{E}_\beta$ and $0 < t < 1$. Then $\varphi = t(g^{-1})^*\psi + (1-t)(g^{-1})^*\rho$ with $(g^{-1})^*\psi, (g^{-1})^*\rho \in \mathfrak{E}_\beta$. Thus, $\varphi = (g^{-1})^*\psi = (g^{-1})^*\rho$. Hence $g^*\varphi = \psi = \rho \in \mathfrak{E}_\beta$ (cf. [127, Appendix]). \square

The adjoint, group \mathbb{Z} action $\text{Ad}(u)$ on \mathfrak{A} by a unitary $u \in \mathfrak{A}$ is defined as an **inner** $*$ -automorphism of \mathfrak{A} as $\text{Ad}(u)(a) = uau^*$ for $a \in \mathfrak{A}$. If $\sigma_t(u) = u$, then we may say that $\text{Ad}(u)$ is σ -invariant. If so, the subgroup $\langle \text{Ad}(u) \rangle$ generated by $\text{Ad}(u)$ is compatible with the \mathbb{R} -action σ_t , and it acts trivially on the sets \mathfrak{E}_β of KMS states.

◇ Note that for any $a \in \mathfrak{A}$ and $t \in \mathbb{R}$,

$$\text{Ad}(u)(\sigma_t(a)) = u\sigma_t(a)u^* = \sigma_t(uau^*) = \sigma_t\text{Ad}(u)(a).$$

Also, if $\varphi \in \mathfrak{E}_\beta$, then

$$\begin{aligned} \text{Ad}(u)^*\varphi(a\sigma_t(b)) &= \varphi(\text{Ad}(u)^{-1}(a\sigma_t(b))) = \varphi(\text{Ad}(u^*)(a)\sigma_t(\text{Ad}(u^*)(b))) \\ &= f_{u^*au, u^*bu}(t), \\ \text{Ad}(u)^*\varphi(\sigma_t(b)a) &= \varphi(\text{Ad}(u)^{-1}(\sigma_t(b)a)) = \varphi(\sigma_t(\text{Ad}(u^*)(b))\text{Ad}(u^*)(a)) \\ &= f_{u^*au, u^*bu}(t+i\beta), \end{aligned}$$

both of which should be equal to $f_{a,b}(t)$ and $f_{a,b}(t+i\beta)$ respectively, if trivial (?) (or if trivial up to such inner automorphisms). \square

• [Endomorphisms]. Let ρ be a $*$ -homomorphism (or $*$ -endomorphism) from a C^* -algebra \mathfrak{A} (or its subalgebra) to itself. Assume that ρ is σ -compatible in that sense. If \mathfrak{A} is unital, then $\rho(1) = \rho(1)\rho(1) (= \rho(1)^*)$ is an idempotent (and a projection) of \mathfrak{A} . Set $p = \rho(1)$. If $\varphi \in \mathfrak{E}_\beta$ is an extremal KMS state on \mathfrak{A} such that $\varphi(p) \neq 0$, then there is a pull back $\rho^*\varphi$ defined well as $\rho^*\varphi = \varphi(p)^{-1}\varphi \circ \rho$.

◇ Note that $1 \geq \varphi(p) = \varphi(p^*p) > 0$. Hence $\varphi(p) = 1$ if and only if $\varphi(1-p) = 0$. Check that for any $a, b \in \mathfrak{A}$,

$$\begin{aligned} \rho^*\varphi(1) &= \varphi(p)^{-1}\varphi(p) = 1, \\ \rho^*\varphi(a\sigma_t(b)) &= \varphi(p)^{-1}\varphi(\rho(a)\sigma_t(\rho(b))) = f_{\varphi(p)^{-1}\rho(a), \rho(b)}(t), \\ \rho^*\varphi(\sigma_t(b)a) &= \varphi(p)^{-1}\varphi(\sigma_t(\rho(b))\rho(a)) = f_{\varphi(p)^{-1}\rho(a), \rho(b)}(t+i\beta) \end{aligned}$$

since $\varphi \in \mathfrak{E}_\beta$. Hence $\rho^*\varphi \in \mathfrak{E}_\beta$. Suppose now that $\rho^*\varphi = t\psi_1 + (1-t)\psi_2$ with $\psi_1, \psi_2 \in \mathfrak{E}_\beta$ and $0 < t < 1$. Then for any $a \in \mathfrak{A}$,

$$\varphi(\rho(a)) = t\varphi(p)\psi_1(a) + (1-t)\varphi(p)\psi_2(a).$$

But $\varphi(p)\psi_j$ may not be a state on \mathfrak{A} since $\varphi(p)\psi_j(1) = \varphi(p)$. If both are states, then $\varphi(p) = 1$. It seems that $\rho^*\varphi \in \mathfrak{E}_\beta$ is provided only that ρ is an $*$ -automorphism of \mathfrak{A} . \square

Now let s be an isometry of a unital \mathfrak{A} , namely, $s^*s = 1 \in \mathfrak{A}$. Suppose that $ss^* = p$ and s is compatible with the time evolution in the sense that $\sigma_t(s) = \lambda^{it}s$ for some $\lambda > 0$.

◊ Note that $\lambda^{it} = e^{it \log \lambda} \in \mathbb{T}$ the 1-torus. □

The adjoint, semigroup \mathbb{N} action $\text{Ad}(s)$ on \mathfrak{A} is defined as an **inner** endomorphism of \mathfrak{A} as $\text{Ad}(s)(a) = sas^*$ for $a \in \mathfrak{A}$. Note that $\text{Ad}(s)^n = \text{Ad}(s^n)$ for $n \in \mathbb{N}$. If the isometry s is σ -invariant in the sense above, then the semi-group generated by $\text{Ad}(s)$ is compatible with the time evolution in the following sense.

◊ Check that for any $a \in \mathfrak{A}$,

$$\text{Ad}(s)(\sigma_t(a)) = \sigma_t(sas^*) = \sigma_t(\text{Ad}(s)(a)). \quad \square$$

For any $\varphi \in \mathfrak{E}_\beta$, then $\text{Ad}(s)^*\varphi \in \mathfrak{E}_\beta$ is defined as $\rho^*\varphi$ with $\rho = \text{Ad}(s)$. But, as checked above, this induced action is trivial (?) (or trivial up to such inner automorphisms).

In defining the induced action on states by endomorphisms, it is necessary to be careful. In fact, there are cases where for KMS_∞ states φ , it only holds that $\varphi(p) = 0$, yet it is still possible to define an interesting action by endomorphisms by a procedure of warming up and cooling down. For this to work, we need sufficiently favorable conditions as that the **warming** up (in temperature $T = \frac{1}{\beta}$) map defined by

$$w_\beta(\varphi)(a) = \text{tr}(e^{-\beta K})^{-1} \text{tr}(\pi_\varphi(a)e^{-\beta K})$$

gives a homeomorphism $w_\beta : \mathfrak{E}_\infty \rightarrow \mathfrak{E}_\beta$ for β sufficiently large. (That $e^{-\beta K}$ may be identified with the similar on the representation space of π_φ .) Then define the induced action by

$$(\rho^*\varphi)(a) = \lim_{\beta \rightarrow \infty} (\rho^*w_\beta(\varphi))(a), \quad \varphi \in \mathfrak{E}_\infty, a \in \mathfrak{A}$$

as the **cooling** down (in temperature) limit.

◊ Check that $w_\beta(\varphi)(1) = 1$, and

$$\begin{aligned} & \frac{w_\beta(\varphi)(a\sigma_{t+h}(b)) - w_\beta(\varphi)(a\sigma_t(b))}{h} \\ &= \text{tr}(e^{-\beta K})^{-1} \text{tr}(\pi_\varphi(a)\pi_\varphi\left(\frac{\sigma_{t+h}(b) - \sigma_t(b)}{h}\right) e^{-\beta K}) \end{aligned}$$

and thus the limit as $h \rightarrow t$ exists if $\sigma_t(b)$ is analytic under the GNS representation π_φ , or if b is σ -analytic (cf. [153, 5.1]). The same is applied for $w_\beta(\varphi)(\sigma_t(b)a)$. Another difficulty may be to find such a holomorphic path $f_{a,b}(t + is)$ with $0 \leq s \leq \beta$ between those analytic functions on the boundary at both $s = 0$ and $s = \beta$. But, we can take

$$w_\beta(\varphi)\left(\frac{\beta - s}{\beta}a\sigma_t(b) + \frac{s}{\beta}\sigma_t(b)a\right)$$

as a possible easy choice for the path. Masaka! (in Japanese, similar to saying Incredible!) □

3.2 The Bost-Connes system

May refer to [48] for the following geometric point of view on the Bost-Connes (BC) system.

We use the notation $\mathrm{Sm}(G, X) = G(\mathbb{Q}) \backslash (G(\mathbb{A}_f) \times X)$ for **Shi-mura** varieties (cf. [122]), where $\mathbb{A}_f = \overline{\mathbb{Z}}^p \otimes \mathbb{Q}$ denotes the finite **adèles** of \mathbb{Q} , with $\overline{\mathbb{Z}}^p = \varprojlim \mathbb{Z}_n$, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. The simplest case is that for $G = GL_1$, so that

$$\mathrm{Sm}(GL_1, \{\pm 1\}) = GL_1(\mathbb{Q}) \backslash (GL_1(\mathbb{A}_f) \times \{\pm 1\}) = \mathbb{Q}_+^* \backslash \mathbb{A}_f^*.$$

Lemma 3.5. *The above quotient as a Shi-mura variety parameterizes the invertible, 1-dimensional \mathbb{Q} -lattices, up to scaling.*

In fact, note that a 1-dimensional \mathbb{Q} -lattice can be written as $(\Lambda, \varphi) = (\lambda\mathbb{Z}, \lambda\rho)$ for some $\lambda > 0$ and some

$$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathbb{Z}_n \cong \overline{\mathbb{Z}}^p.$$

Thus, the set of all 1-dimensional \mathbb{Q} -lattices, up to scaling can be identified with $\overline{\mathbb{Z}}^p$. The invertible 1-dimensional \mathbb{Q} -lattices correspond to the (invertible) elements of $\overline{\mathbb{Z}}^{p,*}$, which in turn is identified with $GL_1^+(\mathbb{Q}) \backslash GL_1(\mathbb{A}_f)$.

The quotient $\mathrm{Sm}(GL_1, \{\pm 1\})$ can be thought of geometrically as the Shimura variety associated to the cyclotomic tower (cf. [48]). This is the tower \mathcal{V} of arithmetic varieties over $V = \mathrm{Spec}(\mathbb{Z})$, with $V_n = \mathrm{Spec}(\mathbb{Z}[\zeta_n])$, where ζ_n is a primitive n -th root of unity. The group $\mathrm{Aut}_V(V_n)$ of deck transformations is $GL_1(\mathbb{Z}/n\mathbb{Z})$, so that the group of deck transformations of the tower is the projective limit:

$$\mathrm{Aut}_V(\mathcal{V}) = \varprojlim_n \mathrm{Aut}_V(V_n) = GL_1(\overline{\mathbb{Z}}^p).$$

If, instead of invertible 1-dimensional \mathbb{Q} -lattices up to scale, we consider the set $\mathrm{Lt}_1/\mathbb{R}_+^*$ of commensurability classes of all 1-dimensional \mathbb{Q} -lattices up to scale, and find a noncommutative version of the Shimura variety $\mathrm{Sm}(GL_1, \{\pm 1\})$, as well as of the corresponding cyclotomic tower. This is described as the quotient

$$\mathrm{Sm}^{nc}(GL_1, \{\pm 1\}) \cong GL_1(\mathbb{Q}) \backslash (\mathbb{A}_f \times \{\pm 1\}) = GL_1(\mathbb{Q}) \backslash (\mathbb{A}^\circ/\mathbb{R}_+^*),$$

where $\mathbb{A}^\circ \cong \mathbb{A}_f \times \mathbb{R}^*$ denotes the adèles of \mathbb{Q} with invertible archimedean component. The corresponding **noncommutative** algebra of coordinates is given by the C^* -algebra crossed product

$$\mathfrak{B}_1 \cong C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^* \cong (C(\overline{\mathbb{Z}}^p) \otimes C_0(\mathbb{Q})) \rtimes \mathbb{Q}_+^*$$

by the group \mathbb{Q}_+^* (cf. [95]). Obtained is an equivalent description of the noncommutative space $\mathrm{Sm}^{nc}(GL_1, \{\pm 1\})$, starting from the description of 1-dimensional \mathbb{Q} -lattices as $(\Lambda, \varphi) = (\lambda\mathbb{Z}, \lambda\rho)$. The commensurability relation is implemented by the action of $\mathbb{N}^\times = \mathbb{Z}_{>0}$ by

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho) & \rho \in n\overline{\mathbb{Z}}^p, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the **noncommutative** algebra of coordinates of the space of commensurability classes of 1-dimensional \mathbb{Q} -lattices up to scaling can be identified with the C^* -algebra crossed product

$$\mathfrak{A}_1 \equiv C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_{\alpha} \mathbb{N}^{\times}$$

by the semi-group \mathbb{N}^{\times} , where there is a C^* -algebra isomorphism as

$$C^*(\mathbb{Q}/\mathbb{Z}) \cong C((\mathbb{Q}/\mathbb{Z})^{\wedge}) \cong C(\overline{\mathbb{Z}^p}),$$

where $\overline{\mathbb{Z}^p}$ is identified with the Pontrjagin dual $(\mathbb{Q}/\mathbb{Z})^{\wedge} = \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{T})$ of the abelian group \mathbb{Q}/\mathbb{Z} . In fact, the C^* -algebra \mathfrak{A}_1 is Morita equivalent to the C^* -algebra \mathfrak{B}_1 (cf. [95]).

As shown in Laca and Raeburn [96] (added), the semi-group C^* -algebra crossed product \mathfrak{A}_1 is isomorphic to the **Bost-Connes Hecke** C^* -algebra \mathfrak{H}_1 considered in [13], which is obtained as a Hecke algebra for an inclusion $\Gamma_0 \subset \Gamma$ of a pair of groups, with $(\Gamma_0, \Gamma) = (S_{\mathbb{Z}}, S_{\mathbb{Q}})$, where S_C denotes the solvable $ax + b$ group with coefficients in C . The pair in this case satisfies that the left Γ_0 -orbits for any element of Γ/Γ_0 are finite, and the same holds for right orbits on the left coset. The ration of the lengths of left and right Γ_0 -orbits determines a canonical time evolution on the C^* -algebra (cf. [13]).

Remark. (Added). Recall from Laca [95] the following. The group C^* -dynamical system $(C_0(\mathbb{A}_f), \mathbb{Q}_+^*, \beta)$ is the minimal automorphic **dilation** of the (Ore) semi-group C^* -dynamical system $(C(\overline{\mathbb{Z}^p}), \mathbb{N}^{\times}, \alpha)$, so that

$$\mathfrak{A}_1 = C(\overline{\mathbb{Z}^p}) \rtimes_{\alpha} \mathbb{N}^{\times} \cong \chi_{\overline{\mathbb{Z}^p}}(C_0(\mathbb{A}_f) \rtimes_{\beta} \mathbb{Q}_+^*) \chi_{\overline{\mathbb{Z}^p}} = \chi_{\overline{\mathbb{Z}^p}} \mathfrak{B}_1 \chi_{\overline{\mathbb{Z}^p}},$$

where $\beta_r(f)(a) = f(r^{-1}a)$ for $a \in \mathbb{A}_f$ and $r \in \mathbb{Q}_+^* = (\mathbb{N}^{\times})^{-1}\mathbb{N}^{\times}$, and the characteristic function $\chi_{\overline{\mathbb{Z}^p}}$ on $\overline{\mathbb{Z}^p}$ that is compact and open in \mathbb{A}_f , belongs to $C_0(\mathbb{A}_f)$ the C^* -algebra of all continuous functions on \mathbb{A}_f vanishing at infinity, where \mathbb{A}_f is the locally compact ring of finite adeles:

$$\prod_p \mathbb{Q}_p \supset \mathbb{A}_f = \{(a_p) \mid a_p \in \mathbb{Z}_p \text{ for all but finitely many primes}\}.$$

Moreover,

$$\mathfrak{H}_1 \cong q_{\overline{\mathbb{Z}^p}} C^*(\mathbb{A}_f \rtimes \mathbb{Q}_+^*) q_{\overline{\mathbb{Z}^p}} \cong \mathfrak{A}_1,$$

where $C^*(\mathbb{A}_f \rtimes \mathbb{Q}_+^*)$ is the group C^* -algebra of the semi-direct product group $\mathbb{A}_f \rtimes \mathbb{Q}_+^*$, and $q_{\overline{\mathbb{Z}^p}}$ is identified with the projection of the group C^* -algebra $C^*(\mathbb{A}_f)$ corresponding to $\chi_{\overline{\mathbb{Z}^p}}$ under the Fourier transform, where $\mathbb{A}_f^{\wedge} \cong \mathbb{A}_f$ as a self-duality.

Such an action β **dilates** the action α if $\beta_n \circ i = i \circ \alpha_n$ for $n \in \mathbb{N}^{\times}$, where $i : C(\overline{\mathbb{Z}^p}) \rightarrow C_0(\mathbb{A}_f)$ is the inclusion map. Such a group C^* -dynamical system by β is said to be **minimal** if the union $\cup_{n \in \mathbb{N}^{\times}} \beta_n^{-1}(i(C(\overline{\mathbb{Z}^p})))$ is dense in $C_0(\mathbb{A}_f) \rtimes_{\beta} \mathbb{Q}_+^*$. If these conditions are satisfied, the system by β is said to be the minimal automorphic dilation of the system by α .

A semi-group S is embeddable into a group $G = S^{-1}S$ if and only if S is an **Ore** semigroup, that is, if S is a cancellative semigroup such that $Sk \cap Sl \neq \emptyset$ for any $k, l \in S$ (cf. [95]). ◀

Remark. (Added). As in Laca [94], the group inclusion of the $ax + b$ -groups with coefficients in \mathbb{Z} and \mathbb{Q} (as well as \mathbb{R}) is

$$S_{\mathbb{Z}} = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subset S_{\mathbb{Q}} = \begin{pmatrix} \mathbb{Q}_+^* & \mathbb{Q} \\ 0 & 1 \end{pmatrix} \quad \left(\subset S_{\mathbb{R}} = \begin{pmatrix} \mathbb{R}_+^* & \mathbb{R} \\ 0 & 1 \end{pmatrix} \text{ act on } \begin{pmatrix} x \\ 1 \end{pmatrix} \right).$$

◊ Note that $S_{\mathbb{Z}} \cong \mathbb{Z} \times 1$ and $S_{\mathbb{Q}} \cong \mathbb{Q} \rtimes \mathbb{Q}_+^*$ as a semi-direct product group, so that the quotient group is also isomorphic to

$$S_{\mathbb{Q}}/S_{\mathbb{Z}} \cong (\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{Q}_+^*.$$

It then follows that the group C^* -algebra of $S_{\mathbb{Q}}/S_{\mathbb{Z}}$ is isomorphic to the C^* -algebra crossed products by the group \mathbb{Q}_+^* (extending the action α of \mathbb{N}^\times):

$$\begin{aligned} C^*(S_{\mathbb{Q}}/S_{\mathbb{Z}}) &\cong C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{Q}_+^* \cong C(\overline{\mathbb{Z}^p}) \rtimes \mathbb{Q}_+^* \\ &\supset \mathfrak{H}_1 \cong C(\overline{\mathbb{Z}^p}) \rtimes_{\alpha} \mathbb{N}^\times = \mathfrak{A}_1. \quad \square \quad \blacktriangleleft \end{aligned}$$

◊ The Morita equivalence between two C^* -algebras is just like the equivalence between two projections (corresponding to respective units). Masaka! \square

The C^* -algebra \mathfrak{H}_1 of the Bost-Connes system has an explicit presentation in terms of certain operators of two types. The first type consists of phase (unitary) operators $e(r)$, parameterized by elements $r \in \mathbb{Q}/\mathbb{Z}$. These phase operators can be represented on the Fock space generated by occupation numbers $|n\rangle$ as the operators defined as $e(r)|n\rangle = \alpha(\zeta_r^n)|n\rangle$, where we denote by $\zeta_{ab} = \zeta_b^a$ (corrected) the abstract roots of unity generating \mathbb{Q}^{cyc} and by $\alpha : \mathbb{Q}^{cyc} \rightarrow \mathbb{C}$ an embedding that identifies \mathbb{Q}^{cyc} with the subfield of \mathbb{C} generated by the concrete roots of unity.

◊ Recall from [74, 3.1.2] the **bracket** (bra-ket) notation by Dirac as follows.

$$\langle \xi_m, A\xi_n \rangle \equiv \langle m|A|n\rangle,$$

where the vectors ξ_m, ξ_n (as states) in the inner product with A an (observable) operator on a (Fock) Hilbert space may be identified with the bra $\langle \xi_m = \langle m$ and the ket $\xi_n = |n\rangle$, respectively, and as well, $A\xi_n = A|n\rangle$. The **Fock** space is defined to be a Hilbert space obtained as the infinite direct sum of tensor products $\otimes^n H$ for $n \in \mathbb{N}$ of a certain Hilbert space H , with the vacuum state generating the orthogonal complement of the direct sum (cf. [150]). \square

Those phase operators are used in the theory of quantum optics and optical coherence to model the phase quantum-mechanically (cf. [100], [102] both not at hand), as well as to model the phasors in quantum computing. They are based on the choice of a certain scale N at which the phase is discretized (No figure). Namely, the quantized optical phase is defined as a state

$$\begin{aligned} |\theta_{m,N}\rangle &= e\left(\frac{m}{N+1}\right) v_N, \quad \left(v_N = \frac{1}{\sqrt{N+1}} \sum_{n=0}^N |n\rangle \right) \\ &= \frac{1}{\sqrt{N+1}} \sum_{n=0}^N \alpha(\zeta_{\frac{mn}{N+1}}) |n\rangle, \quad \left(\zeta_{\frac{mn}{N+1}} = e^{2\pi i \frac{mn}{N+1}} \right) \end{aligned}$$

where v_N is a superposition of occupation states as above (cf. [96, 2.4]). It is then necessary to ensure that the results are consistent over changes of scale.

◊ The removed figure is a picture of phasors with \mathbb{Z}_6 -discretization in the plane, given and parameterized as

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \cdots \\ 1 & e^{2\pi i \frac{1}{6}} & e^{2\pi i \frac{2}{6}} & \cdots & e^{2\pi i \frac{5}{6}} & 1 & \cdots \\ 1 & e^{2\pi i \frac{2}{6}} & e^{2\pi i \frac{4}{6}} & \cdots & e^{2\pi i \frac{10}{6}} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 1 & e^{2\pi i \frac{k}{6}} & e^{2\pi i \frac{2k}{6}} & \cdots & e^{2\pi i \frac{5k}{6}} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix} \quad \square$$

The other operators that generate the Bost-Connes Hecke C^* -algebra \mathfrak{H}_1 can be thought of as implementing the changes of scales in the optical phases in a consistent way. These operators are isometries μ_n , parameterized by positive integers $n \in \mathbb{N}^\times = \mathbb{Z}_{>0}$. The changes of scale are described by the action of μ_n on $e(r)$ as

$$\text{Ad}(\mu_n)e(r) = \mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s), \quad n \in \mathbb{N}^\times, r \in \mathbb{Q}/\mathbb{Z}.$$

◊ For instance, check that for $[x] \in \mathbb{Q}/\mathbb{Z}$, with $0 \leq x < 1$ assumed,

$$\begin{aligned} \left[\frac{1}{2}\right] &= 2\left[\frac{1}{4}\right] = 2\left[\frac{1}{2} + \frac{1}{2^2}\right] = 2\left[\frac{3}{4}\right], \\ \left[\frac{1}{3}\right] &= 3\left[\frac{1}{3^2}\right] = 3\left[\frac{1}{3} + \frac{1}{3^2}\right] = 3\left[\frac{2}{3} + \frac{1}{3^2}\right]. \quad \square \end{aligned}$$

In addition to that compatibility condition, the phase operators $e(r)$ and isometries μ_n satisfy other relations as below. For any $n, k \in \mathbb{N}^\times$ and $r, s \in \mathbb{Q}/\mathbb{Z}$,

$$\begin{aligned} \mu_n^* \mu_n &= 1, & \mu_k \mu_n &= \mu_{kn}, & (kn &= k \vee n \text{ as a lattice or L.C.M.}) \\ e(0) &= 1, & e(r)^* &= e(-r), & e(r)e(s) &= e(r+s). \end{aligned}$$

These and that relations give such a presentation of the Hecke C^* -algebra \mathfrak{H}_1 of the BC system ([13], [94]). It then follows that the Bost-Connes Hecke C^* -algebra \mathfrak{H}_1 is isomorphic to the semi-group C^* -crossed product \mathfrak{A}_1 .

◊ Those relations say that the map $\mathbb{N}^\times \ni n \mapsto \mu_n$ is viewed as an isometric representation of the semi-group \mathbb{N}^\times and that the map $\mathbb{Q}/\mathbb{Z} \ni r \mapsto e(r)$ is viewed as a unitary representation of the group \mathbb{Q}/\mathbb{Z} , and these representations become a covariant representation of both, to extend to define such a semi-group C^* -algebra crossed product by taking a C^* -algebra (norm) completion. \square

In terms of that explicit presentation, the **time** evolution is defined as the form

$$\sigma_t(\mu_n) = n^{it} \mu_n = e^{it \log n} \mu_n \quad \text{and} \quad \sigma_t(e(r)) = e(r).$$

The space $\text{Sm}^{nc}(GL_1, \{\pm 1\}) = GL_1(\mathbb{Q}) \backslash \mathbb{A}^\circ / \mathbb{R}_+^*$ can be compactified by replacing \mathbb{A}° by \mathbb{A} , as in [29], to give the quotient

$$\overline{\text{Sm}^{nc}}(GL_1, \{\pm 1\}) = GL_1(\mathbb{Q}) \backslash \mathbb{A} / \mathbb{R}_+^*.$$

This compactification consists of adding the trivial lattice (with a possibly non-trivial \mathbb{Q} -structure).

The dual space to $\overline{\text{Sm}^{nc}}(GL_1, \{\pm 1\})$, under the duality of type II and type III factors, introduced by Connes (thesis), is a principal \mathbb{R}_+^* -bundle over $\overline{\text{Sm}^{nc}}(GL_1, \{\pm 1\})$, whose noncommutative algebra of coordinates is obtained as the C^* -algebra crossed product of the C^* -algebra corresponding to $\overline{\text{Sm}^{nc}}(GL_1, \{\pm 1\})$ by the time evolution σ_t as an \mathbb{R} -action.

◊ Namely, that is, and contains

$$(C(\mathbb{A}) \rtimes_\beta \mathbb{Q}_+^*) \rtimes_\sigma \mathbb{R} \supset (C_0(\mathbb{A}_f) \rtimes_\beta \mathbb{Q}_+^*) \rtimes_\sigma \mathbb{R} = \mathfrak{B}_1 \rtimes_\sigma \mathbb{R}$$

as a closed ideal. □

The space obtained in this way is the space of adèle classes as the principal \mathbb{R}_+^* -bundle:

$$\begin{array}{ccc} \text{Lt}_1 = GL_1(\mathbb{Q}) \backslash \mathbb{A} & \text{(bundle)} & \longleftarrow \mathbb{R}_+^* \quad \text{(fiber)} \\ \downarrow & & \\ \text{Lt}_1 / \mathbb{R}_+^* = GL_1(\mathbb{Q}) \backslash \mathbb{A} / \mathbb{R}_+^* & \text{(base space)} & \end{array}$$

which gives the spectral realization of zeros of the Riemann zeta function ζ , as in [29]. The passing to this dual space corresponds to considering commensurability classes of 1-dimensional \mathbb{Q} -lattices (without up to scaling).

◊ Consequently, we may denote as

$$\begin{aligned} C^*(\text{Lt}_1 / \mathbb{R}_+^*) &= C^*(GL_1(\mathbb{Q}) \backslash \mathbb{A} / \mathbb{R}_+^*) = C(\mathbb{A}) \rtimes_\beta \mathbb{Q}_+^* = C^*(\overline{\text{Sm}^{nc}}(GL_1, \{\pm 1\})) \\ \text{and } C^*(\text{Lt}_1) &= C^*(GL_1(\mathbb{Q}) \backslash \mathbb{A}) = (C(\mathbb{A}) \rtimes_\beta \mathbb{Q}_+^*) \rtimes_\sigma \mathbb{R}. \end{aligned}$$

As well

$$C^*(\text{Sm}^{nc}(GL_1, \{\pm 1\})) = C^*(GL_1(\mathbb{Q}) \backslash \mathbb{A}^\circ / \mathbb{R}_+^*) = C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*.$$

Moreover, we may denote as well that

$$\begin{aligned} C^*(\mathbb{A}_f \rtimes_\beta \mathbb{Q}_+^*) &= C_0(\mathbb{A}_f) \rtimes_\beta \mathbb{Q}_+^* = \mathfrak{B}_1 \subset C^*(\text{Lt}_1 / \mathbb{R}_+^*), \\ C^*((\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^\times) &= C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^\times = \mathfrak{A}_1 \subset \mathfrak{B}_1, \end{aligned}$$

both of which are reduced from $C^*(\text{Lt}_1 / \mathbb{R}_+^*)$, so may be denoted as $\mathfrak{B}_1 = C_r^*(\text{Lt}_1 / \mathbb{R}_+^*)$ or $\mathfrak{A}_1 = C_r^*(\text{Lt}_1 / \mathbb{R}_+^*)$. □

Structure of KMS states. The Bost-Connes Hecke C^* -algebra $\mathfrak{H}_1 \cong \mathfrak{A}_1$ the semigroup C^* -algebra crossed product of Laca and Raeburn has **irreducible** representations on the Hilbert space $H = l^2(\mathbb{N}^\times)$. These are parameterized

by elements $\alpha \in \overline{\mathbb{Z}}^{p,\wedge} = GL_1(\overline{\mathbb{Z}}^p) = (\overline{\mathbb{Z}}^p)^{-1}$. Any such element α defines an embedding $\alpha : \mathbb{Q}^{cyc} \rightarrow \mathbb{C}$ of the subfield \mathbb{Q}^{cyc} of \mathbb{C} generated by the roots of unity (by the same symbol), and the corresponding representation has the form

$$\pi_\alpha(e(r))\xi_k = \alpha(\zeta_r^k)\xi_k = e^{2\pi i r k} \xi_k \quad \text{and} \quad \pi_\alpha(\mu_n)\xi_k = \xi_{nk}$$

for $r = [r] = r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ and $k \in \mathbb{N}^\times$, where $\{\xi_k\}_{k \in \mathbb{N}^\times}$ is the canonical basis for $l^2(\mathbb{N}^\times)$, with $\xi_k = \chi_k$ the characteristic function at k (cf. [96, 2.4]).

The **Hamiltonian** implementing the time evolution by the action α of \mathbb{N}^\times is given by $H\xi_k = (\log k)\xi_k$. Thus, the **partition** function of the Bost-Connes system is the Riemann zeta function

$$Z(\beta) = \text{tr}(e^{-\beta H}) = \sum_{k=1}^{\infty} \frac{1}{k^\beta} = \zeta(\beta).$$

It is shown by Bost and Connes [13] that KMS states have the following structure, with a phase transition at $\beta = 1$.

Theorem 3.6. (Edited). *Consider on $\mathfrak{A}_1 = \mathfrak{H}_1$ with σ the time evolution.*

• *In the range $0 < \beta \leq 1$, there is a unique KMS $_\beta$ state φ_β . Its restriction to $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ in $C^*(\mathbb{Q}/\mathbb{Z})$ has the form*

$$\varphi_\beta(e(\frac{a}{b})) = b^{-\beta} \prod_{p \text{ prime, } p|b} \frac{1 - p^{\beta-1}}{1 - p^{-1}},$$

where the product corresponds to the prime factorization for b . Moreover, each φ_β is associated to the hyper-finite factor of type III $_1$, that is, \mathcal{R}_∞ of Araki-Woods.

• *For $1 < \beta \leq \infty$, the set \mathcal{E}_β of extremal KMS $_\beta$ states can be identified with $(\overline{\mathbb{Z}}^p)^*$, which has a free and transitive action by (itself) the group also induced by some automorphisms of \mathfrak{A}_1 . The extremal KMS $_\beta$ state $\varphi_{\beta,\alpha}$ corresponding to $\alpha \in (\overline{\mathbb{Z}}^p)^*$ has the form on $C^*(\mathbb{Q}/\mathbb{Z})$*

$$\varphi_{\beta,\alpha}(x) = \frac{1}{\zeta(\beta)} \text{tr}(\pi_\alpha(x)e^{-\beta H}) \left(= \frac{1}{\zeta(\beta)} \sum_{n=1}^{\infty} \frac{\alpha(x)^n}{n^\beta} \right).$$

These states are associated to the type I $_\infty$ factor, that is, $\mathbb{B}(H)$.

• *At $\beta = \infty$, the Galois group $G = \text{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q})$ acts on the values of KMS $_\infty$ states $\varphi \in \mathcal{E}_\infty$ on a certain arithmetic subalgebra $\mathfrak{A}_{1,\mathbb{Q}}$ of \mathfrak{A}_1 . These states have a property that $\varphi(\mathfrak{A}_{1,\mathbb{Q}}) \subset \mathbb{Q}^{cyc}$ and that the class field theory isomorphism $\theta : G \cong (\overline{\mathbb{Z}}^p)^*$ intertwines the Galois action on values with the action of $(\overline{\mathbb{Z}}^p)^*$ by symmetries, namely, $\gamma\varphi(x) = \varphi(\theta(\gamma)x)$ for any $\varphi \in \mathcal{E}_\infty$, $\gamma \in G$, and $x \in \mathfrak{A}_{1,\mathbb{Q}}$, where the **arithmetic** subalgebra $\mathfrak{A}_{1,\mathbb{Q}}$ can be taken as an algebra over \mathbb{Q} , generated by the operators $e(r)$ and μ_n, μ_n^* for $r \in \mathbb{Q}/\mathbb{Z}$ and $n \in \mathbb{N}^\times$.*

Remark. (Added). As mentioned in [13], the critical temperature is at $T = 1$. At lower temperature for $\beta > 1$, the phases of the BC system are parameterized

by all possible embedding of \mathbb{Q}^{cyc} into \mathbb{C} . The Galois group G acts naturally as a group of automorphisms of $\mathfrak{A}_1 = \mathfrak{H}_1$ commuting with the time evolution σ and the spontaneous symmetry breaking occurs for $\beta > 1$.

As shown in [41], that \mathbb{Q} -**algebra** $\mathfrak{A}_{1,\mathbb{Q}}$ can also be obtained as the algebra generated by μ_n, μ_n^* for $n \in \mathbb{N}^\times$ and by homogeneous functions of weight zero on 1-dimensional \mathbb{Q} -lattices obtained as a normalization of the functions

$$\xi_{k,a}(\Lambda, \varphi) = \sum_{y \in \Lambda + \varphi(a)} \frac{1}{y^k} \quad \text{by covolume.}$$

Namely, consider the functions $c^k \xi_{k,a}$, where $c(\Lambda)$ is proportional to the covolume $|\Lambda|$ and satisfies $2\pi\sqrt{-1}c(\mathbb{Z}) = 1$.

The choice of such an arithmetic subalgebra of \mathfrak{A}_1 corresponds to endowing the noncommutative space (as $\mathfrak{A}_{1,\mathbb{Q}}$) with an arithmetic structure. The subalgebra corresponds to the rational functions and the values of KMS_∞ states at elements of this subalgebra should be thought of as values of rational functions at the classical points of the noncommutative space (cf. [48]).

3.3 Noncommutative geometry and the Hilbert 12th problem

The most remarkable arithmetic feature of the result of Bost-Connes recalled above is the Galois action on the ground states of the BC system. First of all, the fact that the Galois action on the values of states would preserve positivity and would give values of other states is an unusual property. Moreover, the values of extremal ground states on elements of the rational subalgebra generate the maximal **Abelian** extension $\mathbb{Q}^{Ab} = \mathbb{Q}^{cyc}$ of \mathbb{Q} . The explicit action of the **Galois** group $\text{Gal}(\mathbb{Q}^{Ab}/\mathbb{Q}) \cong GL_1(\overline{\mathbb{Z}^p})$ is given by automorphisms of the system $(\mathfrak{A}_1, \sigma_t)$ (or the group C^* -algebra crossed product $\mathfrak{A}_1 \rtimes_\sigma \mathbb{R}$). Namely, the class field theory isomorphism intertwines the action of the idèle class group, as symmetry group of the system, that of the Galois group, as permutations of the expectation values of the rational observable operators.

In general, the main theorem in class field theory provides a classification of finite abelian extensions of a local or global field \mathbb{K} in terms of subgroups of a locally compact abelian group canonically associated to the field. This is the multiplicative group $\mathbb{K}^* = GL_1(\mathbb{K})$ in the local non-archimedean case, while in the global case it is the quotient $C_{\mathbb{L}}/D_{\mathbb{K}}$ of the idèle class group $C_{\mathbb{K}}$ by the connected component of the identity.

The **Hilbert** 12th problem can be formulated as the question of providing an explicit set of generators of the maximal **abelian** extension \mathbb{K}^{ab} of a number field \mathbb{K} , inside an algebraic closure $\overline{\mathbb{K}}$, and of the action of the **Galois** group $\text{Gal}(\mathbb{K}^{ab}/\mathbb{K})$. There is the following property that

$$\text{Gal}(\mathbb{K}^{ab}/\mathbb{K}) \cong \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})^{ab}.$$

The result motivating the Hilbert formulation of the explicit class field theory problem is the **Kronecker-Weber** theorem, that is, the explicit class field

theory in the case of $\mathbb{K} = \mathbb{Q}$, already mentioned. In this case, the maximal abelian extension of \mathbb{Q} can be identified with the cyclotomic field \mathbb{Q}^{cyc} . Equivalently, the torsion points of the multiplicative group \mathbb{C}^* , as the roots of unity, generate $\mathbb{Q}^{ab} \subset \mathbb{C}$.

As a remark, the only other case of number fields where such formulation is carried out completely is that of imaginary quadratic fields where the construction relies on the theory of elliptic curves with complex multiplication and on the Galois theory of the fields of modular functions (cf. [148] as a survey).

Remark. (Added). Now may recall from [116] the following fundamental theorem on the **class field** theory.

[The **Artin** reciprocity law]. *For a finite Abel extension K over an algebraic number field \mathfrak{K} that is a finite extension over \mathbb{Q} , there is a non-zero ideal \mathfrak{m} of the ring $\mathcal{O}_{\mathfrak{K}}$ of algebraic integers of \mathfrak{K} such that the **reciprocity** map from $\bigoplus_{S_{\mathfrak{m}}} \mathbb{Z}$ to $\text{Gal}(K/\mathfrak{K})$ maps $\Gamma_{\mathfrak{m}}$ to zero, where $(\bigoplus_{S_{\mathfrak{m}}} \mathbb{Z})/\Gamma_{\mathfrak{m}} \cong Cl(\mathfrak{m})$ the ray class group that is a finite abelian group, where there is a surjective map from $\bigoplus_{S_{\mathfrak{m}}} \mathbb{Z}$ to $Cl(\mathfrak{m})$, induced by $\mathbb{Z} \cong \mathfrak{K}_v^*/\mathcal{O}_v^*$ for $v \in S_{\mathfrak{m}}$ a finite prime point of \mathfrak{K} which does not divide \mathfrak{m} , identified with a fractional ideal relatively prime with \mathfrak{m} .*

There is the maximal ideal \mathfrak{m} satisfying the condition above, called the **conductor** of K , so that induced is the homomorphism from $Cl(\mathfrak{m})$ to $\text{Gal}(K/\mathfrak{K})$.

[Existence theorem]. *Conversely, for a non-zero ideal \mathfrak{m} of $\mathcal{O}_{\mathfrak{K}}$, there is a finite Abel extension $K_{\mathfrak{m}}$ of \mathfrak{K} such that the reciprocity map induces the isomorphism from $Cl(\mathfrak{m})$ to $\text{Gal}(K_{\mathfrak{m}}/\mathfrak{K})$.*

Such Abel extension $K_{\mathfrak{m}}$ for \mathfrak{m} is unique up to isomorphisms over \mathfrak{K} , called the **ray** class field of (the conductor) \mathfrak{m} . If $\mathfrak{m} = \mathcal{O}_{\mathfrak{K}}$, then $K_{\mathfrak{m}}$ is the absolute class field of \mathfrak{K} . A finite Abel extension over \mathfrak{K} with \mathfrak{m} as the conductor is a subfield of $K_{\mathfrak{m}}$. A **class field** of \mathfrak{K} is defined to be a subfield of $K_{\mathfrak{m}}$ corresponding to the image of a subgroup of $Cl(\mathfrak{m})$ under the reciprocity map.

For instance, if $\mathfrak{K} = \mathbb{Q}$ and $\mathfrak{m} = n\mathbb{Z}$, then $K_{\mathfrak{m}} = \mathbb{Q}(\zeta_n)$. If $K = K_{\mathfrak{m}}$, then

$$Cl(\mathfrak{m}) \cong \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_n^*.$$

It then follows [Kronecker-Weber] that the largest Abel extension \mathbb{Q}^{Ab} of \mathbb{Q} is $\mathbb{Q}(\zeta_n \mid n \geq 2)$. ◀

Some generalizations of the original result of Bost-Connes to other global fields as number fields and function fields are obtained by Harari and Leichtnam [79], P. Cohen [21], Arledge, Laca, and Raeburn [4]. A more detailed account of various results related to the BC system and generalizations is given in the section of [41] as further developments. Because of such a close relation to the Hilbert 12th problem, it seems to be clear that obtaining generalizations of the Bost-Connes system to other number fields is a difficult problem, and it is not surprising that these constructions so far have not fully recovered the Galois properties of the ground states of the BC system in the generalized setting.

As the strongest form of such a result in that direction, we can formulate the following:

Theorem 3.7. (Conjecture, edited). *Given a number field \mathbb{K} , we denote by $\mathbb{A}_{\mathbb{K}}$ the ring of **adèles** of \mathbb{K} and by $\mathbb{A}_{\mathbb{K}}^* = GL_1(\mathbb{A}_{\mathbb{K}})$ the group of **idèles** of \mathbb{K} .*

Let $C_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*$ be the group of idèle classes of \mathbb{K} and $D_{\mathbb{K}}$ the connected component of the identity in $C_{\mathbb{K}}$. Construct a C^* -algebra dynamical system $(\mathfrak{A}_{\mathbb{K}}, \sigma, \mathbb{R})$ and an arithmetic subalgebra $\mathfrak{A}_{\mathbb{K}, \mathbb{Q}}$ such that

(1) The (quotient) idèle class group $C_{\mathbb{K}}/D_{\mathbb{K}}$ acts by symmetries on the system $(\mathfrak{A}_{\mathbb{K}}, \sigma, \mathbb{R})$ preserving the subalgebra $\mathfrak{A}_{\mathbb{K}, \mathbb{Q}}$.

(2) The extremal KMS_{∞} states $\varphi \in \mathcal{E}_{\infty}$ evaluated at elements a of $\mathfrak{A}_{\mathbb{K}, \mathbb{Q}}$ satisfy that $\varphi(a) \in \overline{\mathbb{K}}$ the algebraic closure of \mathbb{K} in \mathbb{C} , and the values $\varphi(a) \in \mathbb{C}$ for $a \in \mathfrak{A}_{\mathbb{K}, \mathbb{Q}}$ and $\varphi \in \mathcal{E}_{\infty}$ generate \mathbb{K}^{Ab} .

(3) The class field theory isomorphism $\theta : \text{Gal}(\mathbb{K}^{Ab}/\mathbb{K}) \xrightarrow{\cong} C_{\mathbb{K}}/D_{\mathbb{K}}$ intertwines with the actions, so that for $\alpha \in \text{Gal}(\mathbb{K}^{Ab}/\mathbb{K})$ and $\varphi \in \mathcal{E}_{\infty}$,

$$\begin{array}{ccc} \mathfrak{A}_{\mathbb{K}, \mathbb{Q}} \subset \mathfrak{A}_{\mathbb{K}} & \xrightarrow{\varphi} & \mathbb{K}^{Ab} \subset \mathbb{C} \\ \theta(\alpha) \downarrow & & \downarrow \alpha \\ \mathfrak{A}_{\mathbb{K}, \mathbb{Q}} \subset \mathfrak{A}_{\mathbb{K}} & \xrightarrow{\varphi} & \mathbb{K}^{Ab} \subset \mathbb{C}. \end{array}$$

In general, the arithmetic subalgebra $\mathfrak{A}_{\mathbb{K}, \mathbb{Q}}$ need not be an involutive algebra. The setup described above may provide a possible new approach to the explicit class field theory problem via noncommutative geometry.

Given a number field \mathbb{K} with $[\mathbb{K}, \mathbb{Q}] = n$, there is an embedding from its multiplicative group \mathbb{K}^* into $GL_n(\mathbb{Q})$. Such an embedding induces an embedding from $GL_1(\mathbb{A}_{\mathbb{K}, f})$ into $GL_n(\mathbb{A}_f)$, where $\mathbb{A}_{\mathbb{K}, f} = \mathbb{A}_f \otimes \mathbb{K}$ are the finite adèles of \mathbb{K} .

That suggests that a possible strategy to approach the problem stated above may be to first study quantum statistical mechanical systems corresponding to GL_n -analogues of the Bost-Connes system.

The main result of [41] is the construction of such a system in the case of GL_2 and the analysis of the arithmetic properties of its KMS states. In the case of GL_1 , it is observed that the geometry of modular curves and the algebra of modular forms appear naturally.

As the work of [48], a quantum statistical mechanical system is constructed, and satisfies all the properties listed above for the case where \mathbb{K} is an imaginary quadratic field. The properties of this system are intermediate between those of the original BC system and those of the GL_2 system of [41]. In fact, the construction is geometrically based on replacing the 1-dimensional \mathbb{Q} -lattices of the Bost-Connes system with 1-dimensional \mathbb{K} -lattices. The groupoid of the commensurability relation is then a sub-groupoid of that of the GL_2 system. In fact, 1-dimensional \mathbb{K} -lattices are viewed as a special case of 2-dimensional \mathbb{Q} -lattices with compatible notions of commensurability. Thus, by combining the techniques of the BC system and of the GL_2 system, it is possible to show that the complex multiplication (CM) system defined in that way recovers the full picture of the explicit class field theory of imaginary quadratic fields ([48]), as recalled below later.

There is the case with not yet a complete solution to the explicit class field theory problem, such as the first case where $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ real quadratic fields for some positive integer d . It is natural to ask whether the approach outlined above, based on noncommutative geometry, may provide any new information

on this case. There is a close relation between the real quadratic case and noncommutative geometry, obtained by Manin ([109], [110]). Discussed below is the possible relation between the approach and the GL_2 system.

Remark. Now recall from [116] the following. Suppose that \mathbb{K} is a number field. Let $P = P_f \sqcup P_\infty$ denote the set of all (finite or infinite) primes (divisors) of \mathbb{K} (as equivalence classes of valuations of \mathbb{K}). For any $\mathfrak{p} \in P$, let $\mathbb{K}_{\mathfrak{p}}$ be the completion of \mathbb{K} with respect to \mathfrak{p} (such as \mathbb{Q}_p of \mathbb{Q} with $\mathfrak{p} = p\mathbb{Z}$ for p prime), and let $\mathbb{K}_{\mathfrak{p}}^*$ be the multiplicative group of $\mathbb{K}_{\mathfrak{p}}$. Moreover, let $\mathfrak{O}_{\mathfrak{p}}$ the valuation ring for a finite prime \mathfrak{p} , and let $\mathfrak{U}_{\mathfrak{p}}$ the group of invertibles of $\mathfrak{O}_{\mathfrak{p}}$. Since $\mathfrak{O}_{\mathfrak{p}}$ as an additive group is a compact open subgroup of $\mathbb{K}_{\mathfrak{p}}$, the **adèle** ring $\mathbb{A}_{\mathbb{K}}$ of \mathbb{K} is defined to be the restricted direct product of $\{\mathbb{K}_{\mathfrak{p}}\}$ with respect to $\{\mathfrak{O}_{\mathfrak{p}}\}$. It looks like

$$\prod_{\mathfrak{p} \in P} \mathbb{K}_{\mathfrak{p}} \supset \mathbb{A}_{\mathbb{K}} \supset \mathfrak{O}' \equiv (\prod_{\mathfrak{p} \in P_f} \mathfrak{O}_{\mathfrak{p}}) \sqcup (\prod_{\mathfrak{p} \in P_\infty} \mathbb{K}_{\mathfrak{p}}),$$

with the quotient $\mathbb{A}_{\mathbb{K}}/\mathfrak{O}'$ with discrete topology, to become a locally compact group and ring, where any element $\{x_{\mathfrak{p}}\}$, called **adèle**, of $\mathbb{A}_{\mathbb{K}}$ has any \mathfrak{p} component $x_{\mathfrak{p}}$ in $\mathfrak{O}_{\mathfrak{p}}$ except finitely many \mathfrak{p} component $x_{\mathfrak{p}}$ in $\mathbb{K}_{\mathfrak{p}}$.

As well, since $\mathfrak{U}_{\mathfrak{p}}$ as a multiplicative group is a compact open subgroup of $\mathbb{K}_{\mathfrak{p}}^*$, the **idèle** group $\mathbb{J}_{\mathbb{K}}$ of \mathbb{K} is defined to be the restricted direct sum of $\{\mathbb{K}_{\mathfrak{p}}^*\}$ with respect to $\{\mathfrak{U}_{\mathfrak{p}}\}$. It looks like

$$\prod_{\mathfrak{p} \in P} \mathbb{K}_{\mathfrak{p}}^* \supset \mathbb{J}_{\mathbb{K}} \supset \mathfrak{U}' \equiv (\prod_{\mathfrak{p} \in P_f} \mathfrak{U}_{\mathfrak{p}}) \sqcup (\prod_{\mathfrak{p} \in P_\infty} \mathbb{K}_{\mathfrak{p}}^*),$$

with the similar condition as above, with any element of $\mathbb{J}_{\mathbb{K}}$, called **idèle**. As a group, $\mathbb{J}_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}^*$, but with the topology stronger than the relative topology from the topology on $\mathbb{A}_{\mathbb{K}}$. ◀

3.4 The GL_2 system

In this subsection, described is the main features of the GL_2 analogue of the Bost-Connes system, according to the results of Connes-Marcolli [41].

Let us start with the same geometric approach in terms of Shimura varieties as used above to introduce the BC system (cf. [48], [44]).

The Shimura variety associated to the tower of modular curves is described by the adelic quotient

$$\begin{aligned} \text{Sm}(GL_2, \mathbb{H}^\pm) &= GL_2(\mathbb{Q}) \backslash (GL_2(\mathbb{A}_f) \times \mathbb{H}^\pm) \\ &= GL_2^+(\mathbb{Q}) \backslash (GL_2(\mathbb{A}_f) \times \mathbb{H}) = GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \mathbb{C}^*. \end{aligned}$$

◊ Note that $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ (!) the adèles of \mathbb{Q} , and \mathbb{H}^\pm the union of the upper and lower half-planes in \mathbb{C} is identified with the quotient $GL_2(\mathbb{R})/\mathbb{C}^*$, and $\mathbb{H}^+ = \mathbb{H}^2 = \mathbb{H}$ with $GL_2^+(\mathbb{R})/\mathbb{C}^*$. In fact, $GL_2^+(\mathbb{R}) \approx SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$ with $n = 2$ as Iwasawa decomposition (cf. [158, Example 118]). Possibly, $GL_2(\mathbb{A}_f)$ as well as $GL_2(\mathbb{A})$ in the second line above may be replaced with respective components $GL_2^+(\mathbb{A}_f)$ and $GL_2^+(\mathbb{A})$ with determinant positive. ◻

The **tower** of modular curves has base $V = \mathbb{P}^1$ over \mathbb{Q} and $V_n = X(n)$ the modular curve corresponding to the principal congruence subgroup $\Gamma(n)$. The

automorphisms of the projection $V_n \rightarrow V$ are given by $GL_2(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$, so that the group of deck transformations of the tower \mathcal{V} has the form

$$\text{Aut}_V(\mathcal{V}) = \varprojlim GL_2(\mathbb{Z}_n)/\{\pm 1\} = GL_2(\overline{\mathbb{Z}^p})/\{\pm 1\}.$$

The inverse limit of $\Gamma \backslash \mathbb{H}$ over congruence subgroups Γ in $SL_2(\mathbb{Z})$ gives a connected component of $\text{Sm}(GL_2, \mathbb{H}^\pm)$, while by taking congruence subgroups in $SL_2(\mathbb{Q})$, obtained is the adelic version $\text{Sm}(GL_2, \mathbb{H}^\pm)$.

The simple reason of being necessary to pass to the non-connected case is the following. The varieties in the tower are arithmetic varieties defined over number fields. However, the number field typically changes along the levels of the tower. In fact, V_n is defined over the cyclotomic field $\mathbb{Q}(\zeta_n)$. Passing to non-connected Shimura varieties allows for the definition of a canonical model case where the whole tower is defined over the same number field.

The adelic quotient $\text{Sm}(GL_2, \mathbb{H}^\pm)$ parameterizes invertible 2-dimensional \mathbb{Q} -lattices up to scaling (cf. [122]). Instead of restricting to the invertible case, if we consider commensurability classes of 2-dimensional \mathbb{Q} -lattices up to scaling, then we obtain a noncommutative space whose classical points are the Shimura variety $\text{Sm}(GL_2, \mathbb{H}^\pm)$. More precisely, the following is obtained (cf. [48]):

Lemma 3.8. *The space Lt_2/\mathbb{C}^* of commensurability classes of 2-dimensional \mathbb{Q} -lattices up to scaling is described as the quotient*

$$\text{Sm}^{nc}(GL_2, \mathbb{H}^\pm) \equiv GL_2(\mathbb{Q}) \backslash (M_2(\mathbb{A}_f) \times \mathbb{H}^\pm).$$

Proof. (Edited). Any 2-dimensional \mathbb{Q} -lattice (Λ, φ) can be written as in the form $(\lambda(\mathbb{Z} + \mathbb{Z}\tau), \lambda\rho)$, for some $\lambda \in \mathbb{C}^*$, $\tau \in \mathbb{H}$, and

$$\rho \in M_2(\overline{\mathbb{Z}^p}) = \text{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2).$$

Thus, the space Lt_2/\mathbb{C}^* with $\lambda \in \mathbb{C}^*$ as the scale factor is given by

$$M_2(\overline{\mathbb{Z}^p}) \times \mathbb{H} \quad \text{mod } \Gamma = SL_2(\mathbb{Z}).$$

The commensurability relation is then implemented by a partially defined action of $GL_2^+(\mathbb{Q})$ on that space, given as $g(\rho, z) = (g\rho, g(z))$, where $g(z)$ denote fractional linear transformations as an action.

Commensurability classes of \mathbb{Q} -lattices (Λ, φ) in \mathbb{C} are equivalent to **isogeny** classes of pairs (E, η) of an elliptic curve $E = \mathbb{C}/\Lambda$ and an \mathbb{A}_f -homomorphism

$$\begin{aligned} \eta : \mathbb{Q}^2 \otimes \mathbb{A}_f &\rightarrow \Lambda \otimes \mathbb{A}_f \cong (\Lambda \otimes \overline{\mathbb{Z}^p}) \otimes \mathbb{Q}, \quad \text{with} \\ \Lambda \otimes \overline{\mathbb{Z}^p} &= \varprojlim_n \Lambda/n\Lambda, \quad \Lambda/n\Lambda = E[n] \text{ the } n\text{-torsion of } E. \end{aligned}$$

Since the \mathbb{Q} -lattices are not necessarily invertible, it is not required that η be an \mathbb{A}_f -isomorphism (cf. [122]).

The commensurability relation between \mathbb{Q} -lattices corresponds to the isogeny equivalence of isogeny classes, where two isogeny classes (E, η) and (E', η') are **equivalent** if there is an **isogeny**

$$g : E = \mathbb{C}/\Lambda \rightarrow E' = \mathbb{C}/\Lambda'$$

such that $\eta' = (g \otimes 1) \circ \eta$ on $\mathbb{Q}^2 \otimes \mathbb{A}_f$.

(\diamond Note that such g from E to E' may be identified with its lift on \mathbb{C} sending Λ to Λ' . \square)

It then follows that the isogeny equivalence can be identified with the quotient of $M_2(\mathbb{A}_f) \times \mathbb{H}^\pm$ by the action of $GL_2(\mathbb{Q})$, defined as sending (ρ, z) to $(g\rho, g(z))$. \square

To associate a quantum statistical mechanical system to the space Lt_2/\mathbb{C}^* of commensurability classes of 2-dimensional \mathbb{Q} -lattices up to scaling, it is convenient to use the description of Lt_2/\mathbb{C}^* as $M_2(\overline{\mathbb{Z}}^p) \times \mathbb{H} \bmod \Gamma = SL_2(\mathbb{Z})$. Then considered as a noncommutative algebra of coordinates can be the (**Hecke**) **convolution** algebra $C_c(\mathcal{U}/\Gamma^2)$ of compactly supported, continuous functions on the quotient \mathcal{U}/Γ^2 of the space

$$\mathcal{U} = \{(g, \rho, z) \in GL_2^+(\mathbb{Q}) \times M_2(\overline{\mathbb{Z}}^p) \times \mathbb{H} \mid g\rho \in M_2(\overline{\mathbb{Z}}^p)\}$$

by the action of $\Gamma^2 = \Gamma \times \Gamma$ defined as

$$(\gamma_1, \gamma_2) \cdot (g, \rho, z) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2(z)).$$

\diamond Note that

$$\begin{aligned} (\gamma_1 \gamma_1', \gamma_2 \gamma_2') \cdot (g, \rho, z) &= (\gamma_1 \gamma_1' g (\gamma_2')^{-1} \gamma_2^{-1}, \gamma_2 \gamma_2' \rho, \gamma_2 \gamma_2'(z)) \\ &= (\gamma_1, \gamma_2) \cdot [(\gamma_1', \gamma_2') \cdot (g, \rho, z)]. \quad \square \end{aligned}$$

Endow this algebra $C_c(\mathcal{U}/\Gamma^2)$ denoted by us so with the convolution product and the involution

$$\begin{aligned} (f_1 * f_2)(g, \rho, z) &= \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), s\rho \in M_2(\overline{\mathbb{Z}}^p)} f_1(g s^{-1}, s\rho, s(z)) f_2(s, \rho, z), \\ f^*(g, \rho, z) &= \overline{f(g^{-1}, g\rho, g(z))}. \end{aligned}$$

\diamond Masaka, check that

$$\begin{aligned} (f_1 * (f_2 * f_3))(g, \rho, z) &= \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), s\rho \in M_2(\overline{\mathbb{Z}}^p)} f_1(g s^{-1}, s\rho, s(z)) (f_2 * f_3)(s, \rho, z) = \\ &= \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), s\rho \in M_2(\overline{\mathbb{Z}}^p)} f_1(g s^{-1}, s\rho, s(z)) \sum_{t \in \Gamma \backslash GL_2^+(\mathbb{Q}), t\rho \in M_2(\overline{\mathbb{Z}}^p)} f_2(st^{-1}, t\rho, t(z)) f_3(t, \rho, z) = \\ &= \sum_{t \in \Gamma \backslash GL_2^+(\mathbb{Q}), t\rho \in M_2(\overline{\mathbb{Z}}^p)} \sum_{st \in \Gamma \backslash GL_2^+(\mathbb{Q}), st\rho \in M_2(\overline{\mathbb{Z}}^p)} f_1(g(st)^{-1}, st\rho, st(z)) f_2((st)t^{-1}, t\rho, t(z)) f_3(t, \rho, z) = \\ &= \sum_{t \in \Gamma \backslash GL_2^+(\mathbb{Q}), t\rho \in M_2(\overline{\mathbb{Z}}^p)} \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), s\rho \in M_2(\overline{\mathbb{Z}}^p)} f_1(gt^{-1}s^{-1}, s(t\rho), s(t(z))) f_2(s, t\rho, t(z)) f_3(t, \rho, z) = \\ &= \sum_{t \in \Gamma \backslash GL_2^+(\mathbb{Q}), t\rho \in M_2(\overline{\mathbb{Z}}^p)} (f_1 * f_2)(gt^{-1}, t\rho, tz) f_3(t, \rho, z) = ((f_1 * f_2) * f_3)(g, \rho, z), \end{aligned}$$

and as well

$$(f^*)^*(g, \rho, z) = \overline{f^*(g^{-1}, g\rho, g(z))} = f((g^{-1})^{-1}, g^{-1}g\rho, g^{-1}g(z)) = f(g, \rho, z). \quad \square$$

The **time** evolution on $C_c(\mathcal{U}/\Gamma^2)$ is given by, with $\det(g) > 0$, $t \in \mathbb{R}$,

$$\sigma_t(f)(g, \rho, z) = \det(g)^{it} f(g, \rho, z) = e^{it \log \det(g)} f(g, \rho, z).$$

◇ Note that $\sigma_{t+s}(f) = \sigma_t(\sigma_s(f))$. Check that σ_t is a $*$ -automorphism on $C_c(\mathcal{U}/\Gamma^2)$ as follows. Its being bijective is clear. Also,

$$\begin{aligned} \sigma_t(f_1 * f_2)(g, \rho, z) &= \det(g)^{it} (f_1 * f_2)(g, \rho, z) = \det(gs^{-1}s)^{it} (f_1 * f_2)(g, \rho, z) \\ &= \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), s\rho \in M_2(\overline{\mathbb{Z}}^p)} \det(gs^{-1})^{it} f_1(gs^{-1}, s\rho, s(z)) \det(s)^{it} f_2(s, \rho, z), \\ &= (\sigma_t(f_1) * \sigma_t(f_2))(g, \rho, z), \end{aligned}$$

and as well,

$$\begin{aligned} \sigma_t(f)^*(g, \rho, z) &= \overline{\det(g^{-1})^{it} f(g^{-1}, g\rho, g(z))}, \\ &= \det(g)^{it} \overline{f(g^{-1}, g\rho, g(z))} = \sigma_t(f^*)(g, \rho, z), \end{aligned}$$

with $\det(g)^{it} \det(g^{-1})^{it} = \det(1)^{it} = 1$ in \mathbb{T} . □

For $\rho \in M_2(\overline{\mathbb{Z}}^p)$, let

$$G_\rho = \{g \in GL_2^+(\mathbb{Q}) \mid g\rho \in M_2(\overline{\mathbb{Z}}^p)\},$$

and consider the Hilbert space $H_\rho = l^2(\Gamma \backslash G_\rho)$.

◇ Note that G_ρ may not be a group? Because if $g, h \in G_\rho$, then $(gh)\rho = g(h\rho) \in M_2(\overline{\mathbb{Z}}^p)$ only if $g \in G_{h\rho}$. Also, if $g \in G_\rho$, then $g^{-1}\rho \in M_2(\overline{\mathbb{Z}}^p)$? □

A 2-dimensional \mathbb{Q} -lattice $L = (\Lambda, \varphi) = (\rho, z) \in M_2(\overline{\mathbb{Z}}^p) \times \mathbb{H}^2$ determines a **representation** of the Hecke $*$ -algebra $C_c(\mathcal{U}/\Gamma^2)$ by bounded operators on H_ρ , as setting

$$(\pi_L(f)\xi)(g) = \sum_{s \in \Gamma \backslash G_\rho} f(gs^{-1}, s\rho, s(z))\xi(s).$$

◇ Check that

$$\begin{aligned} (\pi_L(f_1 * f_2)\xi)(g) &= \sum_{s \in \Gamma \backslash G_\rho} (f_1 * f_2)(gs^{-1}, s\rho, s(z))\xi(s) \\ &= \sum_{s \in \Gamma \backslash G_\rho} \sum_{t \in \Gamma \backslash GL_2^+(\mathbb{Q}), t\rho \in M_2(\overline{\mathbb{Z}}^p)} f_1(gs^{-1}t^{-1}, ts\rho, ts(z)) f_2(t, s\rho, s(z))\xi(s) \\ &= \sum_{s' = ts \in \Gamma \backslash G_\rho} f_1(g(ts)^{-1}, ts\rho, ts(z)) \sum_{s \in \Gamma \backslash G_\rho} f_2((ts)s^{-1}, s\rho, s(z))\xi(s) \\ &= \sum_{s' = ts \in \Gamma \backslash G_\rho} f_1(g(ts)^{-1}, (ts)\rho, (ts)(z)) (\pi_L(f_2)\xi)(ts) \\ &= \pi_L(f_1)\pi_L(f_2)\xi(g), \end{aligned}$$

and as well

$$\begin{aligned}
\langle \pi_L(f)^* \xi, \eta \rangle &= \langle \xi, \pi_L(f) \eta \rangle \\
&= \sum_{t \in \Gamma \backslash G_\rho} \xi(t) \overline{\sum_{s \in \Gamma \backslash G_\rho} f(ts^{-1}, s\rho, s(z)) \eta(s)} \\
&= \sum_{s \in \Gamma \backslash G_\rho} \sum_{t \in \Gamma \backslash G_\rho} \overline{f(ts^{-1}, s\rho, s(z))} \xi(t) \overline{\eta(s)} \\
&= \sum_{s \in \Gamma \backslash G_\rho} \sum_{t \in \Gamma \backslash G_\rho} f(st^{-1}, t\rho, t(z)) \xi(t) \overline{\eta(s)} \\
&= \sum_{s \in \Gamma \backslash G_\rho} \pi_L(f^*) \xi(s) \overline{\eta(s)} = \langle \pi_L(f^*) \xi, \eta \rangle,
\end{aligned}$$

with $f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}$. \square

In particular, if the \mathbb{Q} -lattice $L = (\Lambda, \varphi)$ is invertible, then we have (?)

$$H_\rho \cong l^2(\Gamma \backslash M_2^+(\mathbb{Z})).$$

In this case, the **Hamiltonian** implementing the time evolution σ_t is given by the operator as

$$H\xi_m = \log \det(m) \xi_m.$$

Thus, in the special case of invertible \mathbb{Q} -lattices, a positive energy representation is obtained by π_L . In general, for \mathbb{Q} -lattices not commensurable to an invertible one, the corresponding Hamiltonian is not bounded below.

The Hecke algebra $C_c(\mathcal{U}/\Gamma^2)$ admits the maximal C^* -algebra completion $\mathfrak{A}_2 = C^*(\mathcal{U}/\Gamma^2)$, which may be called the **maximal Hecke** C^* -algebra, where the maximal C^* -algebra norm is defined to be the supremum over all representations π_L for $L \in \text{Lt}_2$.

The **partition** function for this GL_2 system as \mathfrak{A}_2 is given by

$$\begin{aligned}
Z(\beta) &= \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \frac{1}{\det(m)} \\
&= \sum_{k=1}^{\infty} \frac{\sigma(k)}{k^\beta} = \zeta(\beta) \zeta(\beta - 1),
\end{aligned}$$

where $\sigma(k) = \sum_{d|k} d$. This suggests the fact that two phase transitions take place at $\beta = 1$ and $\beta = 2$, expected.

Remark. As in [116], note that the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges for $s > 1$ by L. Euler, and is holomorphic for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, and is analytically extended to the complex plane \mathbb{C} as analytic continuation expect $s = 1$ as the only pole of order 1, by Riemann. \blacktriangleleft

The set of components of $\text{Sm}(GL_2, \mathbb{H}^\pm)$ is given by

$$\pi_0(\text{Sm}(GL_2, \mathbb{H}^\pm)) = \text{Sm}(GL_1, \{\pm 1\}).$$

At the level of the classical commutative spaces, this is given by the map

$$\det \times \text{sign} : \text{Sm}(GL_2, \mathbb{H}^\pm) \rightarrow \text{Sm}(GL_1, \{\pm\}),$$

which implies the equality above by passing to the set $\pi_0(X)$ of connected components of a space X .

KMS states of the GL_2 system. The main result of [41] on the structure of KMS states of the GL_2 system is the following:

Theorem 3.9. *The KMS_β states of the GL_2 system as \mathfrak{A}_2 have the following properties:*

- (1) *In the range $\beta \leq 1$, there are no KMS_β states.*
- (2) *In the range $\beta > 2$, the set \mathcal{E}_β of extremal KMS_β states is given by the classical Shi-mura variety as*

$$\mathcal{E}_\beta \cong GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \mathbb{C}^* (\cong \text{Sm}(GL_2, \mathbb{H}^\pm)^\pm \text{ added}).$$

This shows that the extremal KMS_β states at sufficiently low temperature $T = \frac{1}{\beta}$ are parameterized by the invertible 2-dimensional \mathbb{Q} -lattices L , as $\varphi_{\beta,L}$. These extremal KMS_β states obtained are explicitly expressed as

$$\varphi_{\beta,L}(f) = \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \frac{1}{\det(m)^\beta} f(1, m\rho, m(z)),$$

where $L = (\rho, z)$ is an invertible 2- \mathbb{Q} -lattice.

◇ Its being linear is clear. Also, $\varphi_{\beta,L}(1) = 1$. But $C^*(\mathcal{U}/\Gamma^2)$ should be non-unital. It holds that $\|\varphi_{\beta,L}\| = 1$. Its being positive follows from

$$\begin{aligned} \varphi_{\beta,L}(f^* * f) &= \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \frac{1}{\det(m)^\beta} (f^* * f)(1, m\rho, m(z)) = \\ &= \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \frac{1}{\det(m)^\beta} \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), sm\rho \in M_2(\overline{\mathbb{Z}}^p)} f^*(s^{-1}, sm\rho, sm(z)) f(s, m\rho, m(z)) \\ &= \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \frac{1}{\det(m)^\beta} \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), sm\rho \in M_2(\overline{\mathbb{Z}}^p)} \overline{f(s, m\rho, m(z))} f(s, m\rho, m(z)) \geq 0. \end{aligned}$$

Hence, $\varphi_{\beta,L}$ is a state on \mathfrak{A}_2 . For its being KMS_β , observe that

$$\begin{aligned}\varphi_{\beta,L}(f * \sigma_t(h)) &= \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \frac{1}{\det(m)^\beta} \\ &\quad \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), sm\rho \in M_2(\overline{\mathbb{Z}}^p)} f(s^{-1}, sm\rho, sm(z)) \det(s)^{it} h(s, m\rho, m(z)), \\ \varphi_{\beta,L}(\sigma_t(h) * f) &= \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} \frac{1}{\det(m)^\beta} \\ &\quad \sum_{s \in \Gamma \backslash GL_2^+(\mathbb{Q}), sm\rho \in M_2(\overline{\mathbb{Z}}^p)} \det(s^{-1})^{it} h(s^{-1}, sm\rho, sm(z)) f(s, m\rho, m(z)).\end{aligned}$$

Again, a possible choice of the holomorphic path between those may be given as

$$\varphi_{\beta,L}\left(\frac{\beta-s}{\beta}f * \sigma_t(h) + \frac{s}{\beta}\sigma_t(h) * f\right), \quad 0 \leq s \leq \beta. \quad \square$$

As the temperature $T = \frac{1}{\beta}$ rises, and $\beta \rightarrow 2+0$ from above, all the different phases of the system merge (into one state). This is a strong evidence for a fact (or guess?) that in the intermediate range $1 < \beta \leq 2$, the system have only a single KMS_β state.

The symmetry group of $\mathfrak{A}_2 = C^*(\mathcal{U}/\Gamma^2)$ including both automorphisms and endomorphisms can be identified with the group

$$GL_2(\mathbb{A}_f) = GL_2^+(\mathbb{Q})GL_2(\overline{\mathbb{Z}}^p),$$

where the group $GL_2(\overline{\mathbb{Z}}^p)$ acts by automorphisms of \mathfrak{A}_2 , so that

$$\theta_\gamma(f)(g, \rho, z) = f(g, \rho\gamma, z).$$

These symmetries are related to the group $\text{Aut}_V(\mathcal{V}) = GL_2(\overline{\mathbb{Z}}^p)/\{\pm 1\}$ of deck transformations of coverings of modular curves.

◇ Note that

$$\begin{aligned}\theta_{\gamma_1\gamma_2}(f)(g, \rho, z) &= f(g, \rho\gamma_1\gamma_2, z) \\ &= \theta_{\gamma_2}(f)(g, \rho\gamma_1, z) = \theta_{\gamma_1}(\theta_{\gamma_2}(f))(g, \rho, z). \quad \square\end{aligned}$$

The novelty of the GL_2 system with respect to the BC case is that $GL_2^+(\mathbb{Q})$ acts as well by endomorphisms of \mathfrak{A}_2 , so that

$$\theta_m(f)(g, \rho, z) = \begin{cases} f(g, \rho \det(m)^{-1}m, z) & \rho \in mM_2(\overline{\mathbb{Z}}^p), \\ 0 & \text{otherwise.} \end{cases}$$

◇ Check that

$$\begin{aligned}\theta_{m_1 m_2}(f)(g, \rho, z) &= \begin{cases} f(g, \rho \det(m_1 m_2)^{-1} m_1 m_2 z) & \rho \in m_1 m_2 M_2(\overline{\mathbb{Z}}^p), \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \theta_{m_2}(f)(g, \rho \det(m_1)^{-1} m_1, z) & \rho m_1 \in m_2 M_2(\overline{\mathbb{Z}}^p), \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \theta_{m_1}(\theta_{m_2}(f))(g, \rho, z) & \rho \in m_1 M_2(\overline{\mathbb{Z}}^p), \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(But if so, does such happen?) \square

The subgroup \mathbb{Q}^* of $GL_2(\mathbb{A}_f)$ acts by inner endomorphisms, and hence the group S of symmetries of the set \mathcal{E}_β of extremal KMS_β states on \mathfrak{A}_2 has the form $S = \mathbb{Q}^* \backslash GL_2(\mathbb{A}_f)$.

In the case of the set \mathcal{E}_∞ of KMS_∞ states on \mathfrak{A}_2 , defined as weak limits, defining the action of $GL_2^+(\mathbb{Q})$ on the set is more subtle. In fact, the action θ defined above does not directly induce a non-trivial action on \mathcal{E}_∞ . However, there is a non-trivial action induced by the action as ρ on the sets \mathcal{E}_β of KMS_β states for sufficiently large β . This action on $\varphi \in \mathcal{E}_\infty$ is obtained by the warming up and cooling down procedure, as given before. That is,

$$(\rho^* \varphi)(a) = \lim_{\beta=T^{-1} \rightarrow \infty} \frac{\text{tr}(\pi_\varphi(\rho(a))e^{-\beta K})}{\text{tr}(\pi_\varphi(\rho(1))e^{-\beta K})}, \quad a \in \mathfrak{A}_2.$$

Finally, the Galois action on the extremal KMS_∞ states on \mathfrak{A}_2 is described by the following result of [41]:

Theorem 3.10. *There exists an arithmetic subalgebra $\mathfrak{A}_{2,\mathbb{Q}}$ of the unbounded multiplier algebra $M^{ub}(\mathfrak{A}_2)$ of the C^* -algebra \mathfrak{A}_2 , such that the following holds:*

For $\varphi_{\infty,L} \in \mathcal{E}_\infty$ with $L = (\rho, \tau)$ generic, the values on arithmetic elements of $\mathfrak{A}_{2,\mathbb{Q}}$ satisfy $\varphi_{\infty,L}(\mathfrak{A}_{2,\mathbb{Q}}) \subset F_\tau$, where F_τ is the embedding of the modular field F into \mathbb{C} given by evaluation at $\tau \in \mathbb{H}$.

The values of $\varphi_{\infty,L}(\mathfrak{A}_{2,\mathbb{Q}})$ generate F_τ .

There is an isomorphism

$$\theta_{\varphi_{\infty,L}} : \text{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}^* \backslash GL_2(\mathbb{A}_f),$$

that intertwines the Galois action on the values of the KMS_∞ states with the action of symmetries, so that for $\gamma \in \text{Gal}(F_\tau/\mathbb{Q})$, the diagram commutes

$$\begin{array}{ccc} \mathfrak{A}_{2,\mathbb{Q}} \subset M^{ub}(\mathfrak{A}_2) & \xrightarrow{\theta_{\varphi_{\infty,L}}(\gamma)} & \mathfrak{A}_{2,\mathbb{Q}} \subset M^{ub}(\mathfrak{A}_2) \\ \varphi_{\infty,L} \downarrow & & \downarrow \varphi_{\infty,L} \\ \varphi_{\infty,L}(\mathfrak{A}_{2,\mathbb{Q}}) \subset F_\tau \subset \mathbb{C} & \xrightarrow{\gamma} & \varphi_{\infty,L}(\mathfrak{A}_{2,\mathbb{Q}}) \subset F_\tau \subset \mathbb{C}. \end{array}$$

Recall that the **modular** field F in the statement above is defined to be the field of modular functions over \mathbb{Q}^{Ab} , namely that is the union of the fields F_N of modular functions of level N rational, over the cyclotomic field $\mathbb{Q}(\zeta_n)$, that is, such that the q -expansion in powers of $q^{\frac{1}{N}} = e^{\frac{1}{N}2\pi i\tau}$ has all coefficients in $\mathbb{Q}(e^{\frac{1}{N}2\pi i})$.

It has explicit generators given by the Fricke functions ([146], [98]). Let \mathfrak{p} be the Weierstrass \mathfrak{p} -function. It gives the parameterization of the elliptic curve by the quotient $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$:

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \quad z \mapsto (1, \mathfrak{p}(z; \tau, 1), \mathfrak{p}'(z; \tau, 1)) = (1, x, y)$$

(corrected). Then the **Fricke** functions are homogeneous functions of 1-dimensional lattices of weight zero of the form:

$$f_v(\tau) = -2^7 3^5 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)} \mathfrak{p}(\lambda_\tau(v); \tau, 1)$$

parameterized by $v \in \mathbb{Q}^2/\mathbb{Z}^2$, where $\Delta = g_2^3 - 3^3 g_3^2 (\neq 0)$ is the discriminant (up to a scalar) for the quadratic equation with respect to the cubic equation $4x^3 - g_2x - g_3 = 0$, and $\lambda_\tau(v) = v_1\tau + v_2$.

Remark. (Added). Recall from [98] the following. A lattice L in \mathbb{C} is a \mathbb{Z} -rank two subgroup of \mathbb{C} , such that $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ and $\mathbb{C} = \mathbb{R}w_1 + \mathbb{R}w_2$. May assume that $\text{Im}(\frac{w_1}{w_2}) > 0$, that is, $\frac{w_1}{w_2} \in \mathbb{H}$. An **elliptic** function f with respect to L is a meromorphic function on \mathbb{C} which is periodic mod L , so that $f(z + w) = f(z)$ for any $z \in \mathbb{C}$ and $w \in L$. An elliptic function can be viewed as a meromorphic function on the 2-torus \mathbb{C}/L . The **Weierstrass** function is defined to be

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum_{\omega} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum is taken over the set of all non-zero periods for elliptic functions mod L , so that $w = nw_1 + mw_2$ for $(n, m) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

The **Laurent** series expansion of $\mathfrak{p}(z)$ is given by

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)s_{2n+2}(L)z^{2n}, \quad s_m(L) = \sum_{\omega \neq 0} \frac{1}{\omega^m}.$$

By differentiating term by term,

$$\mathfrak{p}'(z) = -\frac{1}{z^3} + \sum_{n=1}^{\infty} (2n+1)(2n)s_{2n+2}(L)z^{2n-1}.$$

If we set $g_2(L) = 60s_4(L)$ and $g_3(L) = 140s_6(L)$, then it holds that

$$\mathfrak{p}'(z)^2 = 4\mathfrak{p}(z)^3 - g_2(L)\mathfrak{p}(z) - g_3(L). \quad \blacktriangleleft$$

Remark. (Added). As in [116], the **Cardano** formula is as follows. To solve $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, first put $b_1 = 9a_0a_1a_2 - 2a_1^3 - 27a_0^2a_3$ and $b_2 =$

$a_1^2 - 3a_0a_2$. Next solve $t^2 - b_1t + b_2^3 = 0$ to have solutions t_+ and t_- . Then the solution for the first equation is obtained as

$$\frac{-a_1 + \zeta_3 t_{\pm}^{\frac{3}{2}} + \zeta_3^2 t_{\mp}^{\frac{3}{2}}}{3a_0}.$$

◊ In the case of $4x^3 - g_2x - g_3 = 0$, we have

$$b_1 = 27 \cdot 4^2 g_3 \quad b_2 = 3 \cdot 4 g_2.$$

The discriminant for the above quadratic equation of t is given by

$$b_1^2 - 4b_2^3 = 3^6 4^4 g_3^2 - 3^3 4^4 g_2^3 = 3^3 4^4 (27g_3^2 - g_2^3) = -3^3 4^4 \Delta.$$

Hence

$$t_{\pm} = 27 \cdot 4^2 g_3 \pm \sqrt{-3^3 4^4 \Delta} = 3 \cdot 4^2 \left\{ 9g_3 \pm \sqrt{3(g_2^3 - 27g_3^2)} \right\}.$$

The solution of the cubic equation is obtained as $\frac{1}{12}(\zeta_3 t_{\pm}^{\frac{3}{2}} + \zeta_3^2 t_{\mp}^{\frac{3}{2}})$. $\square \blacktriangleleft$

For generic τ of $L = (\rho, \tau)$ such that $j(\tau)$ is transcendental, evaluation of the modular functions at the point $\tau \in \mathbb{H}$ gives an embedding of F_{τ} into \mathbb{C} . There is a corresponding isomorphism τ_{τ} from $\text{Gal}(F_{\tau}/\mathbb{Q})$ onto $\text{Aut}(F)$. The isomorphism $\theta_{\varphi_{\infty, L}}$ of $\text{Gal}(F_{\tau}/\mathbb{Q})$ onto $\mathbb{Q}^* \backslash GL_2(\mathbb{A}_f)$ is given by

$$\theta_{\varphi_{\infty, L}}(\gamma) = \rho^{-1} \theta_{\tau} \rho, \quad \gamma \in \text{Gal}(F_{\tau}/\mathbb{Q}).$$

In fact, the automorphism group $\text{Aut}(F)$ has a completely explicit description, due to a result of Shimura [146], so that $\text{Aut}(F)$ can be identified with the quotient $\mathbb{Q}^* \backslash GL_2(\mathbb{A}_f)$.

We are going to explain what is the arithmetic subalgebra $\mathfrak{A}_{2, \mathbb{Q}}$ of \mathfrak{A}_2 in the theorem above, in the following.

We consider continuous functions $f(g, \rho, z) \pmod{\Gamma^2}$ on the quotient \mathcal{U}/Γ^2 , with finite support with respect to the variable $g \in \Gamma \backslash GL_2^+(\mathbb{Q})$.

Define $f_{(g, \rho)}(z) = f(g, \rho, z)$, so that $f_{(g, \rho)} \in C_c(\mathbb{H})$. Let $p_N : M_2(\overline{\mathbb{Z}}^p) \rightarrow M_2(\mathbb{Z}/N\mathbb{Z})$ be the canonical projection. It is said that the function f is of **level** N if $f_{(g, \rho)} = f_{(g, p_N(\rho))}$ for any $(g, \rho) \in GL_2^+(\mathbb{Q}) \times M_2(\overline{\mathbb{Z}}^p) \pmod{\Gamma^2}$. If f is of level N , then f is determined by the functions $f_{(g, m)} \in C_c(\mathbb{H})$ for $m \in M_2(\mathbb{Z}/N\mathbb{Z})$. Note that the invariance as

$$f(g\gamma, \rho, z) = f(g, \gamma\rho, \gamma(z)), \quad \gamma \in \Gamma, (g, \rho, z) \in \mathcal{U}$$

implies the following identity:

$$f_{(g, m)\gamma} = f_{(g, m)}, \quad \gamma \in \Gamma(N) \cap g^{-1}\Gamma g$$

(corrected as action), so that f is invariant under a congruence subgroup.

◊ Note that

$$(g\gamma, \rho, z)\gamma \equiv (1, \gamma) \cdot (g\gamma, \rho, z) = (1g\gamma\gamma^{-1}, \gamma\rho, \gamma(z)) = (g, \gamma\rho, \gamma(z)).$$

Also, with $\gamma = g^{-1}\gamma'g$ with $\gamma' \in \Gamma$,

$$\begin{aligned} f_{(g,m)\gamma}(z) &= f(g\gamma^{-1}, \gamma m, \gamma(z)) = f(gg^{-1}(\gamma')^{-1}g, \gamma m, \gamma(z)) \\ &= f(g, \gamma m, \gamma(z)) = f(g\gamma, m, z) = f(\gamma'g, m, z) \\ &= f(g, m, z) = f_{(g,m)}(z) \end{aligned}$$

(mod $\Gamma \times \Gamma$). □

Define elements f of the **arithmetic** subalgebra $\mathfrak{A}_{2,\mathbb{Q}}$ of $\mathfrak{A}_2 = C^*(\mathcal{U}/\Gamma^2)$, as characterized by the following properties (1) to (3):

- (1) The support of $f(g, \rho, z)$ with respect to g in $\Gamma \backslash GL_2^+(\mathbb{Q})$ is finite.
- (2) Each f has a finite level N with $f_{(g,m)} \in F$ for $m \in M_2(\mathbb{Z}_N)$.
- (3) The following **cyclotomic** condition is satisfied:

$$f_{(g,\alpha(u)m)} = \text{cyc}(u)f_{(g,m)}, \quad \text{diagonal } g \in GL_2^+(\mathbb{Q}), \quad u \in (\overline{\mathbb{Z}^p})^*, \quad \alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

where $\text{cyc} : (\overline{\mathbb{Z}^p})^* \rightarrow \text{Aut}(F)$ denotes the action of the Galois group $(\overline{\mathbb{Z}^p})^* \cong \text{Gal}(\mathbb{Q}^{Ab}/\mathbb{Q})$ on the coefficients of the q -expansion in powers of $q^{\frac{1}{N}} = e^{\frac{1}{N}2\pi i\tau}$.

If we take only the first two conditions (1) and (2) above, it would follow that the algebra $\mathfrak{A}_{2,\mathbb{Q}}$ contains the cyclotomic field $\mathbb{Q}^{Ab} \subset \mathbb{C}$, but this would prevent the existence of states satisfying the desired Galois property. In fact, if the subalgebra contains scalar elements in \mathbb{Q}^{cyc} , then the sought of Galois property would not be compatible with the \mathbb{C} -linearity of states. The cyclotomic condition then forces the spectrum of the elements of $\mathfrak{A}_{2,\mathbb{Q}}$ to contain all Galois conjugates of any root of unity that appears in the coefficients of the q -series, so that elements of $\mathfrak{A}_{2,\mathbb{Q}}$ can not be scalars.

The algebra $\mathfrak{A}_{2,\mathbb{Q}}$ defined by the properties above is a subalgebra of the unbounded multiplier algebra of \mathfrak{A}_2 , which is globally invariant under the group of symmetries $\mathbb{Q}^* \backslash GL_2(\mathbb{A}_f)$.

3.5 Quadratic fields

The first essential step in the direction of generalizing the Bost-Connes system to other number fields and exploring a possible approach to the explicit class field theory problem via noncommutative geometry is the construction of a QSM system that recovers the explicit class field theory for imaginary quadratic fields as $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ for d a positive integer.

Presented in this section is such a construction based on the work of Connes, Marcolli, and Ramachandran [48]. Then discussed are some ideas about the unsolved problem of real quadratic fields, based on the real multiplication program of Manin ([109], [110]).

Imaginary quadratic fields and their lattices. Let $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ be the ring of integers of an imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\tau)$, with the imbedding

of \mathbb{K} into \mathbb{C} with $\tau \in \mathbb{H}$. Let $\text{Nm} : \mathbb{K}^* \rightarrow \mathbb{Q}^*$ denote the norm map. Then $\text{Nm}(x) = \det(g)$ for $g = q_\tau(x)$ the image under the embedding $q_\tau : \mathbb{K}^* \rightarrow GL_2(\mathbb{Q})$ determined by the choice of τ . The image of the embedding q_τ is characterized by the property

$$q_\tau(\mathbb{K}^*) = \{g \in GL_2^+(\mathbb{Q}) \mid g(\tau) = \tau\}.$$

A quantum statistical mechanical (QSM) system for imaginary quadratic fields \mathbb{K} can be constructed by considering commensurability classes of 1-dimensional \mathbb{K} -lattices.

Definition 3.11. For \mathbb{K} an imaginary quadratic field, a 1-dimensional \mathbb{K} -lattice (Λ, φ) is defined to be a finitely generated \mathcal{O} -submodule of \mathbb{C} , such that $\Lambda \otimes_{\mathcal{O}} \mathbb{K} \cong \mathbb{K}$, together with a morphism of \mathcal{O} -modules: $\varphi : \mathbb{K}/\mathcal{O} \rightarrow \mathbb{K}\Lambda/\Lambda$.

A 1-dimensional \mathbb{K} -lattice is **invertible** if φ is an isomorphism of \mathcal{O} -modules.

A 1-dimensional \mathbb{K} -lattice is viewed as a 2-dimensional \mathbb{Q} -lattice. This plays an important role in the notion of commensurability below.

Definition 3.12. Two 1-dimensional \mathbb{K} -lattices (Λ_1, φ_1) and (Λ_2, φ_2) are **commensurable** if $\mathbb{K}\Lambda_1 = \mathbb{K}\Lambda_2$ and $\varphi_1 = \varphi_2$ modulo $\Lambda_1 + \Lambda_2$.

It turns out that two 1-dimensional \mathbb{K} -lattices are commensurable if and only if their underlying \mathbb{Q} -lattices are commensurable.

We can give a more explicit description of the data of 1-dimensional \mathbb{K} -lattices. Namely, the data (Λ, φ) of a 1-dimensional \mathbb{K} -lattice L are equivalent to the data (ρ, s) of an element $\rho \in \overline{\mathbb{Z}}^p \otimes \mathcal{O} \equiv \overline{\mathcal{O}}^p$ and $s \in \mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*$, module the action of $(\overline{\mathcal{O}}^p)^*$ given by $x \cdot (\rho, s) = (x^{-1}\rho, xs)$. Thus, the space of 1-dimensional \mathbb{K} -lattices is identified with

$$\overline{\mathcal{O}}^p \times_{(\overline{\mathcal{O}}^p)^*} (\mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*).$$

Let $\mathbb{A}_{\mathbb{K}}^{\circ} = \mathbb{A}_{\mathbb{K},f} \times \mathbb{C}^*$ denote the subset of adèles of \mathbb{K} with non-trivial archimedean component. There is an identification of the set $\text{Lt}_{1,\mathbb{K}}$ of commensurability classes of 1-dimensional \mathbb{K} -lattices (not up to scale) with the space $\mathbb{A}_{\mathbb{K}}^{\circ}/\mathbb{K}^*$. In turn, the set $\text{Lt}_{1,\mathbb{K}}/\mathbb{C}^*$ of commensurability classes of 1-dimensional \mathbb{K} -lattices up to scaling can be identified with the quotient

$$\overline{\mathcal{O}}^p/\mathbb{K}^* = \mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*,$$

where the left hand side stands for the equivalence classes of elements of $\overline{\mathcal{O}}^p$ module the equivalence relation given by the partially defined action of \mathbb{K}^* .

Then noncommutative algebras of coordinates associated to those quotient spaces can be introduced. The procedure is analogous to that in the Bost-Connes case and the GL_2 case. Consider the convolution algebra $C_c(\text{Lt}_{1,\mathbb{K}})$ and its C^* -algebra completion $C^*(\text{Lt}_{1,\mathbb{K}})$, given as the groupoid algebra and C^* -algebra of the commensurability relation. Regard the elements of these algebras as functions $f(L, L')$ of pairs $(L, L') = ((\Lambda, \varphi), (\Lambda', \varphi')) \in \text{Lt}_{1,\mathbb{K}}^2$, with

the convolution product induced by the equivalence relation. By construction, this groupoid is a sub-groupoid of the groupoid of commensurability classes of 2-dimensional \mathbb{Q} -lattices. If we take the quotient $\text{Lt}_{1,\mathbb{K}}/\mathbb{C}^*$ by scaling, we define $\mathfrak{A}_{1,\mathbb{K}}$ as $C^*(\text{Lt}_{1,\mathbb{K}}/\mathbb{C}^*)$, which is still a groupoid algebra in this case, unlike what happens in the GL_2 case (cf. [48]). The C^* -algebra $\mathfrak{A}_{1,\mathbb{K}}$ is unital as in the case of the BC system.

The time evolution of the GL_2 system induces the natural time evolution on $\mathfrak{A}_{1,\mathbb{K}}$ given by

$$\sigma_t(f) = N^{it} f = e^{it \log N} f, \quad N : \mathbb{K}^* \rightarrow \mathbb{Q}^*.$$

The quantum statistical mechanical (QSM) system $(\mathfrak{A}_{1,\mathbb{K}}, \sigma, \mathbb{R})$ has properties that are intermediated between the BC system and the GL_2 system in some sense.

Note that invertible 1-dimensional \mathbb{K} -lattices $L = (\Lambda, \varphi)$ give rise to positive energy representations of $\mathfrak{A}_{1,\mathbb{K}}$ on the Hilbert space $l^2(G_L)$, where G_L is the set of elements of the groupoid $\text{Lt}_{1,\mathbb{K}}/\mathbb{C}^*$ with source $L = (\Lambda, \varphi)$. The set $\text{Lt}_{1,\mathbb{K}}^*/\mathbb{C}^*$ of invertible 1-dimensional \mathbb{K} -lattices up to scale can be identified with the idèle class group $C_{\mathbb{K}}/D_{\mathbb{K}} = \mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^*$.

◇ Recall that $\mathbb{A}_f = \overline{\mathbb{Z}}^p \otimes \mathbb{Q}$ is the ring of finite adèles of \mathbb{Q} , and

$$\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_f \otimes \mathbb{K} = \overline{\mathbb{Z}}^p \otimes \mathbb{Q} \otimes \mathbb{K} \cong \overline{\mathbb{Z}}^p \otimes \mathbb{K}$$

is the ring of finite adèles of \mathbb{K} , so that $\mathbb{A}_{\mathbb{K},f}$ containing $(\overline{\mathbb{Z}}^p)^* \otimes \mathbb{K}^*$ is the group of finite idèles of \mathbb{K} , and $\mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^*$ is the group of finite idèle classes of \mathbb{K} . \square

For an invertible \mathbb{K} -lattice L , the set G_L can be identified with the set of ideals J of $\mathcal{O} = \mathcal{O}_{\mathbb{K}}$. Then the **Hamiltonian** implementing the time evolution has the form as $He_J = \log \mathbf{n}(J)e_J$, where $\mathbf{n}(J)$ denotes the norm of J . Thus, the partition function of the system $(\mathfrak{A}_{1,\mathbb{K}}, \sigma, \mathbb{R})$ is the J. W. R. **Dedekind** zeta function $\zeta_{\mathbb{K}}(\beta)$ of \mathbb{K} (1877), so that

$$Z(\beta) = \sum_{J \subset \mathcal{O}: \text{ideal}} \frac{1}{\mathbf{n}(J)^\beta} \equiv \zeta_{\mathbb{K}}(\beta) \left(= \prod_{\mathfrak{p}} \frac{1}{1 - \mathbf{n}(\mathfrak{p})^{-\beta}} \right)$$

(cf. [116]). It is expected that this is the natural generalization of the Riemann zeta function $\zeta(\beta) = \zeta_{\mathbb{Q}}(\beta)$ to other number fields.

◇ If $\mathbb{K} = \mathbb{Q}$, then $\mathcal{O} = \mathbb{Z}$ and

$$\zeta_{\mathbb{Q}}(\beta) = \sum_{n\mathbb{Z} \subset \mathbb{Z}: \text{ideal}} \frac{1}{\mathbf{n}(n\mathbb{Z})^\beta} = \sum_{n=1}^{\infty} \frac{1}{n^\beta}. \quad \square$$

The symmetry group of the $\text{Lt}_{1,\mathbb{K}}$ system is the group $\mathbb{A}_{\mathbb{K},f}^*$ of idèle classes, with the subgroup \mathbb{K}^* acting on $\mathfrak{A}_{1,\mathbb{K}}$ by inner automorphisms, so that the induced action on the KMS states on $\mathfrak{A}_{1,\mathbb{K}}$ is given by the idèle class group $\mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^* = C_{\mathbb{K}}/D_{\mathbb{K}}$. As in the case of the GL_2 system, this group of symmetries includes an action by endomorphisms. Only the subgroup $(\overline{\mathcal{O}}^p)^*/\mathcal{O}^*$ with \mathcal{O}^* the group of units of \mathcal{O}^* acts by automorphisms, while the (other) full action of

$\mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^*$ involves endomorphisms. In particular, this shows the appearance of the class number of \mathbb{K} , as in the following commutative diagram:

$$\begin{array}{ccccccc}
1 & \longrightarrow & (\overline{\mathcal{O}^p})^*/\mathcal{O}^* & \longrightarrow & \mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^* & \longrightarrow & Cl(\mathcal{O}) \longrightarrow 1 \\
\parallel & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \parallel \\
1 & \longrightarrow & \text{Gal}(\mathbb{K}^{Ab}/\mathfrak{H}) & \longrightarrow & \text{Gal}(\mathbb{K}^{Ab}/\mathbb{K}) & \longrightarrow & \text{Gal}(\mathfrak{H}/\mathbb{K}) \longrightarrow 1
\end{array}$$

where \mathfrak{H} is the Hilbert class field of \mathbb{K} , that is, the maximal abelian unramified extension of \mathbb{K} . The ideal class group $Cl(\mathcal{O})$ is naturally isomorphic to the Galois group $\text{Gal}(\mathfrak{H}/\mathbb{K})$. The case of class number one is analogous to the BC system as already observed (cf. [79]).

The arithmetic subalgebra $\mathfrak{A}_{2,\mathbb{Q}}$ of \mathfrak{A}_2 for the GL_2 system induces, by restriction to the corresponding sub-groupoid, a natural choice of a rational subalgebra $\mathfrak{A}_{1,\mathbb{K},\mathbb{Q}}$ of $\mathfrak{A}_{1,\mathbb{K}}$ of the system $(\mathfrak{A}_{1,\mathbb{K}}, \sigma, \mathbb{R})$ for the imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$.

Note that, because of the fact that the variable $z \in \mathbb{H}$ in the coordinates of the GL_2 system is now only affecting finitely many values, indexed by the elements of the ideal class group $Cl(\mathcal{O})$, it is obtained that $\mathfrak{A}_{1,\mathbb{K},\mathbb{Q}}$ is a subalgebra of $\mathfrak{A}_{1,\mathbb{K}}$ as in the BC system case, though non involutive, and not just an algebra of unbounded multipliers of $\mathfrak{A}_{1,\mathbb{K}}$, as in the GL_2 case.

With the choice of such an arithmetic subalgebra, it is obtained as a result analogous to the theorem above that (cf. [48]):

Theorem 3.13. *The system $(\mathfrak{A}_{1,\mathbb{K}}, \sigma, \mathbb{R})$ has the following properties:*

- (1) *In the range $0 < \beta = \frac{1}{T} \leq 1$ there is a unique KMS_β state.*
- (2) *For $\beta = \frac{1}{T} > 1$, the set \mathcal{E}_β of extremal KMS_β states is parameterized by invertible 1-dimensional \mathbb{K} -lattices up to scale, so that*

$$\mathcal{E}_\beta \cong \text{Lt}_{1,\mathbb{K}}/\mathbb{C}^* \cong \mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^* \cong C_{\mathbb{K}}/D_{\mathbb{K}} \equiv (\mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*)/(C_{\mathbb{K}})_0,$$

with a free and transitive action of the idèle class group $G = C_{\mathbb{K}}/D_{\mathbb{K}}$ of \mathbb{K} .

- (3) *The set \mathcal{E}_∞ of extremal KMS_∞ states, as weak limits of KMS_β states, is also given by replacing \mathcal{E}_β in the parameterization above. The extremal KMS_∞ states, evaluated on the arithmetic subalgebra $\mathfrak{A}_{1,\mathbb{K},\mathbb{Q}}$, generate the maximal Abel extension \mathbb{K}^{Ab} . The class field theory isomorphism $\theta : \text{Gal}(\mathbb{K}^{Ab}/\mathbb{K}) \rightarrow \mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^*$ intertwines the action $\mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^*$ as symmetries of the system with the Galois action of $\text{Gal}(\mathbb{K}^{Ab}/\mathbb{K})$ on the image of $\mathfrak{A}_{1,\mathbb{K},\mathbb{Q}}$ under the extremal KMS_∞ states $\varphi_{\infty,L}$, so that*

$$\begin{array}{ccc}
\mathfrak{A}_{1,\mathbb{K},\mathbb{Q}} \subset \mathfrak{A}_{1,\mathbb{K}} & \xrightarrow{\varphi_{\infty,L}} & \mathbb{K}^{Ab} \subset \mathbb{C} \\
\theta(\alpha) \downarrow \in \mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^* & & \alpha \downarrow \in \text{Gal}(\mathbb{K}^{Ab}/\mathbb{K}) \\
\mathfrak{A}_{1,\mathbb{K},\mathbb{Q}} \subset \mathfrak{A}_{1,\mathbb{K}} & \xrightarrow{\varphi_{\infty,L}} & \mathbb{K}^{Ab} \subset \mathbb{C}
\end{array}$$

Now note that for $\mathbb{K} = \mathbb{Q}(\tau)$, with $\tau = i\sqrt{d}$, the classes of 1-dimensional \mathbb{K} -lattices L , viewed as 2-dimensional \mathbb{Q} -lattices $L = (\rho, z)$, correspond to only finitely many values of $z \in \mathbb{H} \cap \mathbb{K}$. Moreover, these points no longer satisfy the

generic condition. In fact, for a CM point $z \in \mathbb{H} \cap \mathbb{K}$, evaluation of elements in the modular field F at z no longer gives an embedding. In this case, the image F_z (corrected) in \mathbb{C} a copy of the maximal Abel extension \mathbb{K}^{Ab} of \mathbb{K} , so that $F \rightarrow F_z = \mathbb{K}^{Ab} \subset \mathbb{C}$ (cf. [146]). The explicit action of the Galois group $\text{Gal}(\mathbb{K}^{Ab}/\mathbb{K})$ is obtained through the action of automorphisms of the modular field F via Shimura reciprocity ([146]), as described in the following diagram with exact rows:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{A}_{\mathbb{K},f}^*/\mathbb{K}^* & \xrightarrow{\frac{\bar{u}}{\theta^{-1}}} & \text{Gal}(\mathbb{K}^{Ab}/\mathbb{K}) & \longrightarrow & 1 \\
& & \uparrow & & \parallel & & \parallel \\
1 & \longrightarrow & \mathbb{K}^* & \longrightarrow & GL_1(\mathbb{A}_{\mathbb{K},f}) = \mathbb{A}_{\mathbb{K},f}^* & \xrightarrow{u} & \text{Gal}(\mathbb{K}^{Ab}/\mathbb{K}) & \longrightarrow & 1 \\
\parallel & & N \downarrow \det \circ q_\tau & & \downarrow q_\tau & & \parallel & & \parallel \\
1 & \longrightarrow & \mathbb{Q}^* & \longrightarrow & GL_2(\mathbb{A}_f) & \xrightarrow{\sigma} & \text{Aut}(F) & \longrightarrow & 1
\end{array}$$

where q_τ is the embedding determined by the choice of $\tau \in \mathbb{H}$ with $\mathbb{K} = \mathbb{Q}(\tau)$. Thus, the explicit **Galois** action is given by

$$u(\gamma)(f(\tau)) = f^{\sigma(q_\tau(\gamma))}(\tau), \quad f \in F, f^\alpha = \alpha(f), \alpha \in \text{Aut}(F).$$

That picture provides the intertwining between the symmetries of the quantum statistical mechanical system and the Galois action on the image of states on the elements of the arithmetic subalgebra.

The structure of KMS states at positive temperatures $T = \frac{1}{\beta} > 0$ (which go to 0 as $\beta \rightarrow \infty$) is similar to the Bost-Connes system case (cf. [48] for more details).

Real multiplication. The next fundamental question in the direction of generalizations of the BC system to other number fields is how to approach the more difficult case of real quadratic fields. Given is a brief outline of some ideas on real multiplication of Manin [109], [110], and suggested is how the ideas may be combined with the GL_2 system described above.

Developed by Manin [110] is a theory of real multiplication for noncommutative tori, aimed at providing a setting, within noncommutative geometry, in which to treat the problem of abelian extensions of real quadratic fields on a somewhat similar footing as the known case of imaginary quadratic fields, for which the theory of elliptic curves with complex multiplication provides the right geometric setup.

The first translation in the dictionary developed in [110] between elliptic curves with complex multiplication and noncommutative tori with real multiplication is given by a parallel between lattices Λ and pseudo-lattices PL in \mathbb{C} :

$$(\mathbb{C}, \text{Lattice } \Lambda) \rightleftharpoons (\mathbb{R}, \text{Pseudo-Lattice } PL).$$

A **pseudo** lattice PL means the date (G, ψ) (as $(\mathbb{Z}^2, \psi : \mathbb{Z}^2 \rightarrow \mathbb{R} \subset \mathbb{C})$ up to isomorphism, rotation, and orientation) of a free abelian group G of rank two,

with an injective homomorphism $\psi : G \rightarrow \mathbb{C}$, such that the image lies in an oriented real line.

This aims at generalizing the well known equivalence between the category of elliptic curves as $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and the category of 2-dimensional lattices as $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathbb{H}$, realized by the period functor, to a setting that includes noncommutative 2-tori \mathfrak{A}_θ as a deformation of E_q with $q = \exp(2\pi i\tau) \rightarrow \exp(2\pi i\theta) \in S^1$.

$$\text{Lattices } \Lambda_\tau \Leftrightarrow \text{Elliptic Curves } E_\tau \Longrightarrow \text{NC torus } \mathfrak{A}_\theta.$$

As seen previously, any 2-dimensional lattice has the form $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ for $\tau \in \mathbb{C} \setminus \mathbb{R}$, up to isomorphism, and non-trivial morphisms between such lattices are given by the action of matrices of $M_2(\mathbb{Z})$ by fractional linear transformations. Thus, the moduli space of lattices up to isomorphism is given by the quotient of $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ by $PGL_2(\mathbb{Z})$.

$$\text{Lattices}/M_2(\mathbb{Z}) \approx [\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})]/PGL_2(\mathbb{Z}).$$

A pseudo-lattice has the form $PL_\theta = \mathbb{Z} + \mathbb{Z}\theta$ for $\theta \in \mathbb{R} \setminus \mathbb{Q}$ up to isomorphism, and non-trivial morphisms of pseudo-lattices are given by matrices of $M_2(\mathbb{Z})$, also acting on $\mathbb{P}^1(\mathbb{R})$ by fraction linear transformations. The moduli space of pseudo-lattices is given by the quotient of $\mathbb{P}^1(\mathbb{R})$ by the action of $PGL_2(\mathbb{Z})$. Since the action does give rise to a bad classical quotient, the moduli space should be treated as a noncommutative space.

$$\text{Pseudo-Lattices}/M_2(\mathbb{Z}) \approx \mathbb{P}^1(\mathbb{R})/PGL_2(\mathbb{Z}) \Leftrightarrow \text{NC spaces.}$$

As described in the previous chapter, the resulting space represents a component in the (noncommutative) boundary of the classical moduli space of elliptic curves, which parameterizes the degenerations from lattices to pseudo-lattices, which are invisible to the usual algebro-geometric setting.

The cusp, i.e., the orbit of $\mathbb{P}^1(\mathbb{Q})$, corresponds to the degenerate case where the image in \mathbb{C} of the rank two free abelian group has rank one. This gives another translation in the dictionary, regarding the moduli spaces as

$$X_\Gamma = \mathbb{H}/\Gamma = \mathbb{H}/PSL_2(\mathbb{Z}) \Leftrightarrow C(\mathbb{P}^1(\mathbb{R})) \rtimes PGL_2(\mathbb{Z}),$$

with $\overline{X}_\Gamma = (\mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q}))/\Gamma$ with $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}^+ \subset \mathbb{P}^1(\mathbb{R}) = \mathbb{R}^+ \approx S^1$.

In the correspondence between pseudo-lattices PL and noncommutative tori \mathfrak{A}_θ , the group of invertible morphisms of PL corresponds to isomorphisms of NC tori, realized by strong Morita equivalences. In this context, a morphism is not given as a morphism of algebras, but as a map of the category of modules, obtained by tensoring with bimodules. The notion of Morita equivalences as morphisms fits into the more general context of correspondences for operator algebras as in [27, V. Appendix B] as well as in the algebraic approach to noncommutative spaces of [143].

The category of noncommutative tori is defined by considering as morphisms the isomorphism classes of projective C^* -modules \mathcal{P}_θ (corrected) that are the

ranges of projections of matrix algebras over NC tori. The functor \mathfrak{F} from NC tori to pseudo-lattices PL is then given on objects by

$$\mathfrak{F} = K_0 : \mathbb{T}_\theta^2 = \mathfrak{A}_\theta \mapsto (K_0(\mathfrak{A}_\theta) \cong K_0(\mathcal{A}_\theta), HC_0(\mathcal{A}_\theta), \tau : K_0(\mathfrak{A}_\theta) \rightarrow HC_0(\mathcal{A}_\theta))$$

(cf. [109, 3.3], [110, 1.4]), where \mathcal{A}_θ means the dense subalgebra of the C^* -algebra \mathfrak{A}_θ , $HC_0(\mathcal{A}_\theta) = \mathcal{A}_\theta/[\mathcal{A}_\theta, \mathcal{A}_\theta]$, with τ the universal trace, and the orientation is determined by the cone of positive elements of $K_0(\mathfrak{A}_\theta)$. On morphisms, the functor is given by

$$\mathfrak{F} = \otimes_{\mathfrak{A}_\theta} \mathcal{M}_{\theta, \theta'} : [\mathcal{P}_\theta] \mapsto [\mathcal{P}_\theta \otimes_{\mathfrak{A}_\theta} \mathcal{M}_{\theta, \theta'}],$$

where $\mathcal{M}_{\theta, \theta'}$ are the \mathfrak{A}_θ - $\mathfrak{A}_{\theta'}$ (or \mathcal{A}_θ - $\mathcal{A}_{\theta'}$) bimodules, constructed by Connes in [22]. A crucial point in this definition is the fact that, for noncommutative 2-tori, finite projective modules are classified by the value of a unique normalized trace (cf. [22], [140]).

The functor \mathfrak{F} as K_0 on objects and as the bimodule tensor product on morphisms is weaker than an equivalence of category. For instance, it maps trivially all ring homomorphisms on NC tori, that act trivially on K_0 . However, this correspondence is sufficient to develop a theory of real multiplication for noncommutative tori, parallel to the theory of complex multiplication for elliptic curves (as tori) (cf. [109], [110]).

For rank two lattices Λ or elliptic curves \mathbb{C}/Λ , the typical situation is that $\text{End}(\Lambda) = \mathbb{Z}$, but there are exceptional lattices Λ , for which $\text{End}(\Lambda)$ strictly contains \mathbb{Z} . In this case, there exists a complex quadratic field \mathbb{K} such that Λ is isomorphic to a lattice in \mathbb{K} . More precisely, the endomorphism ring $\text{End}(\Lambda)$ is given by $\mathbb{Z} + f\mathcal{O}$, where \mathcal{O} is the ring of integers of \mathbb{K} and the integer $f \geq 1$ is called the conductor. Such lattices are said to have complex multiplication. The elliptic curve $E_{\mathbb{K}}$ with $E_{\mathbb{K}}(\mathbb{C}) = \mathbb{C}/\mathcal{O}$ is endowed with a complex multiplication map, given on the universal cover by $x \mapsto ax$ for $a \in \mathcal{O}$.

Similarly, there is a parallel situation for pseudo-lattices PL , that $\text{End}(PL) \supsetneq \mathbb{Z}$ happens when there exists a real quadratic field \mathbb{K} such that PL is isomorphic to a pseudo-lattice contained in \mathbb{K} . In this case, it also holds that $\text{End}(PL) = \mathbb{Z} + f\mathcal{O}$. Such pseudo-lattices are said to have real multiplication. The pseudo-lattices PL_θ with real multiplication correspond to values of $\theta \in \mathbb{R} \setminus \mathbb{Q}$ that are quadratic irrationals. These are characterized by having eventually periodic, continued fraction expansion. The real multiplication map is given by tensoring with bimodules, so that in the case of θ with periodic, continued fraction expansion, there is an element $g \in PGL_2(\mathbb{Z})$ such that $g\theta = \theta$, to which an \mathcal{A}_θ - \mathcal{A}_θ bimodule \mathcal{E} can be associated.

An analogue of isogenies for noncommutative tori is obtained by considering morphisms of pseudo-lattices as $PL_\theta \mapsto PL_{n\theta}$, which correspond to a Morita given by \mathfrak{A}_θ viewed as an $\mathfrak{A}_{n\theta}$ - \mathfrak{A}_θ bimodule, where $\mathfrak{A}_{n\theta}$ can be embedded into \mathfrak{A}_θ by sending $u_{n\theta}$ to u_θ^n and $v_{n\theta}$ to v_θ .

◇ Suppose that $v_{n\theta}u_{n\theta} = e^{2\pi i n\theta}u_{n\theta}v_{n\theta}$ in $\mathfrak{A}_{n\theta}$. Then, in \mathfrak{A}_θ ,

$$v_\theta u_\theta^n = e^{2\pi i\theta}u_\theta v_\theta u_\theta^{n-1} = \dots = e^{2\pi i n\theta}u_\theta^n v_\theta. \quad \square$$

By considering isogenies, enriched can be the dictionary between the moduli space of elliptic curves and moduli space of Morita equivalent noncommutative tori. This leads to consider the whole tower of modular curves parameterizing elliptic curves with level structure and the corresponding tower of noncommutative modular curves described in the previous section (cf. [112]).

$$\mathbb{H}/G \rightleftharpoons C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma,$$

for $\Gamma = PGL_2(\mathbb{Z})$ or $PSL_2(\mathbb{Z})$ and G a finite index subgroup of Γ , with $\mathbb{P} = G \backslash \Gamma$ the coset space.

In the problem of constructing the maximal abelian extension of complex quadratic fields, the method is based on evaluating at the torsion points of the elliptic curve $E_{\mathbb{K}}$ a power of the Weierstrass function. This means considering the corresponding values of a coordinate on the projective line $E_{\mathbb{K}}/\mathcal{O}^*$. The analogous object in the noncommutative setting, replacing the projective line, should be a crossed product of functions on \mathbb{K} by the $ax + b$ group with $a \in \mathcal{O}^*$ and $b \in \mathcal{O}$, for a real quadratic field \mathbb{K} (cf. [110]), that is, like

$$C_c(\mathbb{K}) \rtimes (\mathcal{O} \rtimes \mathcal{O}^*).$$

A different method to construct abelian extensions of a complex quadratic field \mathbb{K} is based on **Stark** numbers. Following [110], if f is an injective homomorphism of a free abelian group Λ of rank two to a 1-dimensional complex vector space V , and $\lambda_0 \in \Lambda \otimes \mathbb{Q}$, then consider the zeta function

$$\zeta(\Lambda, \lambda_0, s) = \sum_{\lambda \in \Lambda} \frac{1}{|f(\lambda_0 + \lambda)|^{2s}}.$$

◇ Note that f on Λ may be extended to $f \otimes \text{id}$ on $\Lambda \otimes \mathbb{Q}$. □
It is proved by Stark [147] that the numbers defined by

$$S(\Lambda, \lambda_0) = \exp \zeta'(\Lambda, \lambda_0, 0)$$

(as $\zeta' = \frac{d}{ds} \zeta$) are algebraic units generating certain abelian extensions of \mathbb{K} . The argument in this case is based on a direct computational tool as the Kronecker second limit formula and upon reducing the problem to the theory of complex multiplication. There is no known independent argument for the Stark conjectures, while the analogous question is open for the case of real quadratic fields.

For a real quadratic field \mathbb{K} , instead of zeta functions $\zeta(\Lambda, \lambda_0, s)$ of the form above, the conjectural Stark units are obtained from zeta functions of the following form (as in [110]):

$$\zeta(L, l_0, s) = \text{sgn}(l_0^g) N(\mathfrak{a})^s \sum_{l \in L \bmod L\mathfrak{a}^{-1}} \frac{\text{sgn}(l_0 + l)^g}{|N(l_0 + l)|^s},$$

where $l \mapsto l^g$ means the action of the nontrivial element g in $\text{Gal}(\mathbb{K}/\mathbb{Q})$, and $N(l) = ll^g$, and the element $l_0 \in \mathcal{O}$ is chosen so that the ideals $\mathfrak{a} = (L, l_0)$ and

$l_0 \mathfrak{a}^{-1}$ are coprime with $L \mathfrak{a}^{-1}$, and the summation over $l \in L$ in the formula is restricted by taking only one representative from each coset class $(l_0 + l)\varepsilon$, for units ε satisfying $(l_0 + L)\varepsilon = l_0 + L$, i.e., $\varepsilon \equiv 1 \pmod{L \mathfrak{a}^{-1}}$. Then the Stark numbers are defined to be

$$S(L, l_0) = \exp \zeta'(L, l_0, 0).$$

Developed by Manin in [110] is an approach to the computation of the zeta functions $\zeta(L, l_0, s)$ based on a version of theta functions for pseudo-lattices, which are obtained by averaging theta constants of complex lattices along geodesics with ends at a pair of conjugate quadratic irrationals θ and θ' in $\mathbb{R} \setminus \mathbb{Q}$.

This procedure fits into a general philosophy, according to which recasted can be part of the arithmetic theory of modular curves in terms of the non-commutative boundary as the C^* -algebra crossed product as $C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma$ by studying the limiting behavior as $\tau \rightarrow \theta \in \mathbb{R} \setminus \mathbb{Q}$ along geodesics, or some averaging along such geodesics. In general, a non-trivial quantum system can be obtained when approaching θ along a path that corresponds to a geodesic in the modular curve that spans a limiting cycle, which is the case precisely when θ as the end point is a quadratic irrational. An example of this type of behavior is known as the theory of limiting modular symbols, developed by M and M [112] and [113].

Pseudo-lattices and the GL_2 system.

The noncommutative space of the GL_2 system

$$\mathrm{Sm}^{nc}(GL_2, \mathbb{H}^\pm) = GL_2(\mathbb{Q}) \backslash (M_2(\mathbb{A}_f) \times \mathbb{H}^\pm)$$

also admits a compactification, given by adding the boundary $\mathbb{P}^1(\mathbb{R})$ to \mathbb{H}^\pm , as in the noncommutative compactification of modular curves ([112]),

$$\overline{\mathrm{Sm}}^{nc}(GL_2, \mathbb{H}^\pm) = GL_2(\mathbb{Q}) \backslash (M_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{C})) = GL_2(\mathbb{Q}) \backslash M_2(\mathbb{A}) / \mathbb{C}^*,$$

where $\mathbb{P}^1(\mathbb{C}) = \mathbb{H}^\pm \cup \mathbb{P}^1(\mathbb{R})$.

◇ Note that $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ denotes the adèles of \mathbb{Q} , with $\mathbb{A}_f = \overline{\mathbb{Z}}^p \otimes \mathbb{Q}$ of the finite adèles, and that $\mathbb{H}^\pm \approx GL_2(\mathbb{R}) / \mathbb{C}^*$ in $\mathbb{P}^1(\mathbb{C})$. (cf. [150]). Indeed, in $M_2(\mathbb{R})$, there is an identification as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (z, w) \in \mathbb{C}^2, \quad \text{with } z = (a, c), w = (b, d) \in \mathbb{C},$$

so that $g \in GL_2(\mathbb{R})$ if and only if z and w are linearly independent over \mathbb{R} , and $g \notin GL_2(\mathbb{R})$ if and only if they are linearly dependent on \mathbb{R} . In the latter case, there is $t \in \mathbb{R}$ such that $w = tz$, and the elements $(z, tz) = z(1, t)$ for $z \in \mathbb{C}^*$ correspond to \mathbb{R} in $(\mathbb{C}^2 \setminus \{0_2\}) / \mathbb{C}^* = \mathbb{P}^1(\mathbb{C})$, so that $\mathbb{R} \cup \{0\} \approx S^1 \approx \mathbb{P}^1(\mathbb{R})$, with $0 \in M_2(\mathbb{R})$. In the first case, the component w can not be written as tz for any \mathbb{R} , but can be written as hz for some $h \in \mathbb{H}^\pm$. \square

That compactification corresponds to adding to the space Lt_2 of commensurability classes of 2-dimensional \mathbb{Q} -lattices the set pL_2 of (classes of) pseudo-lattices in the sense of [110], considered together with the \mathbb{Q} -structure. It then

seems that the real multiplication program of Manin may fit in with the boundary of the noncommutative space of the GL_2 system. The crucial question in this respect becomes the construction of an arithmetic algebra associated to the noncommutative modular curves. The results illustrated in the previous section, regarding identifies involving modular forms at the boundary of the classical modular curves and limiting modular symbols, as well as the still mysterious phenomenon of quantum modular forms identified by Zagier, implies the fact that there should exist some class of objects replacing modular forms, when pushed to the boundary.

Regarding the role of modular forms, note that, in the case of the noncommutative compactification of modular curves given above, the dual system can be considered. This is given as a C^* -algebra bundle as

$$\text{Lt}_2 = GL_2(\mathbb{Q}) \backslash M_2(\mathbb{A}).$$

On this dual space, modular forms appear naturally instead of modular functions, and the algebra of coordinates contains the modular Hecke algebra of Connes-Moscovici as an arithmetic subalgebra (cf. [54], [55]).

Thus, the noncommutative (boundary) geometry of the space of \mathbb{Q} -lattices modulo the equivalence relation of commensurability provides a setting that unifies several phenomena involving the interaction between NC geometry and number theory. This include the Bost-Connes system, as well, the NC space underlying the construction of the spectral realization of zeros of the Riemann zeta function as in [29], the modular Hecke algebras of [54], [55], and the noncommutative modular curves of [112].

4 Noncommutative geometry at arithmetic infinity

This section is mostly based on the work of Consani and Marcolli ([58], [59], [60], [61], [63]). Proposed is a model for the dual graph of the maximally degenerate fibers at the archimedean places of an arithmetic surface, in terms of a noncommutative space as a spectral triple, related to the action of a Schottky group on its limit set. This description of ∞ -adic geometry provides a compatible setting that combines the result of Manin [106] on the Arakelov Green function for arithmetic surfaces in terms of hyperbolic geometry and for a cohomological construction of Consani [57] associated to the archimedean fibers of arithmetic varieties, related to the Deninger calculation of the local L -factors as regularized determinants (cf. [68], [69]).

4.1 Schottky uniformization

A compact Riemann surface X of genus g is obtained by gluing each pair of the (oriented and paired) sides of a $4g$ -polygon X^\sim (a polygon in \mathbb{R}^2 with $4g$ sides) along the orientation, as a topological space. The fundament group of X has

the presentation with one relation as

$$\pi_1(X) = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle,$$

where each of the $2g$ generators a_i and b_i correspond to an oriented shift (in X^\sim) (or cycle (or rotation) in X) along an oriented pair of the sides of the $4g$ -polygon X^\sim , identified in X , and $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

◇ Recall that the **fundamental** group of a topological space X is defined to be the group of homotopy classes of continuous maps (or paths) $f(t)$ from $[0, 1]$ to X with some base point of X as $f(0) = f(1)$ as a loop, where the inverse path of $f(t)$ is $f(1-t)$ and the product $f \cdot g$ of two loops $f(t)$ and $g(t)$ is defined by $(f \cdot g)(t) = f(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $(f \cdot g)(t) = g(2t-1)$ for $\frac{1}{2} \leq t \leq 1$ (cf. [116]). \square

In the case of $g = 1$, X is the real 2-dimensional torus \mathbb{T}^2 , with $X^\sim \approx [0, 1]^2$, where $[0, 1] \times \{0\}$ is identified with $[0, 1] \times \{1\}$, and $\{0\} \times [0, 1]$ with $\{1\} \times [0, 1]$. And $\pi_1(X) \cong \mathbb{Z}^2$, with $a_1 b_1 = b_1 a_1$.

◇ Indeed, a homotopy between the corresponding product functions $f_1 \cdot g_1$ and $g_1 \cdot f_1$ is given by moving graphs with changing variables continuously, according to fix the base point. It seems to be difficult to construct a homotopy between the product of multiplicative commutators and the identity, for $g \geq 2$. \square

The parallelogram for $g = 1$ is the fundamental domain of the $\pi_1(X) \cong \mathbb{Z}^2$ action on the complex plane \mathbb{C} , so that $X \approx \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ as an elliptic curve.

For genus at least ≥ 2 , the hyperbolic plane \mathbb{H}^2 admits a tessellation by regular $4g$ -polygons, and the action of the fundamental group by deck transformations is realized by the action of $\pi_1(X)$ as a subgroup G of $PSL_2(\mathbb{R})$ by isometries of \mathbb{H}^2 . The compact Riemann surface X is then endowed with a hyperbolic metric and a Fuchsian uniformization as $X = G \backslash \mathbb{H}^2$.

Another, less well known, type of uniformization of compact Riemann surfaces is Schottky uniformization. Recall briefly some general facts on Schottky groups.

Schottky groups. A **Schottky** group of rank h is a discrete subgroup Γ of the semi-simple Lie group $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm 1_2\}$, which is purely loxodromic and isometric to a free group of rank h . The group $PSL_2(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations as

$$\gamma(z) = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C}), z \in \mathbb{C} \cup \{\infty\},$$

where $\mathbb{P}^1(\mathbb{C})$ is viewed as the Riemann sphere $\mathbb{C} \cup \{\infty\} \approx S^2$ as the boundary (at infinity) of (the compactification of) the real 3-dimensional hyperbolic space $\mathbb{H}^3 \subset \mathbb{R}^3$ with the Riemann metric, and $PSL_2(\mathbb{C})$ is isomorphic to the group $\text{Iso}(\mathbb{H}^3)$ of isometries of \mathbb{H}^3 .

◇ Recall from [158] that

$$S^2 \approx S^3/S^1 \approx P^1(\mathbb{C}) = (\mathbb{C}^2 \setminus \{0_2\})/\sim,$$

where $z \sim w$ if and only if $z = \lambda w$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Also, $\mathbb{H}^3 \approx PSL_2(\mathbb{C})/SU(2)$. As well, $SL_2(\mathbb{C}) \approx SU(2) \times \mathbb{R}^3$. \square

An element $\gamma \in PSL_2(\mathbb{C})$ with $\gamma \neq 1$ is said to be either parabolic, loxodromic, or elliptic, respectively, if it has either 1, 2, or infinitely many fixed points in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ as the compactification as the real 3-dimensional closed ball B_3 . Only in the elliptic case, there are fixed points inside \mathbb{H}^3 . In the other cases, fixed points are on $\mathbb{P}^1(\mathbb{C})$ as the 2-sphere at infinity.

Remark. As well, as in [116], a discrete subgroup of $PSL_2(\mathbb{C})$ is said to be a **Kleinian** group. Any (matrix) element of $PSL_2(\mathbb{C})$, not equal to the unit, is said to be either elliptic, parabolic, or hyperbolic (or **loxodromic**) if the squared trace of the matrix is less than, equal to, or more than 4, respectively. A hyperbolic element has no fixed points in \mathbb{H}^3 and has only one geodesic preserved with two end points fixed in \mathbb{C}^+ as the boundary of the compactification of \mathbb{H}^3 . A parabolic element has no fixed points in \mathbb{H}^3 and has no geodesics preserved, with only one fixed point of \mathbb{C}^+ . A elliptic element has only one geodesic fixed pointwise, and it is a torsion element. A torsion free Kleinian group G acts freely on \mathbb{H}^3 , so that \mathbb{H}^3/G is defined as a real 3-dimensional hyperbolic manifold. There is a finitely generated Kleinian group with torsion, and there is a finite index subgroup of $PSL_2(\mathbb{C})$ without torsion, known as the **Selberg** lemma.

For a Kleinian group G , the limit set Λ_G is defined to be the closure of all element $x \in \mathbb{C}^+$ such that there is a non elliptic element $g \in G$ such that $gx = x$. The complement $\Omega_G = \mathbb{C}^+ \setminus \Lambda_G$ is said to be the region of discontinuity for G . The group G acts properly discontinuously on $\mathbb{H}^3 \cup \Omega_G$. If G is finitely generated, the quotient Ω_G/G becomes a Riemann surface, by **Ahlfors**, and $(\mathbb{H}^3 \cup \Omega_G)/G$ said to be a **Kleinian** manifold ◀

For a Schottky group Γ , the **limit** set Λ_Γ of the action of Γ on $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ is defined to be the smallest non-empty closed Γ -invariant subset of $\mathbb{P}^1(\mathbb{C})$. This limit set can also be described as the closure of the set of the attractive and repelling fixed points $z^\pm(\gamma)$ of the loxodromic elements $\gamma \in \Gamma$.

In the case of rank $h = 1$, the limit set Λ_Γ consists of two points. For rank $h \geq 1$, the limit set is usually a fractal of some Hausdorff dimension as $0 \leq \delta = \dim_H(\Lambda_\Gamma) < 2$.

Remark. Recall from [116] the following. A map f from a metric space (X, d) to itself is said to be a **contraction** map if it has the contractive ratio:

$$\sup_{x \neq y \in X} \frac{d(f(x), f(y))}{d(x, y)} \equiv \text{cr}(f) < 1.$$

For contraction maps f_1, \dots, f_n on a complete metric space X , there uniquely exists a non-empty compact subset K of X such that $K = \cup_{j=1}^n f_j(K)$. This K is said to be the **self-similar** set with respect to the contraction maps f_1, \dots, f_n , as a **fractal**.

For example, let $X = [0, 1]$ and $f_1(x) = \frac{x}{3}$ and $f_2(x) = \frac{x+2}{3}$. For instance,

$$\frac{|f_1(x) - f_1(y)|}{|x - y|} = \frac{1}{3} = \frac{|f_2(x) - f_2(y)|}{|x - y|}.$$

Thus $\text{cr}(f_1) = \frac{1}{3} = \text{cr}(f_2)$. As well, $f_1(X) = [0, \frac{1}{3}]$ and $f_2(X) = [\frac{2}{3}, 1]$, so that $f_1(X) \cup f_2(X) \subsetneq X$. Then

$$\begin{aligned} f_1(f_1(X) \cup f_2(X)) &= [0, \frac{1}{3^2}] \cup [\frac{2}{3^2}, \frac{1}{3}], \\ f_2(f_1(X) \cup f_2(X)) &= [\frac{2}{3}, \frac{2}{3} + \frac{1}{3^2}] \cup [\frac{2}{3} + \frac{2}{3^2}, 1]. \end{aligned}$$

Repeating inductively this process, the self-similar set with respect to these f_1 and f_2 is obtained as the standard (or ternary) Cantor set \mathfrak{C} in $[0, 1]$.

If $X = \mathbb{R}^n$ and f_j for $1 \leq j \leq m$ are similar, contraction maps, and if the open set condition holds as that there is a non-empty open subset U of \mathbb{R}^n such that $\cup_{j=1}^m f_j(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for $i \neq j$, then the **Hausdorff** dimension $\dim_H K$ of K is obtained as

$$\sum_{j=1}^m \text{cr}(f_j)^{\dim_H K} = 1.$$

Consequently, it then follows that $\frac{2}{3^{\dim_H \mathfrak{C}}} = 1$. Therefore, $\dim_H \mathfrak{C} = \frac{\log 2}{\log 3}$. \blacktriangleleft

No figure provided, as taken from Indra's Pearls, by Mumford, Series, and Wright [126].

Denote by Ω_Γ the domain of discontinuity of a Schottky group Γ of rank h , which is the complement of the limit set Λ_Γ in $\mathbb{P}^1(\mathbb{C})$. The quotient $\Gamma \backslash \Omega_\Gamma$ becomes a Riemann surface $X(\mathbb{C})$ of genus h . The covering $\Omega_\Gamma \rightarrow X(\mathbb{C})$ is said to be a **Schottky** uniformization of $X(\mathbb{C})$.

Every compact Riemann surface admits a Schottky uniformization.

Let γ_j for $1 \leq j \leq h$ be generators of a Schottky group Γ of rank h . Set $\gamma_{j+h} = \gamma_j^{-1}$. Then there are $2h$ Jordan curves C_k as a circle on the 2-sphere at infinity as $\mathbb{P}^1(\mathbb{C})$, with pairwise disjoint interiors D_k as an open disk, such that there are elements γ'_k that are given by fractional linear transformations that map the interior D_k of C_k to the exterior of C_j with $|k - j| = h$. The curves C_k give a marking of Γ . The markings are circles in the case of classical Schottky groups. A fundamental domain for the action of a classical Schottky group Γ on $\mathbb{P}^1(\mathbb{C})$ is the region exterior to $2h$ circles.

◊ For a compact Riemann surface X of genus g , its Schottky uniformization as the fundamental domain looks like the 2-sphere with $2g$ holes as open disks, each pair of which is attached along the boundary to make X . (No figure again.) \square

Schottky and Fuchsian. It is noticed that the Fuchsian uniformization for X a compact Riemann surface is the covering $\mathbb{H}^2 \rightarrow X$ as the universal cover, while the Schottky uniformization is the covering $\Omega_\Gamma = \Lambda_\Gamma^c \rightarrow X$, which is far from being simply connected, since it in fact is the complement of a Cantor set.

The relation between Fuchsian and Schottky uniformizations is given by passing to the covering that corresponds to the normal subgroup $\frac{\pi_1(X)}{2}$ of $\pi_1(X)$ generated by half the generators a_1, \dots, a_g , so that

$$\Gamma \cong \pi_1(X) / \frac{\pi_1(X)}{2},$$

with a corresponding covering map

$$\begin{array}{ccc}
 \mathbb{H}^2 \text{ as Fuchsian} & \xrightarrow{J} & \Omega_\Gamma \text{ as Schottky} \\
 \pi_G \downarrow & & \downarrow \pi_\Gamma \\
 X = G \backslash \mathbb{H}^2 = \pi_1(X) \backslash \mathbb{H}^2 & \xlongequal{\quad} & X = \Gamma \backslash \Omega_\Gamma = (\pi_1(X) / \frac{\pi_1(X)}{2}) \backslash \Omega_\Gamma.
 \end{array}$$

At the level of moduli, there is a corresponding map between the Teichmüller space \mathfrak{T}_g and the Schottky space \mathfrak{S}_g , which depends on $3g - 3$ complex moduli.

Remark. Recall from [116] the following. An analytically finite Riemann surface of type (g, h) is defined to be a closed Riemann surface X of genus g , with h holes, obtained by removing h points from X . We may except the cases where $(g, h) = (0, n)$ for $0 \leq n \leq 3$ or $(g, h) = (1, 0)$, for which its moduli space is either trivial or classically known. Then on such X , a hyperbolic metric is defined, so that there is a Fuchsian group Γ such that $X = \mathbb{H}^2/\Gamma$. The **Teichmüller** space $\mathfrak{T}(X)$ for X of type (g, h) is defined to be equivalence classes of pairs (R, f) of Riemann surfaces R of type (g, h) and quasi-conformal mappings $f : X \rightarrow R$, where (R_1, f_1) and (R_2, f_2) are equivalent if $f_2 \circ f_1^{-1}$ is homotopic to some conformal mapping $l : R_1 \rightarrow R_2$.

$$\begin{array}{ccc}
 X & \begin{array}{c} \xrightarrow{f_1} \\ \xleftarrow{f_1^{-1}} \end{array} & R_1 \\
 f_2 \downarrow & & \downarrow l \\
 R_2 & \xlongequal{\quad} & R_2
 \end{array}$$

As well, the **mapping** class group $Mc(X)$ for X of type (g, h) is defined to the group of homotopy classes of quasi-conformal mappings from X to X . The group $Mc(X)$ acts on $\mathfrak{T}(X)$ properly discontinuously, as $[R, f] \mapsto [R, f \circ k^{-1}]$ for $[k] \in Mc(X)$. The quotient space $\mathfrak{T}(X)/Mc(X)$ becomes the moduli space $M_{g,h}$ of equivalence classes of conformal mappings of Riemann surfaces of type (g, h) . ◀

Surface with boundary as simultaneous uniformization. To visualize geometrically the Schottky uniformization of a compact Riemann surface, it may be related to a simultaneous uniformization of the upper and lower half planes, that yields two Riemann surfaces with boundary, joined at the boundary.

A Schottky group Γ that is specified by real parameters so that it is contained in $PSL_2(\mathbb{R})$ is said to be a **Fuchsian** Schottky group. (No figure.) Viewed as a group of isometries of the hyperbolic plane \mathbb{H}^2 , or equivalently of the Poincaré disk, a Fuchsian Schottky group Γ as G produces the quotient $G \backslash \mathbb{H}^2$, topologically as a Riemann surface with boundary.

◊ Namely, such a Fuchsian Schottky simultaneous uniformization looks like

that

$$\begin{array}{ccc}
\mathbb{H}^+ \cup \mathbb{H}^- & \xrightarrow{J} & \Omega_\Gamma^+ \cup \Omega_\Gamma^- \\
\pi_{G=\Gamma} \downarrow & & \downarrow \Gamma \\
G \backslash (\mathbb{H}^+ \cup \mathbb{H}^-) = X_1 \#_{\partial X_1 = \partial X_2} X_2 & \xlongequal{\quad} & \Gamma \backslash (\Omega_\Gamma^+ \cup \Omega_\Gamma^-) = X. \quad \square
\end{array}$$

A Jordan curve C (as a circle) in $\mathbb{P}^1(\mathbb{C}) \approx S^2$ is said to be a **quasi-circle** for Γ if C is invariant under the action of Γ . In particular, such a curve contains the limit set Λ_Γ . It is proved by Bowen that if $X = X(\mathbb{C})$ is a Riemann surface of genus ≥ 2 with Schottky uniformization, then there always exists a quasi-circle for Γ .

Then the complement $\mathbb{P}^1(\mathbb{C}) \setminus C$ is divided into two regions as $\Omega_1 \cup \Omega_2$. For $\pi_\Gamma : \Omega_\Gamma \rightarrow X(\mathbb{C})$ the covering map, let $C^\wedge = \pi_\Gamma(C \cap \Omega_\Gamma)$ in $X(\mathbb{C})$, which is a set of curves on $X(\mathbb{C})$ that disconnect the Riemann surface in the union of two surfaces with boundary, uniformized respectively by Ω_1 and Ω_2 .

There exist conformal maps $\alpha_j : \Omega_j \rightarrow U_j (\approx)$ for $j = 1, 2$, such that $U_1 \cap U_2 = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \approx S^2 \setminus S^1$, with $U_1 \approx \mathbb{H}^+$ and $U_2 \approx \mathbb{H}^-$ in $\mathbb{P}^1(\mathbb{C})$. Moreover, define $G_j = \alpha_j \Gamma \alpha_j^{-1}$, where $\Gamma \sim$ in $SL_2(\mathbb{R})$ is the Γ -stabilizer of each of the two connected components of $\mathbb{P}^1(\mathbb{C}) \setminus C$. Then each $G_j \cong \Gamma$ and $G_j \subset PSL_2(\mathbb{R})$, so that they are Fuchsian Schottky groups.

Let $X_j = U_j/G_j$, which are Riemann surfaces with boundary C^\wedge . The compact Riemann surface $X(\mathbb{C})$ is then obtained as

$$X(\mathbb{C}) = X_1 \#_{\partial X_1 = C^\wedge = \partial X_2} X_2$$

as a connected sum of X_1 and X_2 along the boundary.

In the case where $X(\mathbb{C})$ has a real structure as an involution $\iota : X \rightarrow X$, and the fixed point set $\text{Fx}(\iota)$ of ι as $X(\mathbb{R})$ is non-empty, it in fact holds that $C^\wedge = X(\mathbb{R})$, and the quasi-circle is given by $\mathbb{P}^1(\mathbb{R})$.

Note that in the case of a Fuchsian Schottky group, the Hausdorff dimension $\dim_H \Lambda_\Gamma$ of the limit set Λ_Γ is in fact bounded above by 1, since Λ_Γ is contained in the rectifiable quasi-circle $\mathbb{P}^1(\mathbb{R})$.

Hyperbolic handle-bodies. The action of a rank h Schottky group Γ in $PSL_2(\mathbb{C})$ acting on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations extends to an action by isometries on the real 3-dimensional hyperbolic space \mathbb{H}^3 . For a classical Schottky group, a fundamental domain in \mathbb{H}^3 is given by the region (as an open connected set) external to $2h$ half spheres over the circles (or disks) in $\mathbb{P}^1(\mathbb{C})$ (No figure).

◇ The region (with boundary) looks like a 3-dimensional half cut ball with (smaller) $2h$ half cut balls removed from the cut face. Namely, a sort of ice cream half cut ball after (smaller) $2h$ ice cream half cut balls are removed from the top separately. \square

The quotient space $\mathbb{H}^3/\Gamma = Y_\Gamma$ becomes a **handle-body** of genus h filling (the inside of) the Riemann surface $X(\mathbb{C})$ as a topological space.

The handle-body Y_Γ (with boundary) (as an infinite connected sum of, each which of components, makes a doughnut with handles by the quotient) is a real

hyperbolic 3-manifold of infinite volume, as a metric space, having $X(\mathbb{C})$ as its conformal boundary ∂Y_Γ at infinity.

Denote by \overline{Y}_Γ the compactification of Y_Γ , obtained by adding the conformal boundary at infinity, so that

$$\overline{Y}_\Gamma = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma.$$

In the case of genus zero, we have the 2-sphere $\mathbb{P}^1(\mathbb{C})$ as the conformal boundary at infinity of \mathbb{H}^3 , attached to make the unit ball in the Poincaré model, as

$$\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C}) \approx B^3.$$

In the case of genus one, we have a(n open) solid torus $\mathbb{H}^3/q^\mathbb{Z}$, for $q \in \mathbb{C}^*$ with $|q| < 1$ acting as

$$q(z, y) = (qz, |q|y) \in \mathbb{C} \times (0, \infty) \approx \mathbb{H}^3$$

in the upper half space model, with conformal boundary at infinity as the Jacobi uniformized elliptic curve $\mathbb{C}^*/q^\mathbb{Z}$, attached to make the closed solid torus, as

$$(\mathbb{H}^3/q^\mathbb{Z}) \cup (\mathbb{C}^*/q^\mathbb{Z}).$$

In this case, the limit set Λ_Γ consists of two points set $\{0, \infty\}$, the domain of discontinuity is \mathbb{C}^* as the complement of Λ_Γ in $\mathbb{P}^1(\mathbb{C})$, and a fundamental domain is the annulus

$$\{z \in \mathbb{C} \mid |q| < |z| \leq 1\}$$

as the intersection of the (open) exterior of a closed disk with radius $|q|$ and the (closed) exterior of the complement of a closed disk with radius 1.

The relation of Schottky uniformization to the usual Euclidean uniformization of the complex tori as $X = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is given by $q = \exp(2\pi i\tau)$.

We have a return soon later to the case of genus one, to discuss a physical interpretation of the result of Manin on the Green function. However, for our purposes, the most interesting case is when genus is more than 1. In this case, the limit set Λ_Γ becomes a Cantor set with an interesting dynamics for the action of Γ . That is nothing but the dynamics by the Schottky group on its limit set, that does generate an interesting noncommutative space.

Geodesics in the handle-body Y_Γ . The hyperbolic handle-body Y_Γ has infinite volume, but it contains a region of finite volume, which is a deformation retract of Y_Γ . This is called the convex **core** of Y_Γ and is obtained by taking the geodesic hull of the limit set Λ_Γ in \mathbb{H}^3 and then as the quotient by Γ . Identify different classes of infinite geodesics in Y_Γ .

[**Closed** geodesics]. Since Γ is purely loxodromic, for any $\gamma \in \Gamma$, there exist two fixed points $\{z^+(\gamma), z^-(\gamma)\} \subset \mathbb{P}^1(\mathbb{C})$. The geodesics in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ with ends as two such points $\{z^\pm(\gamma)\}$, for some $\gamma \in \Gamma$, correspond to closed geodesics in the quotient $Y_\Gamma = \mathbb{H}^3/\Gamma$.

[**Bounded** geodesics]. The images in Y_Γ of geodesics in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ having both ends on the limit set Λ_Γ are geodesics that remain confined within the convex core of Y_Γ .

[**Unbounded** geodesics]. These are the geodesics in Y_Γ that eventually wander off the convex core towards $X(\mathbb{C}) = \partial Y_\Gamma$ the conformal boundary at infinity. They correspond to geodesics in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ with at least one end as a point of Ω_Γ the domain of discontinuity of Γ as Λ_Γ^c in $\mathbb{P}^1(\mathbb{C})$.

In the case of genus **one**, there is a unique primitive closed geodesic, namely the image in the quotient of the geodesic in \mathbb{H}^3 connecting 0 and ∞ . The bounded geodesics are those corresponding to geodesics in \mathbb{H}^3 originating at 0 or ∞ .

The most interesting case is that of genus more than 1, where the bounded geodesics form a complicated tangle inside Y_Γ . This is a generalized solenoid as a topological space. Namely, it is locally the product space of a line (or line segment) and a Cantor set.

◊ A **solenoid** is a usual spiral coil. This is a real line or line segment as a topological space. But it has a projection to the circle S^1 (in one direction of the coil core), with fibers a finite set as the winding number of the coil. \square

4.2 Dynamics and noncommutative geometry

Since the uniformizing group Γ is a free group, there is a simple way of obtaining a coding of the bounded geodesics in Y_Γ . The set of such geodesics can be identified with $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$, by specifying the end points in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ module the action of Γ .

Let $\{\gamma_j\}_{j=1}^g$ be a choice of generators of Γ and set $\gamma_{g+j} = \gamma_j^{-1}$ for $1 \leq j \leq g$. We introduce a **subshift** of finite type as follows. Let

$\mathfrak{S} =$

$$\{w = \{a_k\}_{k \in \mathbb{Z}} \equiv \cdots a_{-k} \cdots a_{-1} a_0 a_1 \cdots a_k \cdots \mid a_k \in \{\gamma_j\}_{j=1}^{2g}, a_{k+1} \neq a_k^{-1}, k \in \mathbb{Z}\}$$

be the set of doubly infinite words w in the generators and their inverses, with the admissibility condition, so that no cancellations occur in each word. Define the map T on \mathfrak{S} as the invertible **shift** (to the left, one by one)

$$T(\cdots a_{-2} a_{-1} a_0 a_1 a_2 \cdots) = \cdots a_{-1} a_0 a_1 a_2 a_3 \cdots$$

so that $T(w) = T(\{a_k\}_{k \in \mathbb{Z}}) = \{a_{k+1}\}_{k \in \mathbb{Z}} \in \mathfrak{S}$ as our notation.

Then the discrete **dynamical** system $(\mathfrak{S}, T, \mathbb{Z})$ defined above can be passed to its suspension flow, to obtain the **mapping** torus as

$$\mathfrak{S}_T = (\mathfrak{S} \times [0, 1]) / \sim,$$

where $(x, 0) \in \mathfrak{S} \times [0, 1]$ is identified with $(T(x), 1)$.

◊ This quotient space of the suspension flow on $\mathfrak{S} \times [0, 1]$ or $\mathfrak{S} \times \mathbb{R}$ (extended periodically) may be exactly denoted as, for instance, $I\mathfrak{S}_{IT}$ or $(S\mathfrak{S})_{ST}$. \square

That space is a (generalized) solenoid as a topological space, that is, a bundle

over S^1 with fiber a Cantor set, as

$$\begin{array}{ccc} \mathfrak{S} \times (0, 1) & \xrightarrow[\subset]{} & \mathfrak{S}_T = I\mathfrak{S}_{IT} = (S\mathfrak{S})_{ST} \longleftarrow 2^{\mathbb{Z}} \approx \Pi^\infty\{0, 1\} \\ & & \downarrow \\ & & S^1 = [0, 1]/\sim \end{array}$$

where $\mathfrak{S} \times (0, 1) \approx \mathfrak{S} \times \mathbb{R}$ is the **suspension** of \mathfrak{S} , on which T acts trivially, and $2^{\mathbb{Z}}$ is a Cantor set as a fiber.

Remark. As in [11], the **mapping** torus C^* -algebra for T is defined as

$$M_T = \{f \in C([0, 1], C(\mathfrak{S})) \mid f(1) = T(f(0))\}$$

where $T \in \text{Aut}(C(\mathfrak{S}))$ as an $*$ -automorphism of $C(\mathfrak{S})$ is identified with that homeomorphism T on \mathfrak{S} . Then there is the following short exact sequence of C^* -algebras:

$$0 \rightarrow SC(\mathfrak{S}) \xrightarrow{i} M_T \xrightarrow{q} C(\mathfrak{S}) \rightarrow 0$$

with $SC(\mathfrak{S}) \cong C_0(\mathfrak{S} \times (0, 1)) \cong C_0(\mathbb{R}) \otimes C(\mathfrak{S})$ is the **suspension** of $C(\mathfrak{S})$. Then the following six-term exact sequence is deduced:

$$\begin{array}{ccccc} K_0(SC(\mathfrak{S})) \cong K_1(C(\mathfrak{S})) & \xrightarrow{i_*} & K_0(M_T) & \xrightarrow{q_*} & K_0(C(\mathfrak{S})) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(C(\mathfrak{S})) & \xleftarrow{q^*} & K_1(M_T) & \xleftarrow{i^*} & K_1(SC(\mathfrak{S})) \cong K_0(C(\mathfrak{S})) \end{array}$$

and moreover, $K_j(M_T) \cong K_{j+1}(C(\mathfrak{S}) \rtimes_T \mathbb{Z})$ for $j = 0, 1 \pmod{2}$. This may follow from comparing with the Pimsner-Voiculescu six-term exact sequence given below and the Five (or Six) Lemma. \square

Homotopy quotient. The space \mathfrak{S}_T defined above has a natural interpretation in noncommutative geometry as the homotopy quotient in the sense of Baum-Connes [9] of the noncommutative space given by the C^* -algebra crossed product $C(\mathfrak{S}) \rtimes_T \mathbb{Z}$ describing the shift action T on the totally disconnected space \mathfrak{S} . The noncommutative space as the C^* -algebra parameterizes bounded geodesics in the handle-body Y_Γ . The homotopy quotient is given by $\mathfrak{S} \times_{\mathbb{Z}} \mathbb{R} = \mathfrak{S}_T$.

The K-theory of the C^* -algebra crossed product by \mathbb{Z} can be computed by the Pimsner-Voiculescu six-term exact sequence, with induced maps with $*$:

$$\begin{array}{ccccc} K_0(C(\mathfrak{S})) & \xrightarrow[\delta_*]{(1-T)_*} & K_0(C(\mathfrak{S})) & \xrightarrow{i_*} & K_0(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) & \xleftarrow{i_*} & K_1(C(\mathfrak{S})) & \xleftarrow[\delta_*]{(1-T)_*} & K_1(C(\mathfrak{S})) \end{array}$$

where $i : C(\mathfrak{S}) \rightarrow C(\mathfrak{S}) \rtimes_T \mathbb{Z}$ is the canonical inclusion, and $\delta = 1 - T$. Since the space \mathfrak{S} is totally disconnected, we have that $K_1(C(\mathfrak{S}))$ is zero and

$K_0(C(\mathfrak{S})) \cong C(\mathfrak{S}, \mathbb{Z})$ of locally constant, integer valued functions on \mathfrak{S} . Therefore, the following exact sequence of groups is obtained:

$$0 \rightarrow K_1(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \rightarrow C(\mathfrak{S}, \mathbb{Z}) \xrightarrow[\delta_*]{(1-T)_*} C(\mathfrak{S}, \mathbb{Z}) \rightarrow K_0(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \rightarrow 0,$$

so that

$$\begin{aligned} K_1(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) &\cong \ker(\delta_*) \cong \mathbb{Z}, \\ K_0(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) &\cong \operatorname{coker}(\delta_*) \cong C(\mathfrak{S}, \mathbb{Z})/\operatorname{im}(\delta_*) \end{aligned}$$

with $C(\mathfrak{S}, \mathbb{Z})/\ker(\delta_*) \cong \operatorname{im}(\delta_*)$. In the language of dynamical systems, the kernel and cokernel are respectively the invariant and the coinvariant of the invertible shift T (cf. [15], [131]).

In terms of the homotopy quotient, that exact sequence can be described more geometrically in terms of the Thom isomorphism and the assembly μ -map, as, for $* = 0, 1 \pmod{2}$, (added as)

$$\begin{array}{ccc} K_{*+1}(M_T) (\cong K_{*+1}(M_{T^\wedge})) & \xrightarrow{\cong} & K_*(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \\ \cong \downarrow & & \uparrow \mu \\ K_{*+1}(C(\mathfrak{S}_T)) \cong K^{*+1}(\mathfrak{S}_T) & \xrightarrow[\operatorname{Ch}]{\cong} & H^{*+1}(\mathfrak{S}_T, \mathbb{Z}) \end{array}$$

where the Connes' Thom isomorphism (or the Takai duality involving the crossed product C^* -algebra by the dual action T^\wedge to T) may be used in the non-trivial isomorphism in the first horizontal line, and the map Ch in the second horizontal line is the Chern character, which is also a non-trivial isomorphism because of being torsion free of the K-theory groups (cf. [11]), and the assembly map in this case means the non-trivial isomorphism between the topological K-theory for spaces such as $K^{*+1}(\mathfrak{S}_T)$ and the K-theory for C^* -algebras as $K_*(C(\mathfrak{S}) \rtimes_T \mathbb{Z})$.

◇ Indeed, the **Takai** duality implies that

$$(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \rtimes_{T^\wedge} \mathbb{T} \cong C(\mathfrak{S}) \otimes \mathbb{K}.$$

It says that the dual crossed product of, the crossed product of a C^* -algebra \mathfrak{A} by an action of an Abelian locally compact group G , by the dual action of the dual group G^\wedge is stably isomorphic to the C^* -algebra \mathfrak{A} . On the other hand, with $\mathbb{Z} \subset \mathbb{R}$ acting trivially, with $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$,

$$(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \rtimes_{T^\wedge} \mathbb{R} \cong M_{T^\wedge},$$

where this mapping torus C^* -algebra M_{T^\wedge} is that of T^\wedge on $(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \rtimes_{T^\wedge} \mathbb{T}$ (cf. [11]). The Connes' Thom isomorphism for crossed product C^* -algebras by \mathbb{R} (to be assumed trivial in K-theory) implies that $K_{*+1}(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \cong K_*(M_{T^\wedge})$.

It seems that the left vertical isomorphism is non-trivial as well. It may follow from the same K-theory group diagram as for M_{T^\wedge} noncommutative as well as $C(\mathfrak{S}_T) = C((S\mathfrak{S})_{S_T})$ commutative, with bridge by the Five (or Six) Lemma. \square

Consequently, we have

$$K_1(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \cong H^0(\mathfrak{S}_T) \cong \mathbb{Z} \quad \text{and} \quad K_0(C(\mathfrak{S}) \rtimes_T \mathbb{Z}) \cong H^1(\mathfrak{S}_T).$$

The first (Alexander or Čech) cohomology group $H^1(\mathfrak{S}_T)$ can be identified with the homotopy group $[\mathfrak{S}_T, U(1)]$ given by the homotopy classes of mappings $f \mapsto \exp(2\pi i t f(x))$ for $f \in C(\mathfrak{S}, \mathbb{Z})/\delta(C(\mathfrak{S}, \mathbb{Z}))$ (corrected).

Filtration. The identification of $H^1(\mathfrak{S}_T, \mathbb{Z})$ the cohomology group of \mathfrak{S}_T with the K-theory group $K_0(C(\mathfrak{S}) \rtimes_T \mathbb{Z})$ of the crossed product C^* -algebra for the action T on \mathfrak{S} endows $H^1(\mathfrak{S}_T, \mathbb{Z})$ with a filtration as follows.

Theorem 4.1. *The first cohomology of \mathfrak{S}_T is a direct limit $\varinjlim F_n$ of free abelian groups F_n of ranks $\text{rank}(F_0) = 2g$ and $\text{rank}(F_n) = 2g(2g-1)^{n-1}(2g-2)$ (corrected) for $n \geq 1$, with $F_n \subset F_{n+1}$ for $n \geq 0$.*

In fact, it follows from the Pimsner-Voiculescu six-term exact sequence that the abelian group $K_0(C(\mathfrak{S}) \rtimes_T \mathbb{Z})$ can be identified with the cokernel of the map $1 - T$ acting as $f \mapsto f - f \circ T$ on the \mathbb{Z} -module $C(\mathfrak{S}, \mathbb{Z}) \cong K_0(C(\mathfrak{S}))$. Then the filtration is given by the submodules of functions depending only on the $a_0 \cdots a_n$ coordinates in the doubly infinite words describing points in \mathfrak{S} . Namely, it then follows that

$$H^1(\mathfrak{S}_T, \mathbb{Z}) \cong C(\mathfrak{S}, \mathbb{T})/\delta(C(\mathfrak{S}, \mathbb{Z})) = \mathcal{P}/\delta\mathcal{P},$$

where \mathcal{P} denotes the \mathbb{Z} -module of locally constant \mathbb{Z} -valued functions that depend only on future coordinates. These functions can be identified with functions on the limit set Λ_Γ , since each point in Λ_Γ is described by an infinite to the right, admissible sequence in the generators γ_j and their inverses. Thus, $\mathcal{P} \cong C(\Lambda_\Gamma, \mathbb{Z})$.

The module \mathcal{P} has a filtration by the submodules \mathcal{P}_n of functions of the first $n+1$ coordinates. Then $\text{rank}(\mathcal{P}_n) = 2g(2g-1)^n$.

◊ The \mathbb{Z} -module \mathcal{P}_0 has a basis corresponding to the set of generators and inverses of Γ . The \mathbb{Z} -module \mathcal{P}_1 has a basis corresponding to the set of words of \mathfrak{S} of length two. \square

Set $F_0 = \mathcal{P}_0$ and $F_n = \mathcal{P}_n/\delta\mathcal{P}_{n-1}$ for $n \geq 1$. It can be shown that there are induced injections from F_n to F_{n+1} and that $H^1(\mathfrak{S}_T) \cong \varinjlim F_n$.

◊ If $f = \chi_{\gamma_j}$ the characteristic function at γ_j (or on all the words starting from γ_j), then

$$\delta f = \chi_{\gamma_j} - \chi_{\gamma_j} \circ T = \begin{cases} 1 & \text{at } \gamma_j \\ -1 & \text{at } T^{-1}\gamma_j \\ 0 & \text{otherwise} \end{cases}$$

which is a function at $\gamma_j \gamma_j = \gamma_j(T^{-1}\gamma_j)$ as two points (as $a_0 a_1$).

If $a_0 a_1$ is the part of the corresponding admissible word of \mathfrak{S} , then it may correspond to the function $f_{a_0 a_1} = \chi_{a_0} - \chi_{a_1} \circ T$. Then

$$\delta(f_{a_0 a_1}) = f_{a_0 a_1} - f_{a_0 a_1} \circ T = \chi_{a_0} - \chi_{a_0} \circ T - \chi_{a_1} \circ T + \chi_{a_1} \circ T^2,$$

which should be defined as $f_{a_0 a_1 a_1}$.

If $a_0 a_1 a_2$ is the part of the corresponding admissible word of \mathfrak{S} , then it may correspond to the function

$$\chi_{a_0} - \chi_{a_0} \circ T - \chi_{a_1} \circ T - \chi_{a_2} \circ T^2. \quad \square$$

Moreover, we have $\text{rank}(F_n) = \theta_n - \theta_{n-1}$ (corrected), where θ_n is the number of admissible words of length $n + 1$. All the \mathbb{Z} -module F_n and the quotients F_n/F_{n-1} are torsion free (cf. [131]).

◊ For instance,

$$\text{rank}(F_1) = \text{rank}(\mathcal{P}_1) - \text{rank}(\mathcal{P}_0) = 2g(2g - 1) - 2g = 2g(2g - 2).$$

As well,

$$\text{rank}(F_2) = \text{rank}(\mathcal{P}_2) - \text{rank}(\mathcal{P}_1) = 2g(2g - 1)^2 - 2g(2g - 1) = 2g(2g - 1)(2g - 2).$$

Similarly,

$$\text{rank}(F_3) = 2g(2g - 1)^3 - 2g(2g - 1)^2 = 2g(2g - 1)^2(2g - 2). \quad \square$$

Hilbert space and grading. It is convenient to consider the complex vector space defined as $\mathcal{P}_{\mathbb{C}} = C(\Lambda_{\Gamma}, \mathbb{Z}) \otimes \mathbb{C}$ and the corresponding exact sequence computing the cohomology with \mathbb{C} as coefficients, as

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{P}_{\mathbb{C}} \xrightarrow{\delta_*} \mathcal{P}_{\mathbb{C}} \rightarrow H^1(\mathfrak{S}_T, \mathbb{C}) \rightarrow 0$$

obtained by tensoring with \mathbb{C} of that sequence for $C(\mathfrak{S}, \mathbb{Z})$ identified with \mathcal{P} .

The complex vector space $\mathcal{P}_{\mathbb{C}}$ may be contained in the complex Hilbert space $L^2 = L^2(\Lambda_{\Gamma}, d\mu)$, where μ is the Patterson-Sullivan measure on the limit set Λ_{Γ} , satisfying

$$\gamma^* d\mu = |\gamma'|^{\dim_H \Lambda_{\Gamma}} d\mu, \quad \text{with } \dim_H \Lambda_{\Gamma} \text{ the Hausdorff dimension.}$$

This defines such a Hilbert space L^2 , together with a filtration by finite dimensional subspaces $\mathcal{P}_n \otimes \mathbb{C}$. In this setting, it is natural to consider a corresponding **grading** operator, defined as an infinite direct sum

$$D = \sum_{n=0}^{\infty} n \Pi_n^{\wedge} = \oplus_n n \Pi_n^{\wedge}, \quad \text{on } \oplus_{n=0}^{\infty} (\mathcal{P}_{n, \mathbb{C}} \cap \mathcal{P}_{n-1, \mathbb{C}}^{\perp}) = L^2,$$

as the infinite direct sum Hilbert space, where Π_n denotes the orthogonal projection from L^2 onto $\mathcal{P}_n \otimes \mathbb{C} = \mathcal{P}_{n, \mathbb{C}}$ and $\Pi_n^{\wedge} = \Pi_n - \Pi_{n-1}$, with $\mathcal{P}_{n-1, \mathbb{C}}^{\perp}$ the orthogonal complement of $\mathcal{P}_{n-1, \mathbb{C}}$ in L^2 , and $\Pi_{-1} = 0$ and $\mathcal{P}_{-1, \mathbb{C}} = \{0\}$.

The Cuntz-Krieger algebra. There is a noncommutative space that encodes the dynamics of the Schottky group Γ on its limit set Λ_{Γ} . Consider the $2g \times 2g$ matrix $A = (a_{ij})$ with entries of $\{0, 1\}$ such that $a_{ij} = 1$ for $|i - j| \neq g$ and $a_{ij} = 0$ otherwise. The **adjacency** (or admissibility, ad) matrix A corresponds to the admissibility condition for sequences in \mathfrak{S} .

◇ For instance, if $g = 2$, then

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \gamma_1 & \gamma_1 & \emptyset & \gamma_1 \\ \gamma_2 & \gamma_2 & \gamma_2 & \emptyset \\ \emptyset & \gamma_1^{-1} & \gamma_1^{-1} & \gamma_1^{-1} \\ \gamma_2^{-1} & \emptyset & \gamma_2^{-1} & \gamma_2^{-1} \end{pmatrix}$$

where the columns from the left to the right correspond to the admissible one-words in the right of $\gamma_1, \gamma_2, \gamma_1^{-1}$, and γ_2^{-1} , respectively. \square

The **Cuntz-Krieger** (CK) C^* -algebra \mathcal{O}_A associated to the ad matrix A is defined to be the universal unital C^* -algebra generated by (mutually orthogonal) partial isometries s_j for $1 \leq j \leq 2g$ satisfying the (CK) relations

$$\sum_{j=1}^{2g} s_j s_j^* = 1 \quad \text{and} \quad s_k^* s_k = \sum_{j=1}^{2g} a_{kj} s_j s_j^*.$$

◇ The first equation says that the range projections of s_j add up to the unit. The second says that the initial (or domain) projection of each s_k is such a sum of the range projections of s_j . \square

Recall that a partial isometry is an operator s satisfying $s = ss^*s$.

◇ A **partial** isometry s on a Hilbert space H is defined to be an isometry on $\ker(s)^\perp$ the orthogonal complement to the kernel $\ker(s)$ a closed subspace of H (cf. [83]). Let $p_{\ker(s)^\perp}$ be the orthogonal projection from H onto $\ker(s)^\perp$. For $\xi = x \oplus y \in \ker(s) \oplus \ker(s)^\perp = H$,

$$\langle s^* s \xi, \xi \rangle = \langle sy, sy \rangle = \|sy\|^2 = \|y\|^2 = \langle y, y \rangle = \langle p_{\ker(s)^\perp} \xi, p_{\ker(s)^\perp} \xi \rangle.$$

It then follows that $s^* s = p_{\ker(s)^\perp}$. Moreover, for $\xi = x \oplus y \in H$ as above,

$$ss^* s \xi = s(s^* s \xi) = sy = s \xi.$$

Hence, $s = ss^*s$.

Conversely, if $s^* s$ is a projection $p = p^2 = p^*$, then $\ker(s) = \ker(p)$, so that s is an isometry on $\ker(s)^\perp$ equal to the range of p .

As well, suppose that $s = ss^*s$. Then $(s^* s)^2 = s^*(ss^*s) = s^*s = (s^* s)^*$. \square

The Cuntz-Krieger C^* -algebra is related to the Schottky group by the following:

Proposition 4.2. *There is a C^* -algebra isomorphism*

$$\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma.$$

Up to the stabilization by tensoring with the C^* -algebra \mathbb{K} of compact operators, there is another crossed product description as

$$\mathcal{O}_A \otimes \mathbb{K} \cong \mathfrak{F}_A \rtimes_T \mathbb{Z},$$

where \mathfrak{F}_A is an AF-algebra, defined as an inductive limit of finite dimensional C^* -algebras.

◊ It should be a nice question how to prove those isomorphisms above. The proof is done below, as faithfully represented. \square

Consider the cochain complex of Hilbert spaces as the following bottom line to up line completion diagram (edited):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & L^2(\Lambda_\Gamma) & \xrightarrow{\delta_*} & L^2(\Lambda_\Gamma) & \longrightarrow & \overline{H^1(\mathfrak{S}_T, \mathbb{C})} & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{P}_{\mathbb{C}} & \xrightarrow{\delta_*} & \mathcal{P}_{\mathbb{C}} & \longrightarrow & H^1(\mathfrak{S}_T, \mathbb{C}) & \longrightarrow & 0
\end{array}$$

where $\mathcal{P}_{\mathbb{C}} = \mathcal{P} \otimes \mathbb{C} = C(\Lambda_\Gamma, \mathbb{Z}) \otimes \mathbb{C}$.

◊ Note that the diagram follows from the density of the complex vector space $\mathcal{P}_{\mathbb{C}}$ in the L^2 -space. The quotient $\overline{H^1}$ of L^2 by $\delta_* L^2$ means the L^2 -closure of H^1 . \square

Proposition 4.3. *The Cuntz-Krieger C^* -algebra \mathcal{O}_A has a faithful representation on the Hilbert space $L^2 = L^2(\Lambda_\Gamma, d\mu)$.*

The proof is as follows. Let $d_H = \dim_H(\Lambda_\Gamma)$ the Hausdorff dimension. Consider the operators on L^2 defined as, for $1 \leq j \leq 2g$,

$$P_j f = \chi_{\gamma_j} f \quad \text{and} \quad T_j f = |(\gamma_j^{-1})'|^{d_H/2} f \circ \gamma_j^{-1},$$

where $\{\gamma_j\}_{j=1}^{2g}$ are the generators of Γ and their inverses. For any $\gamma \in \Gamma$, define

$$T_\gamma f = |\gamma'|^{d_H/2} f \circ \gamma.$$

Define $S_i = \sum_{j=1}^{2g} a_{ij} T_i^* P_j$ for $1 \leq i \leq 2g$. Then S_i are partial isometries on L^2 satisfying the Cuntz-Krieger relations for the matrix A as the subshift T on \mathfrak{S} of finite type. This extends to the representation of \mathcal{O}_A on L^2 .

◊ Note that $P_j P_k = 0$ if $j \neq k$ and $= P_j$ if $j = k$. Also, for $f_1, f_2 \in L^2$,

$$\begin{aligned}
\langle T_j^* f_1, f_2 \rangle &= \langle f_1, T_j f_2 \rangle = \sum_{\gamma \in \Lambda_\Gamma} f_1(\gamma) \overline{|(\gamma_j^{-1})'|^{d_H/2} f_2(\gamma_j^{-1} \gamma)} \\
&= \sum_{\eta = \gamma_j^{-1} \gamma \in \Lambda_\Gamma} \overline{|(\gamma_j^{-1})'|^{d_H/2} f_1(\gamma_j \eta) f_2(\eta)}
\end{aligned}$$

(with some weight in the inner product?). Check that

$$\begin{aligned}
\sum_{i=1}^{2g} S_i S_i^* &= \sum_{i=1}^{2g} \left[\sum_{j=1}^{2g} a_{ij} T_i^* P_j \sum_{k=1}^{2g} a_{ik} P_k T_i \right] \\
&= \sum_{i=1}^{2g} \left[\sum_{j=1}^{2g} \sum_{k=1}^{2g} a_{ij} a_{ik} T_i^* P_j P_k T_i \right] = \sum_{i=1}^{2g} \left[\sum_{j=1}^{2g} a_{ij} T_i^* P_j T_i \right].
\end{aligned}$$

Moreover, compute that for $f \in L^2$,

$$\begin{aligned} T_i^* P_j T_i f &= T_i^*(P_j T_i f) = \overline{|\gamma_i^{-1}'|^{d_H/2}} (P_j T_i f) \circ \gamma_i \\ &= \overline{|\gamma_i^{-1}'|^{d_H/2}} (\chi_{\gamma_j} \circ \gamma_i)(T_i f) \circ \gamma_i \\ &= \overline{|\gamma_i^{-1}'|^{d_H/2}} |\gamma_i^{-1}'|^{d_H/2} (\chi_{\gamma_j} \circ \gamma_i) f. \end{aligned}$$

There may be more reason for this to be completed. Indeed, possibly, in the definition of T_j , the factor may be replaced with $|(\gamma_j^{-1})'|^{d_H/2}$, so that

$$\overline{|\gamma_j^{-1}'|^{d_H/2}} |\gamma_j^{-1}'|^{d_H/2} = \exp(-i \frac{d_H}{2} \log |(\gamma_j^{-1})'|) \exp(i \frac{d_H}{2} \log |(\gamma_j^{-1})'|) = 1.$$

Note as well that the χ_{γ_j} should mean the characteristic function on all the words starting from γ_j . It then follows that

$$\sum_{i=1}^{2g} S_i S_i^* = \sum_{i=1}^{2g} \sum_{j=1}^{2g} a_{ij} (\chi_{\gamma_j} \circ \gamma_i) = \sum_{j=1}^{2g} a_{jj} (\chi_{\gamma_j} \circ \gamma_j) = \sum_{j=1}^{2g} (\chi_{\gamma_j} \circ \gamma_j) = 1.$$

Note that the operation $\circ \gamma_j$ should be the projection to the set of all the words starting from γ_j .

Check also that $S_k^* S_k = \sum_{j=1}^{2g} a_{kj} S_j S_j^*$. Indeed,

$$S_k^* S_k = \sum_{i=1}^{2g} a_{ki} P_i T_k \sum_{l=1}^{2g} a_{kl} T_k^* P_l = \sum_{i=1}^{2g} \sum_{l=1}^{2g} a_{ki} a_{kl} P_i T_k T_k^* P_l$$

Moreover, compute that for any $f \in L^2$ (corrected),

$$\begin{aligned} P_i T_k T_k^* P_l f &= \chi_{\gamma_i} T_k T_k^* P_l f = \chi_{\gamma_i} |(\gamma_k^{-1})'|^{d_H/2} T_k^* P_l f \circ \gamma_k^{-1} \\ &= \chi_{\gamma_i} |(\gamma_k^{-1})'|^{d_H/2} \overline{|\gamma_k^{-1}'|^{d_H/2}} P_l f \circ \gamma_k \circ \gamma_k^{-1} = \chi_{\gamma_i} \chi_{\gamma_l} f = P_i P_l f. \end{aligned}$$

Therefore,

$$S_k^* S_k = \sum_{i=1}^{2g} \sum_{l=1}^{2g} a_{ki} a_{kl} P_i P_l = \sum_{i=1}^{2g} a_{ki} P_i = \sum_{i=1}^{2g} a_{ki} S_i S_i^*,$$

with

$$S_i S_i^* f = \sum_{j=1}^{2g} a_{ij} T_i^* P_j T_i f = \sum_{j=1}^{2g} a_{ij} (\chi_{\gamma_j} \circ \gamma_i) f = a_{ii} (\chi_{\gamma_i} \circ \gamma_i) f = P_i f$$

with $a_{ii} = 1$, and $\chi_{\gamma_i} \circ \gamma_i = \chi_{\gamma_i}$. Note that the operation $\circ \gamma_i$ should be the projection to the set of all the words starting from γ_i . \square

The spectral triple for Schottky groups. On the direct sum $H = L^2 \oplus L^2$ of the Hilbert space $L^2 = L^2(\Lambda_\Gamma)$, consider the diagonal action of the CK algebra

\mathcal{O}_A by the representation obtained above and also the Dirac operator D defined as $D = D_{L^2 \oplus 0} \otimes D_{0 \oplus L^2}$:

$$D = \begin{pmatrix} 0 & D_{0 \oplus L^2} \\ D_{L^2 \oplus 0} & 0 \end{pmatrix}, \quad D_{L^2 \oplus 0} = \sum_{n=1}^{\infty} n \Pi_n^\wedge, \quad D_{0 \oplus L^2} = - \sum_{n=1}^{\infty} n \Pi_n^\wedge$$

(corrected and improved), where each Π_n is the orthogonal projection with respect to $P_{n, \mathbb{C}}$ ($n \geq 0$) of $L^2 = \overline{\bigcup_{n=0}^{\infty} P_{n, \mathbb{C}}}$, with $\Pi_n^\wedge = \Pi_n - \Pi_{n-1}$ for $n \geq 1$. The choice of the sign in the formula above is not optimal from the point of view of the K-homology class, determined by the triple (\mathcal{O}_A, H, D) . A better choice would be $F = 1 \otimes 1$ on H . This would require in turn a modification of $|D|$. The construction along these lines is considered by Marcolli, with Alina Vdovina and Gunther Cornelissen. In this setting, the reason for the above choice as D as in [61] is the last formula in this section relating to the logarithm of Frobenius at arithmetic infinity.

Theorem 4.4. *For a Schottky group Γ with $\dim_H \Lambda_\Gamma < 1$ and rank h , a non finitely summable, but θ -summable spectral triple is defined as the date (\mathcal{O}_A, H, D) , for $H = L^2 \oplus L^2(\Lambda_\Gamma, d\mu)$ with the diagonal action of \mathcal{O}_A by the faithful representation defined as $S_j = \sum_{k=1}^{2h} a_{jk} T_j^* P_k$ for $1 \leq j \leq 2h$, and with the Dirac operator D as above.*

◇ The CK-algebra \mathcal{O}_A in the statement may be replaced with its dense smooth subalgebra \mathcal{O}_A^∞ . \square

The key point of this result is to show the compatibility relation between the CK-algebra and the Dirac operator, namely that the commutators $[D, a]$ are bounded operators for any element of the involutive dense subalgebra \mathcal{O}_A^∞ of \mathcal{O}_A , generated algebraically by the partial isometries S_j subject to the CK relations.

This follows by estimating the norm of the commutators $\|[D, S_j]\|$ and $\|[D, S_j^*]\|$, in terms of the Poincaré series of the Schottky group with $d_H < 1$, such that

$$\sum_{\gamma \in \Gamma} |\gamma'|^s, \quad s = 1 > d_H = \dim_H \Lambda_\Gamma,$$

where the Hausdorff dimension d_H becomes the exponent of convergence of the Poincaré series.

◇ Compute that

$$\begin{aligned} [D, S_j] &= D \begin{pmatrix} S_j & 0 \\ 0 & S_j \end{pmatrix} - \begin{pmatrix} S_j & 0 \\ 0 & S_j \end{pmatrix} D \\ &= \begin{pmatrix} 0 & D_{0 \oplus L^2} S_j - S_j D_{0 \oplus L^2} \\ D_{L^2 \oplus 0} S_j - S_j D_{L^2 \oplus 0} & 0 \end{pmatrix}. \end{aligned}$$

Moreover, in particular,

$$S_j D_{0 \oplus L^2} = \sum_{k=1}^{2h} a_{jk} T_j^* P_k \left(- \sum_{n=1}^{\infty} n \Pi_n^\wedge \right) = - \sum_{n=1}^{\infty} n \sum_{k=1}^{2h} a_{jk} T_j^* P_k (\Pi_n - \Pi_{n-1}).$$

It then follows that

$$[D_{0 \oplus L^2}, S_j] = - \sum_{n=1}^{\infty} n \sum_{k=1}^{2h} a_{jk} [T_j^* P_k, \Pi_n - \Pi_{n-1}]$$

A possible estimate of the norm of this commutator is given by the upper bounded as $2h \sum_{n=1}^{\infty} n$, but which diverges. Thus, it is necessary to have that the norm of the commutators as the direct summands converges to zero at infinity, of order less than $O(\frac{1}{n^2})$ as $n \rightarrow \infty$. Possibly, it is necessary to consider elements of \mathcal{O}_A^∞ which are smooth, rapidly decreasing, or compactly supported, but with respect to (Π_n^\wedge) as a sort of basis. \square

The dimension of the $(i = \sqrt{-1})n$ -th (corrected) eigenspace of D is $2g(2g - 1)^{n-1}(2g - 2)$ for $n \geq 1$, $2g$ for $n = 0$, and $2g(2g - 1)^{-n-1}(2g - 2)$ for $n \leq -1$, so that the spectral triple is not finitely summable, since $|D|^z$ is not of trace class. But it is θ -summable, since the operator $\exp(-tD^2)$ is of trace class, for all $t > 0$.

\diamond For instance, suppose that $D(\xi \oplus \eta) = 0 \oplus 0$ on $H = \oplus^2 L^2$. Then $\sum_{n=1}^{\infty} n \Pi_n^\wedge \eta = 0$ and $\sum_{n=1}^{\infty} n \Pi_n^\wedge \xi = 0$. If we have the dimension of the kernel of D equal to $2g$, the summations in the definition of D should start with $n = 1$. Moreover, note that for $\xi_n \in \Pi_n^\wedge L^2$,

$$\begin{pmatrix} 0 & -n\Pi_n^\wedge \\ n\Pi_n^\wedge & 0 \end{pmatrix} \begin{pmatrix} \xi_n \\ -i\xi_n \end{pmatrix} = \begin{pmatrix} in\xi_n \\ n\xi_n \end{pmatrix} = in \begin{pmatrix} \xi_n \\ -i\xi_n \end{pmatrix}, \quad \text{and} \\ \begin{pmatrix} 0 & -n\Pi_n^\wedge \\ n\Pi_n^\wedge & 0 \end{pmatrix} \begin{pmatrix} \xi_n \\ i\xi_n \end{pmatrix} = \begin{pmatrix} -in\xi_n \\ n\xi_n \end{pmatrix} = -in \begin{pmatrix} \xi_n \\ i\xi_n \end{pmatrix}. \quad \square$$

Using the description of the noncommutative space \mathcal{O}_A as the crossed product C^* -algebra $\mathfrak{F}_A \rtimes_T \mathbb{Z}$ of an AF \mathfrak{F}_A by the action of the shift T , it may be able to find a 1-summable spectral triple, where the dense subalgebra involved in it should not contain any of the group elements. In fact, by the result of Connes [26], the group Γ is a free group, and hence its growth is too fast to support finitely summable spectral triples on its group ring.

4.3 Arithmetic infinity as archimedean primes

An algebraic number field \mathbb{K} , which is an extension of \mathbb{Q} such that $[\mathbb{K}, \mathbb{Q}] = n$, admits $n = r_1 + 2r_2$ embeddings as $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$. These can be subdivided into r_1 real embeddings as $\mathbb{K} \hookrightarrow \mathbb{R}$ and r_2 pairs of complex conjugate embeddings. The embeddings α are said to be the **archimedean** primes of the number field \mathbb{K} . The set of archimedean primes is often referred to as **arithmetic** infinity, as a terminology, borrowed from the case of the unique embedding as $\mathbb{Q} \hookrightarrow \mathbb{R}$, which is called the **infinite** prime.

\diamond An **extension** of a field K is a field L with an inclusion $K \subset L$ with K as a subfield. The extension degree $[L, K]$ is defined to be the dimension of L as a vector space over K , since $KL \subset L$.

For instance, $[\mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}] = 4 = r_1$ with $r_2 = 0$. Indeed, as vector spaces,

$$\begin{aligned} \mathbb{Q}(\sqrt{2}, \sqrt{3}) &\cong \mathbb{Q}1 \oplus \mathbb{Q}\sqrt{2} \oplus \mathbb{Q}\sqrt{3} \oplus \mathbb{Q}\sqrt{6} \\ &\cong \mathbb{Q}(\sqrt{2})1 \oplus \mathbb{Q}(\sqrt{2})\sqrt{3} \quad \text{with} \\ [\mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}] &= [\mathbb{Q}(\sqrt{2}), \mathbb{Q}][\mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{2})] = 2^2. \end{aligned}$$

There is the identity embedding as $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \hookrightarrow \mathbb{R}$. As well, the others are obtained by sending respectively $(\sqrt{2}, \sqrt{3}) \mapsto (-\sqrt{2}, \sqrt{3})$, $(\sqrt{2}, -\sqrt{3})$, or $(-\sqrt{2}, -\sqrt{3})$.

As well, $[\mathbb{Q}(i), \mathbb{Q}] = 2$ with $r_2 = 1$. Indeed, $\mathbb{Q}(i) \cong \mathbb{Q}1 \oplus \mathbb{Q}\sqrt{-1}$, and there is the identity embedding as $\mathbb{Q}(i) \hookrightarrow \mathbb{C}$, and the other is obtained by sending $i \mapsto -i$ (cf. [129]). \square

A general strategy in arithmetic geometry is to adapt the tools of classical algebraic geometry to the arithmetic setting. In particular, the (spectrum) $\text{Sp}(\mathbb{Z})$ of primes over \mathbb{Q} is the analog of the affine line in arithmetic geometry. It then becomes clear that some compactification is necessary, at least in order to have a well behaved form of intersection theory in arithmetic geometry. Namely, we need to pass from the affine (Spec) $\text{Sp}(\mathbb{Z})$ to the projective case. The compactification is obtained by adding the infinite prime to the set of finite primes. Then a goal of arithmetic geometry becomes developing a setting that treats the infinite prime and the finite primes in the equal footing.

More generally, for a number field \mathbb{K} , with $O_{\mathbb{K}}$ as its ring of integers, the set $\text{Sp}(O_{\mathbb{K}})$ of primes is compactified by adding the set of archimedean primes, as

$$\overline{\text{Sp}(O_{\mathbb{K}})} = \text{Sp}(O_{\mathbb{K}}) \cup \{\alpha : \mathbb{K} \hookrightarrow \mathbb{C}\}.$$

Arithmetic surfaces. Let X be a smooth projective algebraic curve defined over \mathbb{Q} . Then, obtained by clearing denominators is an equation with coefficients in \mathbb{Z} . This determines a **scheme** $X_{\mathbb{Z}}$ over $\text{Sp}(\mathbb{Z})$, as

$$X_{\mathbb{Z}} \otimes_{\mathbb{Z}} \text{Sp}(\mathbb{Q}) = X,$$

where the closed fiber of $X_{\mathbb{Z}}$ at a prime $p \in \text{Sp}(\mathbb{Z})$ is the reduction of X mod p . Thus, an algebraic curve viewed as an arithmetic variety becomes a 2-dimensional fibration over the affine line $\text{Sp}(\mathbb{Z})$.

Also consider reductions of X defined over \mathbb{Z} modulo p^n for some prime $p \in \text{Sp}(\mathbb{Z})$. The corresponding limit as $n \rightarrow \infty$ defines a **p -adic** completion of $X_{\mathbb{Z}}$. This can be thought of as an infinitesimal neighbourhood of the fiber at p .

The corresponding picture is more complicated at arithmetic infinity, since there is no suitable notion of reduction mod ∞ , available to define the closed fiber. On the other hand, there is the analog of the p -adic completion at hand. This is given by the Riemann surface $X(\mathbb{C})$ as a smooth projective algebraic curve over \mathbb{C} , determined by the equation of the algebraic curve X over \mathbb{C} , under the embedding of $\mathbb{Q} \subset \mathbb{C}$, as

$$X(\mathbb{C}) = X \otimes_{\mathbb{Q}} \text{Sp}(\mathbb{C}),$$

with the absolute value $|\cdot|$ at the infinite prime, replacing the p -adic valuations.

Similarly, for \mathbb{K} a number field with $[\mathbb{K} : \mathbb{Q}] = n$ and $O_{\mathbb{K}}$ its ring of integers, a smooth proper algebraic curve X over \mathbb{K} determines a smooth minimal **model** $X_{O_{\mathbb{K}}}$, which defines an arithmetic surface $\mathcal{X}_{O_{\mathbb{K}}}$ over $\mathrm{Sp}(O_{\mathbb{K}})$. The closed fiber $X_{\mathfrak{p}}$ of $\mathcal{X}_{O_{\mathbb{K}}}$ over a prime $\mathfrak{p} \in O_{\mathbb{K}}$ is given by the reduction mod \mathfrak{p} .

When $\mathrm{Sp}(O_{\mathbb{K}})$ is compactified by adding the archimedean primes, obtained are $n = r_1 + 2r_2$ Riemann surfaces $X_{\alpha}(\mathbb{C})$, which are obtained from the equation defining X over \mathbb{K} under the n embeddings $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$. The corresponding r_1 Riemann surfaces of n have the real involution.

Then, the picture of an arithmetic surface $X_{\overline{\mathrm{Sp}(O_{\mathbb{K}})}}$ over $\overline{\mathrm{Sp}(O_{\mathbb{K}})}$ with inclusions is as follows:

$$\begin{array}{ccccccc}
 X_{\mathfrak{p}} & \xrightarrow{\subset} & \mathcal{X}_{O_{\mathbb{K}}} = X_{\mathrm{Sp}(O_{\mathbb{K}})} & \xrightarrow{\subset} & X_{\overline{\mathrm{Sp}(O_{\mathbb{K}})}} & \xleftarrow{\supset} & ? \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathfrak{p} & \xrightarrow{\subset} & \mathrm{Sp}(O_{\mathbb{K}}) & \xrightarrow{\subset} & \overline{\mathrm{Sp}(O_{\mathbb{K}})} & \xleftarrow{\supset} & \{\alpha : \mathbb{K} \hookrightarrow \mathbb{C}\}
 \end{array}$$

where there is no explicit geometric description of the closed fibers as (?) over the archimedean primes α .

An arithmetic surface over $\mathrm{Sp}(\mathbb{Z})$ of primes looks like a bundle of curves or crossed ones over primes, whose limit at infinity is what as (?). (No figure).

It is able to formally enlarge the group of divisors on the arithmetic surface by adding formal real linear combinations as $\sum_{\alpha} \lambda_{\alpha} F_{\alpha}$ of irreducible closed vertical fibers F_{α} at infinity, where the fibers F_{α} are only treated as formal symbols, and no geometric model of such fibers is provided. A remarkable fact says that Hermitian geometry on the Riemann surfaces $X_{\alpha}(\mathbb{C})$ is sufficient to specify the contribution of such divisors to intersection theory on the arithmetic surface, even without an explicit knowledge of the closed fiber.

The main ideal of Arakelov geometry is that it is sufficient to work with the infinitesimal neighbourhood $X_{\alpha}(\mathbb{C})$ of the fibers F_{α} , to have well defined intersection indices.

By analogy, in the case of the classical geometry of a degeneration of algebraic curves over a disk with a special fiber at 0, the analogous statement would be that the geometry of the special fiber is completely determined by the generic fibers. This is a strong statement as the form of the degeneration. For instance, blowing up points at the special fiber may be not seen by just looking at the generic fibers. Investigating this analogy leads to expect that the fiber at infinity should behave like the totally degenerate case. This is the case as the maximal degeneration, where all the components of the closed fiber are the same as \mathbb{P}^1 and the geometry of the degeneration is completely encoded by the dual graph, which describes in a purely combinatorial way how the components as \mathbb{P}^1 are joined. The dual graph has a vertex for each components of the closed fiber and an edge for each double point.

The local intersection multiplicities of two finite, horizontal, irreducible divisors D_1, D_2 on $X_{O_{\mathbb{K}}}$ is given by

$$[D_1, D_2] = [D_1, D_2]_f + [D_1, D_2]_{\infty},$$

where the first term counts the contribution from the finite places as what happens over $\mathrm{Sp}(O_{\mathbb{K}})$ and the second term is the contribution of the archimedean primes, at the part of the intersection that happens over arithmetic infinity. While the first term is computed in algebro-geometric terms, from the local equations for the divisors D_i at a point p , the second term is defined as a sum of values of Green functions g_α on the Riemann surfaces $X_\alpha(\mathbb{C})$ as

$$[D_1, D_2]_\infty = - \sum_{\alpha} \epsilon_\alpha \sum_{\beta, \gamma} g_\alpha(p_{1,\beta}^\alpha, p_{2,\gamma}^\alpha),$$

at points $p_{1,\beta}^\alpha$ and $p_{2,\gamma}^\alpha$ of $X_\alpha(\mathbb{C})$ for $1 \leq \beta \leq [\mathbb{K}(D_1), \mathbb{K}]$ and $1 \leq \gamma \leq [\mathbb{K}(D_2), \mathbb{K}]$ respectively, for finite extensions $\mathbb{K}(D_j)$ of \mathbb{K} determined by D_j for $j = 1, 2$, where $\epsilon_\alpha = 1$ for α real embeddings and $\epsilon_\alpha = 2$ for α complex embeddings.

For a more detailed account of these notions of Arakelov geometry, may refer to [65] and [99].

Further evidence for the similarity between the archimedean and the totally degenerate fibers comes from an explicit computation of the Green function at the archimedean places, as derived by Manin [106], in terms of a Schottky uniformization of the Riemann surface $X_\alpha(\mathbb{C})$. Such a uniformization has an analog at a finite prime, in terms of p -adic Schottky groups, only in the totally degenerate case. Another source of evidence comes from a cohomological theory of the local factors at archimedean primes, as developed by Consani [57], in which shown is that the resulting description of the local factor as regularized determinant at the archimedean primes resembles closely the case of the totally degenerate reduction at a finite prime.

Presented soon are both results in the light of the noncommutative space as a spectral triple (\mathcal{O}_A, H, D) introduced in the previous section. As shown by Consani and Marcolli in [58], [59], [60], and [61], the noncommutative geometry of this space is naturally related to both the result of Manin on the Arakelov Green function and the cohomological construction of Consani.

4.4 Arakelov geometry and hyperbolic geometry

In this subsection, given is a detailed account of the result of Manin [106] on the relation between the Arakelov Green function on a Riemann surface $X(\mathbb{C}) = \Gamma \backslash \Omega_\Gamma$ with Schottky uniformization $\Omega_\Gamma = \Lambda_\Gamma^c$ in $\mathbb{P}^1(\mathbb{C}) \approx S^2$ and geodesics in the real 3-dimensional hyperbolic handle-body $Y_\Gamma = \mathbb{H}^3/\Gamma$. The exposition in the following follows the seminal paper [106] closely.

Arakelov Green function. Given a divisor $A = \sum_{x \in X} m_x(x)$ with support denoted as $|A|$ on a smooth compact Riemann surface $X(\mathbb{C}) = X$, and also a choice of a positive real analytic 2-form $d\mu$ on $X(\mathbb{C})$, the **Green** function $g_A = g_{A,\mu}$ is defined to be a real analytic function on $X(\mathbb{C})$, uniquely determined by the following conditions:

- **Laplace** equation: $\partial\bar{\partial}g_A = \pi i(\mathrm{deg}(A)d\mu - \delta_A)$ is satisfied, with the δ -current $\delta_A : \varphi \mapsto \sum_{x \in X} m_x \varphi(x)$.

- Singularities: $g_A - m_x \log |z|$ is locally real analytic, for z a local coordinate in a neighbourhood of x .
- Normalization: $\int_X g_A d\mu = 0$ is satisfied.

If $B = \sum_{y \in X} n_y(y)$ is another divisor, such that $|A| \cap |B| = \emptyset$, then the **expression** defined as $g_\mu(A, B) = \sum_u n_y g_{A, \mu}(y)$ is symmetric and biadditive in A, B . In general, such an expression as g_μ depends on μ , where the choice of μ is equivalent to the choice of a real analytic Riemannian metric on X , compatible with the complex structure. However, in the special case of degree zero divisors as $\deg A = \deg B = 0$, the expression $g_\mu(A, B)$ are conformal invariants, denoted as $g(A, B)$, named as the **Arakelov Green** function.

In the case of the Riemann sphere $\mathbb{P}^1(\mathbb{C}) \approx S^2 \approx \mathbb{C} \cup \{\infty\}$, if w_A is a meromorphic function with $\text{Div}(w_A) = A$, then we have

$$g(A, B) = \log \prod_{y \in |B|} |w_A(y)|^{n_y} = \text{Re} \int_{\gamma_B} \frac{dw_A}{w_A},$$

where γ_B is a 1-chain with boundary B .

In the case of degree zero divisors A, B on a Riemann surface of higher genus, the formula as above for $g(A, B)$ can be generalized, replacing the logarithmic differential dw_A/w_A with w_A a differential of the third kind, as a meromorphic differential with non-vanishing residues, with purely imaginary and residues as m_x at x , as given as

$$g(A, B) = \text{Re} \int_{\gamma_B} w_A.$$

Thus, $g(A, B)$ can be explicitly computed from a basis of differentials of the third kind with purely imaginary periods.

Remark. Recall from [116] the following. A **differential** on a Riemann surface R is a 1-form as $w = udx + vdy$ with $z = x + iy$ locally, with $w^* = -vdx + udy$ conjugate differential, with $(w^*)^* = -w$. If $dw = 0$, the w is said to be **closed**. Then, for $\alpha \in H_1(R)$ the homology, the integral $\int_\alpha w$ is defined uniquely to be the **period** along a 1-cycle representing the class α . If there is a function F on R such that $dF = w$, then w is said to be **exact**. if $w^* = -iw$, then w is said to be **pure**. Then $w = f(z)dz$ locally. If f is holomorphic, then w is said to be **holomorphic** or **analytic**, which is equivalent to that w is closed and pure. If f is meromorphic (as holomorphic or with only poles), then w is **meromorphic** or **Abelian** differential. An Abelian differential on a closed (or compact) Riemann surface have the sum of residues equal to zero. Abelian differentials are divided into being of the **first**, **second**, and **third** kind, respectively, if f is holomorphic, f has zero residues for poles, and otherwise. The indefinite integral $\int_{p_0}^p w$ is said to be **Abelian** integral, where p_0 is not a pole. In particular, an Abel integral on a closed Riemann surface of genus 1 is said to be **elliptic** integral.

As well, a **divisor** on a Riemann surface R is defined to be a formal sum with coefficients in \mathbb{Z} as $D = \sum_{p \in R} n_p p$, where the **support** of D is a discrete set of $p \in R$ with nonzero $n_p \in \mathbb{Z}$. Define the **degree** of D as $\deg D = \sum_{p \in R} n_p$.

A divisor on D on R is said to be a **divisor** for f or $w = f(z)dz$ if the support of D consists of zeros and poles of f , and if p is a zero, n_p is its degree (i.e., $f(z) = (z-p)^{n_p}h(z)$ locally, with h holomorphic with $h(p) \neq 0$) and if p is a pole, then $-n_p \geq 1$ is its degree (i.e., $f(z) = \frac{h(z)}{(z-p)^{-n_p}}$ locally, similarly as above). On a closed (or compact) Riemann surface of genus $g \geq 1$, a divisor of a meromorphic function is said to be **principal**, and a divisor of an Abelian differential is said to be **canonical**. For D principal, $\deg D = 0$. For D canonical, $\deg D = 2g - 2$.

◀

Cross ratio and geodesics. The basic step leading to the result of Manin [106], expressing the Arakelov Green function in terms of geodesics in the hyperbolic handle-body $Y_\Gamma = \mathbb{H}^3/\Gamma$ is a simple classical fact in hyperbolic geometry. Namely, it is the fact that the (logarithmic) cross ratio of four points in $\mathbb{P}^1(\mathbb{C}) \approx S^2 \approx \mathbb{C} \cup \{\infty\}$ can be expressed in terms of geodesics in the interior \mathbb{H}^3 as

$$\log |[a, b, c, d]| = -\text{ord}(a * [c, d], b * [c, d]),$$

where $[a, b, c, d] = -(a-b)(c-d)(a-d)^{-1}(b-c)^{-1}$ is the **cross ratio** of points $a, b, c, d \in \mathbb{P}^1(\mathbb{C})$, and $\text{ord}(\cdot, \cdot)$ means the **oriented distance** in \mathbb{H}^3 , and $a * [c, d]$ indicates the on the geodesic $[c, d]$ in \mathbb{H}^3 with end points $c, d \in \mathbb{P}^1(\mathbb{C})$, obtained as the intersection of $[c, d]$ with the unique geodesic from a that intersects with $[c, d]$ at a right angle. (No original figure).

Differentials and Schottky uniformization. The next important step in the result of Manin [106] is to show that if $X(\mathbb{C}) = \Gamma \backslash \Omega_\Gamma$ is a Riemann surface with a Schottky uniformization, then obtained is a basis of differentials of the third kind with purely imaginary periods, by taking suitable averages over the group Γ of expressions involving the cross ratio of points of $\mathbb{P}^1(\mathbb{C})$.

Denote by $C(\gamma)$ a set of representatives of $\Gamma/\gamma^{\mathbb{Z}}$, and by $C(\rho, \gamma)$ a set of representatives $\rho^{\mathbb{Z}} \backslash \Gamma/\gamma^{\mathbb{Z}}$, and by $S(\gamma)$ the conjugacy class of γ in Γ .

Let w_A be a meromorphic function on $\mathbb{P}^1(\mathbb{C})$ with divisor $A = (a) - (b)$, such that the support $|A|$ is contained in the complement of an open neighbourhood of Λ_Γ .

For a fixed choice of a base point $z_0 \in \Omega_\Gamma$, the series

$$\nu_{(a)-(b)} = \sum_{\gamma \in \Gamma} d \log[a, b, \gamma z, \gamma z_0], \quad \text{with} \quad [a, b, \gamma z, \gamma z_0] = \frac{(a-b)(\gamma z - \gamma z_0)}{-(a - \gamma z_0)(b - \gamma z)},$$

gives the lift to Ω_Γ of a differential of the third kind on the Riemann surface $X(\mathbb{C})$, endowed with the choice of Schottky uniformization. These differentials have residues ± 1 at the image of a, b in $X(\mathbb{C})$, and they have vanishing a_k periods, where a_k, b_k for $1 \leq k \leq g$ are the generators of the (first) homology of $X(\mathbb{C})$.

Similarly, obtained are the lifts of differentials of the first kind on $X(\mathbb{C})$, by

considering the series

$$\omega_\gamma = \sum_{h \in \mathcal{C}(\gamma)} d \log[hz^+(\gamma), hz^-(\gamma), z, z_0],$$

$$\text{with } [hz^+(\gamma), hz^-(\gamma), z, z_0] = \frac{(hz^+(\gamma) - hz^-(\gamma))(z - z_0)}{-(hz^+(\gamma) - z_0)(hz^-(\gamma) - z)},$$

where $z^\pm(\gamma) \in \Lambda_\Gamma$ denote the pair of the attractive and repelling fixed points of $\gamma \in \Gamma$.

The series as $\nu_{(a)-(b)}$ and ω_γ converge absolutely on compact subsets of Ω_Γ , whenever $\dim_H \Lambda_\Gamma < 1$. Moreover, they do not depend on the choice of the base point as $z_0 \in \Omega_\Gamma$.

In particular, given $\{\gamma_k\}_{k=1}^g$ a choice of generators of the Schottky group Γ , obtained is a basis of holomorphic differentials ω_{γ_k} , satisfying $\int_{a_k} \omega_{\gamma_l} = 2\pi\sqrt{-1}\delta_{kl}$.

Then use a linear combination of the holomorphic differentials ω_{γ_k} to correct the meromorphic differentials $\nu_{(a)-(b)}$ in such a way that the resulting meromorphic differentials have purely imaginary b_k -periods. Let $x_l(a, b)$ denote coefficients such that the differential of the third kind defined as

$$\omega_{(a)-(b)} = \nu_{(a)-(b)} - \sum_{l=1}^g x_l(a, b)\omega_{\gamma_l}$$

(corrected) have purely imaginary b_k -periods. The coefficients $x_l(a, b)$ satisfy the system of equations

$$\begin{aligned} \sum_{l=1}^g x_l(a, b) \operatorname{Re} \int_{b_k} \omega_{\gamma_l} &= \operatorname{Re} \int_{b_k} \nu_{(a)-(b)} = \sum_{h \in S(\gamma_k)} \log |[a, b, z^+(h), z^-(h)]| \\ &= - \sum_{h \in S(\gamma_k)} \operatorname{ord}(a * [z^+(h), z^-(h)], b * [z^+(h), z^-(h)]) \end{aligned}$$

(specified). Thus, it is obtained ([106], cf. also [157]) the Arakelov Green function $g(A, B)$ for $X(\mathbb{C})$ with Schottky uniformization can be computed as

$$\begin{aligned} g((a) - (b), (c) - (d)) &= \operatorname{Re} \int_{\gamma_{(c)-(d)}} \omega_{(a)-(b)} \\ &= \operatorname{Re} \int_{\gamma_{(c)-(d)}} [\nu_{(a)-(b)} - \sum_{l=1}^g x_l(a, b)\omega_{\gamma_l}] \\ &= \operatorname{Re} \int_{\gamma_{(c)-(d)}} \nu_{(a)-(b)} - \sum_{l=1}^g x_l(a, b) \operatorname{Re} \int_{\gamma_{(c)-(d)}} \omega_{\gamma_l} \\ &= \sum_{h \in \Gamma} \log |[a, b, hc, hd]| - \sum_{l=1}^g x_l(a, b) \sum_{h \in S(\gamma_l)} \log |[z^+(h), z^-(h), c, d]|. \end{aligned}$$

◊ Note that

$$[a, b, c, d] = \frac{(a-b)(c-d)}{-(a-d)(b-c)} = \frac{(c-d)(a-b)}{-(c-b)(d-a)} = [c, d, a, b]. \quad \square$$

It is noticed that the result above seems to indicate that there is a choice of Schottky uniformization involved as an additional datum for Arakerov geometry at arithmetic infinity. However, as remarked previously, at least in the case of archimedean primes that are real embeddings as the case of arithmetic infinity for \mathbb{Q} , the Schottky uniformization is determined by the real structure, by splitting $X(\mathbb{C})$ along the real locus $X(\mathbb{R})$, when the latter is non-trivial.

Green function and geodesics. By combining the basic formula by the oriented distance with the formula calculated above for the Arakerov Green function on a Riemann surface with Shottky uniformization, we can replace each term appearing in the formula with the corresponding term which computes the oriented geodesic length of a certain arc of geodesic in Y_Γ as

$$g((a) - (b), (c) - (d)) = - \sum_{h \in \Gamma} \text{ord}(a * [hc, hd], b * [hc, hd]) \\ + \sum_{l=1}^g x_l(a, b) \sum_{h \in S(\gamma_l)} \text{ord}(z^+(h) * [c, d], z^-(h) * [c, d]).$$

The coefficients $x_l(a, b)$ can also be expressed in terms of geodesics, using the system equations given above.

4.5 Quantum gravity and black holes as Intermezzo

The anti-de Sitter space a-dS_{d+1} of real dimension $d+1$ is a highly symmetric space-time, satisfying Einstein's equations with constant curvature $R < 0$. Physically, it describes an empty space with a negative cosmological constant. In order to avoid time-like closed geodesics, it is customary to pass to the universal cover a-dS_{d+1}^\sim , the boundary at infinity of which (or a-dS_{d+1}) is a compactification of d -dimensional Minkowski space time Mk^d . When passing to Euclidean signature, the space a-dS_{d+1}^\sim becomes the real $d+1$ -dimensional hyperbolic space \mathbb{H}^{d+1} .

The real 4-dimensional anti-de Sitter space with $d = 3$ is a well known example as space-time in general relativity. The space a-dS_{3+1} has the form $S^1 \times \mathbb{R}^3$ the cylinder as a topological space, while it is realized metrically as the hyperboloid in \mathbb{R}^5 defined as

$$-u^2 - v^2 + x^2 + y^2 + z^2 = 1 \quad \text{with} \quad ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$$

as the metric element. The universal cover a-dS_{3+1}^\sim is \mathbb{R}^4 as a topological space. In the context of quantum gravity, it is especially interesting to consider the case of the real 3-dimensional a-dS_{2+1} and its Euclidean counterpart as the real 3-dimensional hyperbolic space \mathbb{H}^3 .

◊ In this case, the **Minkowski** space time $\text{Mk}^d \approx \mathbb{R}^d \approx \mathbb{R} \times \mathbb{R}^{d-1}$ with the metric as the diagonal sum

$$G = (g_{\mu\nu}) = -1 \oplus 1_{d-1}.$$

Note also that, with $G = -1_2 \oplus 1_3$, in the anti-de Sitter space a-dS₄,

$$ds^2 = ds \cdot ds = \langle (-1_2 \oplus 1_3)(du, dv, dx, dy, dz)^t, d(u, v, x, y, z) \rangle. \quad \square$$

The holographic principle postulates the existence of an explicit correspondence between gravity on a bulk space which is asymptotically a-dS_{d+1}[~], like a space obtained as a global quotient of a-dS_{d+1}[~] by a discrete group of isometries, and field theory on its conformal boundary at infinity.

The above given, relation between the *log* of the cross ratio as $[a, b, c, d]$ for $a, b, c, d \in \mathbb{P}^1(\mathbb{C})$ and the oriented distance $\text{ord}(\cdot, \cdot)$ in terms of geodesics in \mathbb{H}^3 , which identifies the Green function $g((a) - (b), (c) - (d))$ on $\mathbb{P}^1(\mathbb{C})$ with the oriented length of a geodesic arc in \mathbb{H}^3 , can be thought of as an instance of the holography principle, when we interpret one side as geodesic propagator on the bulk space in a semi-classical approximation, and the other side as the Green function as the two-point correlation function of the boundary field theory. Note that, because of the prescribed behavior of the Green function at the singularities given by the points of the divisor, the four-point invariant as $g((a) - (b), (c) - (d))$, when a and b converge to c, d respectively, gives the two-point correlator with a logarithmic divergence which is intrinsic and does not depend on a choice of cut-off functions, unlike the way in regularization is often used in the physics literature.

It is shown in [111] that the above given, (another) relation between Arakelov Green functions and configurations of geodesics in the hyperbolic handle-body Y_Γ , proved by Manin in [106], provides in fact precisely the correspondence prescribed by the holography principle, for a class of $(2+1)$ -dimensional space-time known as Euclidean Krasnov black holes. These as holes include the Bañados-Teitelboim-Zanelli (BTZ) black holds, as an important class of space-times in the context of $(2+1)$ -dimensional quantum gravity.

The Bañados-Teitelboim-Zanelli black hole. Consider the case of a hyperbolic handle-body as Y_Γ of genus one, as a solid (open) torus (as a donut), with conformal boundary at infinity given by an elliptic curve.

Recall that elliptic curves can be described via the Jacobi uniformization, as already encountered in the previous sections, in the context of noncommutative elliptic curves. Let $X_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$ be such a description of an elliptic curve, where q is a hyperbolic element of $PSL_2(\mathbb{C})$ with fixed points as $\{0, \infty\}$ on the 2-sphere as $\mathbb{P}^1(\mathbb{C})$ at infinity, for \mathbb{H}^3 , so that $q \in \mathbb{C}^*$ with $|q| < 1$. The action of q on $\mathbb{P}^1(\mathbb{C})$ extends to an action on $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ by

$$q \cdot (z, y) = (qz, |q|y) \in \mathbb{C} \times (0, \infty) \approx \mathbb{H}^3.$$

It follows that the quotient space $Y_q = \mathbb{H}^3/q^{\mathbb{Z}}$ by this action is a solid (open) torus as a topological space, which is compactified at infinity by the conformal boundary $X_q(\mathbb{C})$ as ∂Y_q (as a torus).

The space Y_q is well known in the physics literature as the Euclidean Bañados-Teitelboim-Zanelli **black hole**, where the parameter $q \in \mathbb{C}^*$ is written as the form

$$q = \exp\left(\frac{2\pi(i|r_-| - r_+)}{l}\right) \quad \text{with} \quad r_{\pm}^2 = \frac{1}{2}\left(Ml \pm \sqrt{M^2l^2 + J}\right),$$

where M and J are the mass and angular momentum of the rotating black hole, and $-\frac{1}{l^2}$ is the cosmological constant. The corresponding black hole in Minkowskian signature may be illustrated, but no figure provided (as in [3]).

In the case of the elliptic curve as $X_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$, the formula of Alvarez-Gaumé, Moore, and Vafa [2] gives the operator product expansion of the path integral for bosonic field theory as

$$g(z, 1) = \log\left(|q|^{2^{-1}B_2\left(\frac{\log|z|}{\log|q|}\right)}|1 - z|\prod_{n=1}^{\infty}|1 - q^n z||1 - q^n z^{-1}|\right).$$

This is in fact the Arakelov Green function on $X_q(\mathbb{C})$. In terms of geodesics in the Euclidean BTZ black hole (BH), that becomes

$$g(z, 1) = -\frac{1}{2}l(\gamma_0)B_2\left(\frac{l_{\gamma_0}(\bar{z}, \bar{1})}{l(\gamma_0)}\right) + \sum_{n \geq 0} l_{\gamma_1}(\bar{0}, \bar{z}_n) + \sum_{n \geq 1} l_{\gamma_1}(\bar{0}, \bar{z}_n^{-1}),$$

where $B_2(v) = v^2 - v + \frac{1}{6}$ is the second Bernoulli polynomial, and we use the notation as that $\bar{x} = x * [0, \infty]$, $\bar{z}_n = q^n z * [1, \infty)$, and $\bar{z}_n^{-1} = q^n z^{-1} * [1, \infty)$ in \mathbb{H}^3 as well as X_q , as in [106]. (No respective figures.) These terms describe gravitational properties of the Euclidean BTZ **BH**. For instance, $l(\gamma_0)$ measures the black hole entropy. The whole expression is a combination of geodesic propagators.

Krasnov black holes. The problem of computing the bosonic field propagator on an algebraic curve $X_{\mathbb{C}}$ (such as $X(\mathbb{C}) = \Gamma \backslash \Omega_{\Gamma}$) can be solved by providing differentials of the third kind with purely imaginary periods as

$$\omega_{(a)-(b)} = \nu_{(a)-(b)} - \sum_{l=1}^g x_l(a, b)\omega_{\gamma_l},$$

and thus it can be related directly to the problem of computing the Arakelov Green function.

Differentials as above then determine all the higher correlation functions as

$$G(z_1, \dots, z_m, w_1, \dots, w_l) = \sum_{j=1}^m \sum_{k=1}^l q_k \langle \varphi(z_k, \bar{z}_k) \varphi(w_j, \bar{w}_j) \rangle q'_j,$$

for q_k a system of charges at positions z_k interacting with charges q'_j at positions w_j from the basic two-point correlator $G_{\mu}(a - b, z)$ given by the Green function expressed in terms of the differentials $\omega_{(a)-(b)}(z)$. By using a Schottky uniformization, obtained are the differentials $\omega_{(a)-(b)}$ as given above.

The bulk space corresponding to the conformal boundary $X(\mathbb{C})$ is given by the hyperbolic handle-body Y_Γ . As in the case of the BTZ black hole, it is possible to interpret these real hyperbolic 3-manifolds as analytic continuations to Euclidean signature of Minkowskian black holes that are global quotients of $a\text{-dS}_{2+1}$. This is not just the effect of the usual rotation from Minkowskian to Euclidean signature, but a more refined form of analytic continuation which is adapted to the action of a Schottky group, and which is introduced by Kirill Krasnov [92], [93] in order to deal with this class of space-times.

The above given formula for the Green function $g(\cdot, \cdot)$ in terms of oriented distances of geodesics as in Y_Γ gives the explicit bulk and boundary correspondence of the holography principle for this class of space times, so that each term in the Bosonic field propagator for $X_\mathbb{C}$ is expressed in terms of geodesics in the Euclidean **Krasnov** black hole as $Y_\Gamma = \mathbb{H}^3/\Gamma$.

4.6 Dual graph and noncommutative geometry

The result of Manin on the Arakelov Green function and hyperbolic geometry suggests a geometric model for the dual graph of the mysterious fiber at arithmetic infinity. In fact, the result discussed above on the Green function has an analog, due to Drinfel'd and Manin [73] in the case of a finite prime with a totally split fiber. This is the case where the p -adic completion admits a Schottky uniformization by a p -adic Schottky group.

Remark. Record from [116] as well as [123], [128] the following. Let R be a Dedekind ring (or a Noether integral domain) as a subring of a (quotient) field $K(= \frac{R}{R^*}$ of R) and \mathfrak{p} its maximal ideal. For any nonzero $a \in K$, define $v(a) = n$ if the ideal (a) is the product of \mathfrak{p}^n for $n \in \mathbb{Z}$ with an ideal, relatively prime with \mathfrak{p} (or if $a = p^n \frac{c}{b}$ for $b, c \in R$ and $n \in \mathbb{Z}$, where p is a prime element of R and b, c are not divided by p). Then v is a normal, discrete, non-Archimedean, **additive valuation** (av) of K , which is said to be a \mathfrak{p} -adic exponential valuation. Namely, v is a map from K to $G \cup \{\infty\}$, with G an ordered additive group as \mathbb{Z} , such that (av 1) $v(a) = \infty \Leftrightarrow a = 0$, (av 2) $v(ab) = v(a) + v(b)$ for nonzero $a, b \in K$, (av 3) $v(a + b) \geq \min\{v(a), v(b)\}$ for any $a, b \in K$. The valuation **ring** of v is defined to be the subring R_v of all $a \in K$ such that $v(a) \geq 0$. There is the unique maximal ideal \mathfrak{m}_v of all $a \in K$ such that $v(a) > 0$, defined as the valuation **ideal** of v . The quotient ring R_v/\mathfrak{m}_v is said to be the **residue class field** of v .

◇ The (av 2) implies that $v(-1) = v(-1) + v(1)$, so that $v(1) = 0$. □

The \mathfrak{p} -adic valuation of K is a non-Archimedean (n-A), **multiplicative valuation** (mv) w of K defined as $w(a) = c^{-v(a)} \in \mathbb{R}$ (non-negative) for some constant $c > 1$ and a \mathfrak{p} -adic exponential valuation v , satisfying that (mv 1) $w(a) = 0 \Leftrightarrow a = 0$, (mv 2) $w(ab) = w(a)w(b)$, (mv 3) $w(a + b) \leq w(a) + w(b)$, and moreover (n-A) $w(a + b) \leq \max\{w(a), w(b)\}$, for $a, b \in K$ (and otherwise, Archimedean).

◇ By (mv 2), $w(1) = w(1)^2$ in \mathbb{R} . Hence $w(1) = 1$ or 0 . Thus, $w(1) = 1$ by (mv 1). Also, $w(1) = w(-1)^2$. Thus, $0 \leq w(-1) = 1$. Moreover, $1 = w(1) = w(aa^{-1}) = w(a)w(a^{-1})$ for any $a \in K^*$.

The trivial multiplicative valuation on K is defined by $w(a) = 1$ for any nonzero $a \in K$. \square

A non-Archimedean, multiplicative valuation w is said to be discrete if $w(K^*)$ as a subgroup of \mathbb{R}^* by (mv 2) is isomorphic to \mathbb{Z} .

\diamond Check that $w(0) = c^{-v(0)} = c^{-\infty} = 0$. Also,

$$w(ab) = c^{-v(ab)} = c^{-v(a)-v(b)} = c^{-v(a)}c^{-v(b)} = w(a)w(b),$$

$$w(a+b) = c^{-v(a+b)} \leq c^{-\min\{v(a),v(b)\}} \leq \max\{w(a),w(b)\} \leq w(a) + w(b). \quad \square$$

The **valued** group of w is defined to be the group of all $w(a)$ for any nonzero $a \in K$. A multiplicative valuation w on K is said to be **Archimedean** if for any nonzero $a, b \in K$, there is $n \in \mathbb{Z}$ such that $w(na) \geq w(b)$. Otherwise, it is said to be non-Archimedean (**n-A**).

If w on K is Archimedean, then there is an embedding σ of K into \mathbb{C} , and w is equivalent to the pull back as $|\sigma(a)|$. If w on K (with $c > 1$) is non-Archimedean, then define $v(a) = -\log_b w(a)$ with $b > 1$, which is a rank 1 (additive) valuation. The converse also holds.

With w a multiplicative valuation (with $c > 1$), define a **metric** on a field K by $d(a, b) = w(a - b)$ for $a, b \in K$.

\diamond Note that $d(a, b) = 0$ if and only if $w(a - b) = 0$ if and only if $a - b = 0$. As well, $d(b, a) = w(b - a) = w(-1)w(a - b) = d(a, b)$. Moreover,

$$d(a, b) = w(a - b) = w(a - c + c - b) \leq w(a - c) + w(c - b) = d(a, c) + d(c, b). \quad \square$$

If K is complete with respect to this metric, w is said to be **complete**. If not so, there is the unique completion K' of K such that K' is an extension of K with w' as an extension of w , so that K is dense in K' and K' is complete with respect to w' .

The **p-adic** completion $K_{\mathfrak{p}}$ of K is defined to be the completion of K with respect to the **p-adic** completion. In particular, if K is an algebraic number field, $K_{\mathfrak{p}}$ is said to be the **p-adic** algebraic number field.

If $R = \mathbb{Z}$ and $\mathfrak{p} = (p) = p\mathbb{Z}$ for a prime p , the corresponding p -adic valuation is defined as w or v . Also, the p -adic number field \mathbb{Q}_p is defined to be the completion of \mathbb{Q} with respect to the p -adic valuation. Any nonzero element $a \in \mathbb{Q}_p$ as a p -adic number has the form

$$a = \sum_{n=r}^{\infty} a_n p^n, \quad a_n \in \mathbb{Z}/p\mathbb{Z}, r \in \mathbb{Z}, a_r \neq 0,$$

so that $v(a) = r$. The valuation ring of v on \mathbb{Q}_p is denoted as \mathbb{Z}_p of p -adic integers. \blacktriangleleft

\diamond May check that

$$v(ab) = v\left(\sum_{n=r}^{\infty} a_n p^n \sum_{m=s}^{\infty} b_m p^m\right) = v\left(\sum_{n=r}^{\infty} \sum_{m=s}^{\infty} a_n b_m p^{n+m}\right) = r + s = v(a) + v(b),$$

$$v(a+b) = v\left(\sum_{n=r}^{\infty} a_n p^n + \sum_{m=s}^{\infty} b_m p^m\right) \geq \max\{r, s\} = \max\{v(a), v(b)\}.$$

Schottky-Mumford curves. Let K be a given finite extension of \mathbb{Q}_p and let \mathcal{O} in K its ring of integers, and \mathfrak{m} in \mathcal{O} the maximal ideal, and $\mathfrak{k} = \mathcal{O}/\mathfrak{m}$ the residue field, that is a finite field of cardinality $q = \text{card}(\mathcal{O}/\mathfrak{m})$.

It is well known that a curve X over a finite extension K of \mathbb{Q}_p , which is \mathfrak{k} -split degenerate for \mathfrak{k} the residue field, admits a p -adic uniformization by a p -adic Schottky group Γ acting on the Bruhat-Tits tree Δ_K .

The **Bruhat-Tits** (BT) tree Δ_K is obtained by considering free \mathcal{O} -modules M of rank 2, with the equivalence relation that $M_1 \sim M_2$ if there is $\lambda \in K^*$ such that $M_1 = \lambda M_2$. The set Δ_K^0 of vertices of the BT tree consists of the equivalence classes $[M]$ under the relation \sim . The distance between two classes of the set Δ_K^0 is defined by $d([M_1], [M_2]) = |l - k|$, where if $M_1 \supset M_2$, then

$$M_1/M_2 \cong \mathcal{O}/\mathfrak{m}^l \oplus \mathcal{O}/\mathfrak{m}^k \quad \text{for some } l, k \in \mathbb{N}.$$

To form the BT tree, an edge of the set Δ_K^1 of edges is defined to be a (directed or not) arrow from $[M_2]$ to $[M_1]$ if $M_2 \subset M_1$ with $d([M_1], [M_2]) = 1$. Then the BT tree $\Delta_K = (\Delta_K^0, \Delta_K^1)$ becomes a connected, locally finite tree with $q + 1$ edges departing from each vertex.

The group $PGL_2(K)$ acts on Δ_K (from the left) transitively by isometries.

The Bruhat-Tits tree Δ_K is the analog of the real 3-dimensional hyperbolic space \mathbb{H}^3 at the infinite primes. The set of ends of Δ_K is identified with $\mathbb{P}^1(K)$, just like that $\mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3$ in the case at infinity.

A p -adic **Schottky** group Γ is defined to be a discrete subgroup of $PGL_2(K)$ which consists of hyperbolic elements γ in K , for which the eigenvalues have different valuation, and Γ is isomorphic to a free group with g generators. Denote by $\Lambda_{\Gamma, K}$ the limit set in $\mathbb{P}^1(K)$, which is the closure of the set of points of $\mathbb{P}^1(K)$ that are fixed points of some $\gamma \in \Gamma \setminus \{1\}$. As in the case at infinity, it holds that $\text{card}(\Lambda_{\Gamma, K}) < \infty$ if and only if $\Gamma = \langle \gamma \rangle^{\mathbb{Z}}$ for some $\gamma \in \Gamma$, in the genus one case. Denote by $\Omega_{\Gamma}(K)$ the complement of $\mathbb{P}^1(K)$ in $\Lambda_{\Gamma, K}$, as the domain of discontinuity of Γ .

In the case of genus more than 1, the quotient $X_{\Gamma, K} = \Omega_{\Gamma}(K)/\Gamma$ is a **Schottky-Mumford** curve, with p -adic Schottky uniformization. In the case of genus 1, it is a Mumford (elliptic) curve, with the Jacobi-Tate uniformization. (No original figure).

A path in the Bruhat-Tits tree Δ_K which is infinite in both directions, with no back-tracking, is said to be an **axis** of Δ_K . Any two points $z_1, z_2 \in \mathbb{P}^1(K)$ uniquely define an axis connecting z_1 and z_2 as endpoints in $\partial\Delta_K$. The unique axis of Δ_K whose ends are the fixed points of a hyperbolic element γ is said to be the **axis** of γ . The element γ acts on its axis as a translation. Denote by Δ'_{Γ} the smallest subtree of Δ_{Γ} containing all the axes of elements of Γ (which may be called as the axis **tree**).

The subtree Δ'_{Γ} is Γ -invariant, with $\Lambda_{\Gamma, K}$ as the set of ends. The quotient Δ'_{Γ}/Γ is a finite graph, which is the dual graph of the closed fiber of the minimal smooth model over \mathcal{O} of $X_{\Gamma, K}$ (as a \mathfrak{k} -split degenerate semi-stable curve). With no figures, there are correspondences among the special (closed) fibers as curves, the dual graphs such as two circles a, b as edges attached with a point

or an edge c , and the axis trees Δ'_Γ generated by $\{a, b\}$ or $\{a, b, c\}$, for all the possible cases of maximal degenerations special for genus 2.

For each $n \geq 0$, also consider the subtree or subgraph $\Delta_{K,n}$ of the BT tree Δ_K defined by setting the set of vertices and the set of edges as

$$\Delta_{K,n}^0 = \{\text{vertex } v \in \Delta_K^0 \mid d(v, \Delta'_\Gamma) \equiv \inf\{d(v, v') \mid v' \in (\Delta'_\Gamma)^0\} \leq n\},$$

where $d(v, v')$ is the distance on Δ_K^0 , with $v = [M], v' = [M']$, and

$$\Delta_{K,n}^1 = \{\text{edge } w \in \Delta_K^1 \mid \text{source } s(w), \text{ range } r(w) \in \Delta_{K,n}^0\}.$$

In particular, $\Delta_{K,0} = \Delta'_\Gamma$.

For all $n \in \mathbb{N}$, the subtree or subgraph $\Delta_{K,n}$ is invariant under the action of a p -adic Schottky group Γ on Δ_K , and the quotient $\Delta_{K,n}/\Gamma$ is a finite graph, which is the dual graph of the reduction $X_{\Gamma,K} \otimes \mathcal{O}/\mathfrak{m}^{n+1}$.

For a more detailed account of Schottky-Mumford curves, see [104] and [124].

Model of the dual graph. The (revised) dictionary between the case of Mumford curves and the case at arithmetic infinity is now summarized as in the following:

Table 6: The dictionary between the BT tree and the Hyperbolic 3-geometry

Notion	Tree geometric dynamics	Hyperbolic 3GD
Space	BT tree graph $\Delta_K = (\Delta_K^0, \Delta_K^1)$	H 3-space \mathbb{H}^3
Boundary	$\mathbb{P}^1(K) = \partial\Delta_K$	$\mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3 \approx S^2$
Path (mini)	Connecting edges in Δ_K	Geodesics in \mathbb{H}^3
Group action	Schottky $\Gamma \subset PGL_2(K)$	Schottky $\Gamma \subset PSL_2(\mathbb{C})$
Discontinuity of Γ by Γ	Schottky-Mumford curve $X_{\Gamma,K} = \Omega_\Gamma(K)/\Gamma = \Lambda_{\Gamma,K}^c/\Gamma$	Riemann surface $X_\Gamma = \Omega_\Gamma/\Gamma = \Lambda_\Gamma^c/\Gamma$
Solid quotient	BT Graph Δ_K/Γ	Handle-body $Y_\Gamma = \mathbb{H}^3/\Gamma$
Core part	Axis subtree $\Delta'_\Gamma \subset \Delta_\Gamma$	Convex core in \mathbb{H}^3
Bounded quo.	Finite dual graph Δ'_Γ/Γ	Bounded geodesics in Y_Γ

Since bounded geodesics in Y_Γ can be identified with infinite geodesics in \mathbb{H}^3 with endpoints on $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ module the action of Γ , these are parameterized by the complement of the diagonal in $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$. This quotient is identified with the quotient of the totally disconnected space \mathfrak{S} of admissible, doubly infinite words with generators of a free group Γ and their inverses as generating characters, by the action of the invertible shift T . Thus obtained is the following model for the dual graph of the fiber at infinity.

- The solenoid $\mathfrak{S}_T = (\mathfrak{S} \times [0, 1]) / \sim$ as the mapping torus for T is a geometric model of the dual graph of the fiber at infinity of an arithmetic surface.

- The Cuntz-Krieger C^* -algebra \mathcal{O}_A as a noncommutative space, representing the algebra of coordinates on the quotient Λ_Γ/Γ , corresponds to the set of vertices of the dual graph as the set of components of the fiber at infinity,

while the C^* -algebra crossed product $C(\mathfrak{S}) \rtimes_T \mathbb{Z}$ as a noncommutative space, corresponding to the quotient $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$, gives the set of edges of the dual graph.

Moreover, given by using noncommutative geometry is a notion of reduction mod ∞ , analogous to the reduction maps mod p^m defined by the subgraphs $\Delta_{K,n}$ of the BT tree Δ_K in the case of Mumford curves. In fact, the reduction map corresponds to the paths connecting ends of the graph $\Delta_{K,n}/\Gamma$ to the corresponding vertices of the graph $\Delta'_\Gamma/\Gamma = \Delta_{K,0}/\Gamma$. Then the analog at arithmetic infinity consists of geodesics in Y_Γ which are the images of geodesics in \mathbb{H}^3 starting at some point $x_0 \in \mathbb{H}^3 \cup \Omega_\Gamma$ and having the other end as a point of Λ_Γ . These are parameterized by the set $\Lambda_\Gamma \times_\Gamma (\mathbb{H}^3 \cup \Omega_\Gamma)$. Thus, in terms of NC geometry, the reduction mod ∞ corresponds to a compactification of the homotopy quotient $\Lambda_\Gamma \times_\Gamma \mathbb{H}^3$ with $\mathbb{H}^3 = E\Gamma$ and $B\Gamma = \mathbb{H}^3/\Gamma = Y_\Gamma$. Hence, we can view Λ_Γ/Γ as the quotient of a foliation on the homotopy quotient with contractible leaves as \mathbb{H}^3 . Then the reduction mod ∞ is given by the assembly map μ as

$$\mu : K^{*+1}(\Lambda_\Gamma \times_\Gamma \mathbb{H}^3) \rightarrow K_*(C(\Lambda_\Gamma) \rtimes \Gamma), \quad * = 0, 1.$$

This shows that the spectral triple (\mathcal{O}_A, H, D) as a noncommutative (Riemann) space is closely related to the geometry of the fiber at arithmetic infinity of an algebraic variety. Then, may ask a question that what arithmetic information is captured by the Dirac operator D of the spectral triple. Let us see in the next section that as proved in [61], the Dirac operator gives another important arithmetic invariant, namely the local L -factor at the Archimedean prime.

4.7 Arithmetic varieties and L -factors

The **L**-function is an important invariant of arithmetic varieties. This is written as a product of contributions from the finite primes and the archimedean primes, as

$$\prod_{\mathfrak{p} \in \overline{\text{Sp}(\mathcal{O}_K)}} L_{\mathfrak{p}}(H^m(X), s).$$

For a detailed account on the subject, may refer to [144]. Let us only try to convey some basic ideas.

The reason why necessary is it to consider also the contribution of the archimedean primes can be seen in the case of the affine line $(\text{Spec}) \text{Sp}(\mathbb{Z})$, where the **Riemann** zeta function is involved and written as the **Euler** product as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}.$$

◇ That is holomorphic for $\text{Re}(s) > 1$ is analytically extended to a meromorphic function on \mathbb{C} with only pole $s = 1$ of order 1. Namely, as the **Laurent** expansion, locally, but for $s \in \mathbb{C} \setminus \{1\}$,

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} c_k (s-1)^k, \quad \gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right)$$

with γ the **Euler** constant and c_k the generalized **Euler** constants. \square

However, to have the functional equation as below, we need to consider the product

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\frac{1}{\pi^{\frac{s}{2}}} = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\frac{1}{\pi^{\frac{1-s}{2}}}$$

included with a contribution of the archimedean prime, expressed in terms of the **Gamma** function, with factorials

$$\Gamma(s) = \int_0^\infty \frac{t^s}{te^t} dt = (s-1)\Gamma(s-1), \quad \Gamma(n) = (n-1)!, \quad \Gamma(1) = 1.$$

An analogy with ordinary geometry suggests to think of the functional equation as a sort of Poincaré duality, which holds for a compact manifold, hence for which, we need to compactify arithmetic varieties by adding the archimedean primes and the corresponding archimedean fibers.

Looking at an arithmetic variety over a finite prime $\mathfrak{p} \in \text{Sp}(O_{\mathbb{K}})$, the fact that the reduction lives over a residue field of positive characteristic implies that there is a special operator as the geometric Frobenius $Fr_{\mathfrak{p}}^*$, acting on a suitable cohomology theory as étale cohomology, induced by the **Frobenius** automorphism $\varphi_{\mathfrak{p}}$ of $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ (as a p^n -power map).

◊ Note that $\overline{\mathbb{F}_p} \cong \cup_{1 \leq n < \infty} \mathbb{F}_{p^n}$ as an algebraic closure of the finite field \mathbb{F}_p with order a prime p , with $[\mathbb{F}_{p^n}, \mathbb{F}_p] = n$ (cf. [123]). \square

The **local** L -factors of the L -function first given as the product at finite primes \mathfrak{p} encode the action of the geometric Frobenius $Fr_{\mathfrak{p}}^*$ as in the form, as in [144]

$$L_{\mathfrak{p}}(H^m(X), s) = \det(1 - Fr_{\mathfrak{p}}^* N(\mathfrak{p})^{-1} | H^m(\overline{X}, \mathbb{Q}_l)^{I_{\mathfrak{p}}})^{-1},$$

where we consider the action of the geometric Frobenius $Fr_{\mathfrak{p}}^*$ on the inertia invariants $H^m(\overline{X}, \mathbb{Q}_l)^{I_{\mathfrak{p}}}$ of the étale cohomology, as explained in the following. We use the notation N for the norm map in the local L -factors (converted).

An introduction to étale cohomology as well as a precise definition of these arithmetic structures are beyond the scope of this survey. In fact, our primary concern would only be the contribution of the archimedean primes to the L -function, where the construction is based on the ordinary de Rham cohomology. Thus, we only give a quick and somewhat heuristic explanation of the local L -factors. May refer to [144], [155] for a detailed and rigorous account and for the precise hypotheses under which the following holds.

Let $X = X(\mathbb{Q})$ be a smooth projective algebraic variety of any dimension defined over \mathbb{Q} . Set $\overline{X} = X \otimes \text{Sp}(\overline{\mathbb{Q}}) \equiv \overline{X}(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ denotes an algebraic closure. For l a prime, the (**étale**) cohomology $H^*(\overline{X}, \mathbb{Q}_l)$ is a finite-dimensional \mathbb{Q}_l -vector space, satisfying

$$H^j(X(\mathbb{C}), \mathbb{C}) \cong H^j(\overline{X}, \mathbb{Q}_l) \otimes \mathbb{C}.$$

◊ The (countable!) algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} may be given as the set of all algebraic numbers over \mathbb{Q} (such as $\sqrt{2}$ and $\sqrt{3}$), up to a \mathbb{Q} -isomorphism. Then

$\mathbb{Q} \subset \overline{\mathbb{Q}}$ but $\overline{\mathbb{Q}} \subsetneq \mathbb{C}$, because there are transcendental numbers in \mathbb{C} such as e and π (and γ ?) (cf. [116], [129]). \square

The **absolute** Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^*(\overline{X}, \mathbb{Q}_l)$.

Similarly, we can consider the (étale) cohomology $H^*(\overline{X}, \mathbb{Q}_l)$ for $X = X(\mathbb{K})$ defined over a number field $\mathbb{K} \supset \mathbb{Q}$, with $\overline{X} = X \otimes \text{Sp}(\overline{\mathbb{K}}) \equiv \overline{X}(\overline{\mathbb{K}})$. For $\mathfrak{p} \in \text{Sp}(O_{\mathbb{K}})$ of maximal ideals of $O_{\mathbb{K}}$ the ring of integers in \mathbb{K} and l a prime such that $(l, q) = 1$, where q is the cardinality of the residue field $O_{\mathbb{K}}/\mathfrak{p}$ at \mathfrak{p} , the (physical law of) **inertia** invariants are defined to be

$$H^*(\overline{X}, \mathbb{Q}_l)^{I_{\mathfrak{p}}} \subset H^*(\overline{X}, \mathbb{Q}_l)$$

as the part of the l -adic cohomology, on which the **inertia** group at \mathfrak{p} acts trivially, and the inertia group is defined as the kernel of the short exact sequence of groups with $r_{\mathfrak{p}}$ as the quotient map:

$$1 \rightarrow I_{\mathfrak{p}} = \ker(r_{\mathfrak{p}}) \rightarrow D_{\mathfrak{p}} = \{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma(\mathfrak{p}) = \mathfrak{p}\} \xrightarrow{r_{\mathfrak{p}}} \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \rightarrow 1.$$

The Frobenius automorphism $\varphi_{\mathfrak{p}}$ of $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ (as a p^n -power map) lifts to $\varphi_{\mathfrak{p}} \in D_{\mathfrak{p}}/I_{\mathfrak{p}}$ by the same symbol, which induces the geometric **Frobenius** $Fr_{\mathfrak{p}}^* = (\varphi_{\mathfrak{p}}^{-1})^*$ acting on $H^*(\overline{X}, \mathbb{Q}_l)^{I_{\mathfrak{p}}}$. Thus, the **local** L -factor given above can be written equivalently as

$$L_{\mathfrak{p}}(H^m(X), s) = \prod_{\lambda \in \text{sp}(Fr_{\mathfrak{p}}^*)} \frac{1}{(1 - \lambda q^{-s})^{\dim H^m(X_{\Gamma})_{\lambda}^{I_{\mathfrak{p}}}}},$$

where $H^m(X_{\Gamma})_{\lambda}^{I_{\mathfrak{p}}}$ denotes the eigen-space of the Frobenius with eigenvalue $\lambda \in \text{sp}(Fr_{\mathfrak{p}}^*)$ the (operator) (point) spectrum.

\diamond The notation $H^m(X_{\Gamma})_{\lambda}^{I_{\mathfrak{p}}}$ may better be replaced with $H^m(\overline{X}, \mathbb{Q}_l)_{\lambda}^{I_{\mathfrak{p}}}$. \square

For our purpose, what is most important to retain from the discussion above is that the local L -factors depend on the data $(H^*(\overline{X}, \mathbb{Q}_l)^{I_{\mathfrak{p}}}, Fr_{\mathfrak{p}}^*)$ of a vector space and a linear operator on it, which have a cohomological interpretation.

Archimedean L -factors. Since the étale cohomology satisfies the compatibility as

$$H^j(\overline{X}, \mathbb{Q}_l) \otimes \mathbb{C} \cong H^j(X(\mathbb{C}), \mathbb{C}),$$

if we again resort to the general philosophy, according to which we can work with the smooth complex manifold $X(\mathbb{C})$ and gain information on the closed fiber at arithmetic infinity, then we are led to expect that the contribution of the archimedean primes to the L -function may be expressed in terms of the cohomology $H^*(X(\mathbb{C}), \mathbb{C})$, or equivalently in terms of de Rham cohomology.

In fact, it is shown by Serre [144] that the expected contribution of the archimedean primes depends on the **Hodge** structure as

$$H^m(X(\mathbb{C})) = \bigoplus_{p+q=m} H^{p,q}(X(\mathbb{C}))$$

and is again expressed in terms of Gamma functions, as in the case of the functional equation for $\zeta(s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}$. Namely, given is the product of Gamma functions according to the **Hodge** numbers $h^{p,q}$ (as the dimension of $H^{p,q}(X(\mathbb{C}))$)

for the (another) local L-factor:

$$L(H^m(X(\mathbb{C})), s) = \begin{cases} \prod_{p < q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \prod_p \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}, & \text{for the real embedding} \\ \prod_{p,q} \Gamma_{\mathbb{C}}(s - \min\{p, q\})^{h^{p,q}}, & \text{for the complex one} \end{cases}$$

where $h^{p,\pm}$ is the dimension of the $\pm(-1)^p$ -eigenspace of the involution on $H^{p,p}$ induced by the real structure, and

$$\Gamma_{\mathbb{C}}(s) \equiv (2\pi)^{-s} \Gamma(s) \quad \text{and} \quad \Gamma_{\mathbb{R}}(s) \equiv \sqrt{2}^{-1} \sqrt{\pi}^{-s} \Gamma(2^{-1}s).$$

One of the general ideas in arithmetic geometry is the always seeking a unified picture of what happens at the finite and at the infinite primes. In particular, it should be a suitable reformulation of the local L -factors $L_{\mathfrak{p}}(H^m(X(\mathbb{K})), s)$ and $L(H^m(X(\mathbb{C})), s)$ as the determinant or the product, where both formulae can be expressed in the similar way.

By seeking a unified description of local L -factors at finite and infinite primes, both L -factors as above are expressed as infinite determinants, by Deninger in [68], [69], [70].

Recall that the **Ray-Singer** determinant of an operator T with pure (imaginary) point spectrum $\lambda \in \text{sp}(T)$ with finite multiplicities m_{λ} is defined to be

$$\det_{\infty}(s - T) \equiv \exp\left(-\frac{d}{dz} \zeta_T(s, z)|_{z=0}\right), \quad \zeta_T(s, z) = \sum_{\lambda \in \text{sp}(T)} \frac{m_{\lambda}}{(s - \lambda)^z}$$

as the zeta function of the operator T . Described in [107] are suitable conditions for the convergence of these expressions in the case of the local L -factors.

It is shown by Deninger that the local L -factor $L_{\mathfrak{p}}(H^m(X), s)$ as the product can be written equivalently in the form

$$L_{\mathfrak{p}}(H^m(X), s)^{-1} = \det_{\infty}(s - \Theta_q),$$

for an operator $s - \Theta_q$ with spectrum given as

$$\text{sp}(s - \Theta_q) = \left\{ s - \alpha_{\lambda} + \frac{2\pi i n}{\log q} \mid n \in \mathbb{Z}, \lambda \in \text{sp}(Fr_{\mathfrak{p}}^*) \right\}$$

with multiplicities d_{λ} and $q^{\alpha_{\lambda}} = \lambda$.

Moreover, the (another) local L -factor $L(H^m(X(\mathbb{C})), s)$ at infinity can be written similarly in the form

$$L(H^m(X(\mathbb{C})), s) = \det_{\infty}((2\pi)^{-1}(s - \Phi)|_{\mathcal{H}^m})^{-1},$$

where \mathcal{H}^m is an infinite dimensional vector space and Φ is a linear operator with spectrum $\text{sp}(\Phi) = \mathbb{Z}$ with finite multiplicities. This operator is regarded as a logarithm of Frobenius at arithmetic infinity.

Given the two formulae of Deninger for the local L -factors by \det_∞ , it is natural to ask for a cohomological interpretation of the data (\mathcal{H}^m, Φ) , somewhat analogous to the data $(H^*(\overline{X}, \mathbb{Q}_l)^{I_p}, Fr_p^*)$.

Arithmetic surfaces: L -factor and Dirac operator. As a return, let us now consider the special case of arithmetic surfaces, in the case of genus $g \geq 2$. At an archimedean prime, consider the Riemann surface $X_\alpha(\mathbb{C})$ with a Schottky uniformization $X(\mathbb{C}) = \Omega_\Gamma/\Gamma$. In the case of a real embedding $\alpha : \mathbb{K} \rightarrow \mathbb{C}$, we can assume that the choice of a Schottky uniformization corresponds to the real structure, obtained by cutting $X(\mathbb{C})$ along the real locus $X(\mathbb{R})$.

Now consider the spectral triple (\mathcal{O}_A, H, D) associated to the Schottky group Γ acting on its limit set Λ_Γ , where $H = \oplus^2 L^2(\Lambda_\Gamma)$ and $\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma$ and

$$D = \sum_{n=1}^{\infty} n \Pi_n^\wedge \otimes \left(- \sum_{n=1}^{\infty} n \Pi_n^\wedge \right).$$

Since the spectral triple is not finitely summable (but θ -summable), we can not define zeta functions of the spectral triple as in the form $\text{tr}(a|D|^z)$. However, we can consider the restriction of D to a suitable subspace of H (if any), on which the trace is finite.

In particular, consider the zeta function with respect to the (Dirac) operator D (restricted)

$$\zeta_{\pi_{\mathcal{V}}, D}(s, z) = \sum_{\lambda \in \text{sp}(D)} \text{tr}(\pi_{\mathcal{V}} \Pi(\lambda, D)) \frac{1}{(s - \lambda)^z}$$

(i inserted), where $\Pi(\lambda, D) (= p(\lambda, D))$ is the projection on the eigenspace for an eigenvalue λ of D , and $\pi_{\mathcal{V}}$ is the orthogonal projection of H on the subspace \mathcal{V} of $0 \oplus L^2 \subset H$ defined by

$$\Pi_n \pi_{\mathcal{V}} \Pi_n = \sum_{j=1}^g s_j^n (s_j^*)^n \quad \text{on } L^2,$$

where Π_n are the projections onto $\mathcal{P}_{n, \mathbb{C}}$ in L^2 and are also the spectral projections of the Dirac operator D , with $\Pi(-n, D) = \Pi_n$ for $n \geq 0$.

In the case of an arithmetic surface, as the interesting local factor. the first cohomology $L(H^1(X), s)$ is computed ([59], [61]) as in the following, from the zeta function $\zeta_{\pi_{\mathcal{V}}, D}$ of the spectral triple (\mathcal{O}_A, H, D) .

Theorem 4.5. *The local L -factor given as $L(H^m(X(\mathbb{C})), s)$ above is computed as*

$$L(H^1(X(\mathbb{C})), s) = \exp \left(- \frac{d}{dz} \zeta_{\pi_{\mathcal{V}}, (2\pi)^{-1} D}((2\pi)^{-1} s, z) \Big|_{z=0} \right)^{-1}.$$

In the case of a real embedding, the same holds, with the projection $\pi_{\mathcal{V}, \overline{F_\infty}}$ onto the $+1$ eigenspace of the involution $\overline{F_\infty}$ induced by the real structure on \mathcal{V} .

In particular, this result shows that, for the special case of arithmetic surfaces with $X(\mathbb{C})$ of genus $g \geq 2$, the pair $(\mathcal{V} \subset L^2(\Lambda_\Gamma), D|_{\mathcal{V}})$ is a possible geometric construction of the pair (\mathcal{H}^1, Φ) of (\mathcal{H}^m, Φ) . Moreover, the Dirac operator of the spectral triple has an arithmetic meaning, as that it recovers the logarithm of Frobenius as $D|_{\mathcal{V}} = \Phi$.

Looking more closely at the subspace \mathcal{V} of the Hilbert space H , we see that it has a simple geometric interpretation in terms of the geodesics in the handle-body Y_Γ . Recall that the filtered subspace $\mathcal{P}_\mathbb{C} = C(\Lambda_\Gamma, \mathbb{Z}) \otimes \mathbb{C}$ of $L^2 = L^2(\Lambda_\Gamma, d\mu)$ describes 1-cochains on the mapping torus \mathfrak{S}_T , with $\mathcal{P}_\mathbb{C}/\delta_*\mathcal{P}_\mathbb{C} = H^1(\mathfrak{S}_T)$.

The mapping torus \mathfrak{S}_T is viewed as a copy of the tangle of bounded geodesics inside Y_Γ . Among these geodesics there are g fundamental closed geodesics that correspond to the generators of Γ , which correspond to geodesics in \mathbb{H}^3 connecting the fixed points $\{z^\pm(\gamma_j)\}$ for $j = 1, \dots, g$. Topologically, these are the g core handles of the handle-body Y_Γ , which generate the homology $H_1(Y_\Gamma)$.

Consider the cohomology $H^1(\mathfrak{S}_T)$. Suppose that there are elements of $H^1(\mathfrak{S}_T)$ supported on those fundamental closed geodesics. But this is impossible, because 1-cochains on \mathfrak{S}_T are defined by functions in $C(\mathfrak{S}, \mathbb{Z})$, that are supported on some clopen subsets covering the totally disconnected set \mathfrak{S} , which contains no isolated points. However, it is possible to choose a sequence of 1-cochains on \mathfrak{S}_T whose supports are much smaller clopen subsets containing the infinite word in \mathfrak{S} corresponding to one of the g fundamental geodesics. The complex finite dimensional subspace $\mathcal{V}_n = \mathcal{V} \cap \mathcal{P}_{n,\mathbb{C}} \subset \mathcal{P}_{n,\mathbb{C}}$ with $\dim_{\mathbb{C}} \mathcal{V}_n = 2g$ gives representatives of such cohomology classes in $H^1(\mathfrak{S}_T)$, where we get $2g$ instead of g because we take into account the two possible choices of orientation.

Thus, it gives us the cohomological interpretation of the space $\mathcal{H}^1 = \mathcal{V}$ of the pair (\mathcal{H}^1, Φ) , in the case of arithmetic surfaces. Moreover, the Schottky uniformization also provides us with a way of expressing the cohomology as $\mathcal{H}^1 = \mathcal{V}$ in terms of the de Rham cohomology $H^1(X(\mathbb{C}))$. In fact, already seen in the calculation of the Green function, under the hypothesis that $\dim_H(\Lambda_\Gamma) < 1$, to each generator γ_j of the Schottky group Γ , we can associate a holomorphic differential ω_{γ_j} on the Riemann surface $X(\mathbb{C})$. Namely, the map

$$\gamma_j \mapsto \omega_{\gamma_j} = \sum_{h \in C(\gamma_j)} d \log [hz^+(\gamma_j), hz^-(\gamma_j), z, z_0]$$

induces an identification as

$$\mathcal{V} \cong \bigoplus_{n \in \mathbb{Z}, n \geq 0} H^1(X(\mathbb{C})).$$

See in the next section that, in fact, the right hand side above is a particular case of a more general construction that works for arithmetic varieties in any dimension and that gives a cohomological interpretation of (\mathcal{H}^m, Φ) .

4.8 Archimedean cohomology

Given by Consani in [57], for general arithmetic varieties in any dimension, a cohomological interpretation of the pair (\mathcal{H}^m, Φ) in the Deninger calculation

of the archimedean L -factors as regularized determinants. Its construction is motivated by the analogy between geometry at arithmetic infinity and the classical geometry of a degeneration over the disk. Introduced by her is a double complex of differential forms with an endomorphism \mathbf{n} representing the logarithm of the monodromy around the special fiber at arithmetic infinity, which is modelled on a resolution of the complex of nearby cycles in the geometric case. The definition of the complex of nearby cycles and of its resolution is rather technical, on which the following construction is modelled. What is easy to visualize it geometrically is the related complex of the vanishing cycles of a geometric degeneration. (No original figure).

◊ But it looks like that a surface with genus two degenerates to the corresponding (2-dimensional) graph with 3 vertexes and 6 edges, which looks like that two (solid) triangles are attached at respective vertexes. \square

Describe the construction of [57] using the notation of [63]. And construct the cohomology theory underlying (\mathcal{H}^m, Φ) in the following three steps. In the following, assume that $X = X(\mathbb{C})$ is a complex compact Kähler manifold.

◊ A complex manifold X with J an almost **complex** structure such that a map $J : TM \rightarrow TM$ the tangent bundle over M with $J^2 = -\text{id}$ the identity map, and with a Riemann metric g as an **Hermite** metric, so that $g(Jx, Jy) = g(x, y)$, is said to be a **Kähler** manifold if $d\omega = 0$, where $\omega(x, y) = g(Jx, y)$ is a 2-form (cf. [116]). \square

As the **step 1**, we begin by considering a doubly infinite graded **complex** as

$$C^* = \Omega^*(X) \otimes \mathbb{C}[u, u^{-1}] \otimes \mathbb{C}[h, h^{-1}],$$

where $\Omega^*(X)$ is the de Rham complex of differential forms on X , while u and h are formal variables, with u of degree two and h of degree zero. On this complex, consider the differentials

$$d'_C = hd \quad \text{and} \quad d''_C = \sqrt{-1}(\bar{\partial} - \partial),$$

with total differential $\delta_C = d'_C + d''_C$ (with the exterior derivative $d = \partial + \bar{\partial}$ as the decomposition by the Dolbeault operator $\bar{\partial}$ with respect to a Hermite metric). Define an **inner** product as

$$\langle \alpha \otimes u^r \otimes h^k, \beta \otimes u^s \otimes h^t \rangle = \langle \alpha, \beta \rangle \delta_{r,s} \delta_{k,t},$$

where the usual **Hodge** inner product of forms is given by

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \overline{*C(\beta)\beta}, \quad \text{with } C(\eta) = i^{p-q} \text{ for } \eta \in \Omega^{p,q}(X)$$

(corrected), where $\alpha \wedge \overline{* \beta} = \langle \alpha, \beta \rangle * 1$ with $*$ the Hodge (complex) operator defined so, with $*1$ as the volume form (cf. [116]).

As the **step 2**, we use the **Hodge** filtration as

$$F^p \Omega^m(X) = \bigoplus_{p'+q=m, p' \geq p} \Omega^{p',q}(X)$$

to define linear subspaces of C^* of the form

$$\mathfrak{C}^{m,2r} = \bigoplus_{p+q=m, k \geq \max\{0, 2r+m\}} F^{m+r-k} \Omega^m(X) \otimes u^r \otimes h^k$$

and the \mathbb{Z} -graded vector space

$$\mathfrak{C}^* = \bigoplus_{*=m+2r} \mathfrak{C}^{m,2r}.$$

As the **step 3**, we pass to the real vector space by considering $\mathfrak{T}^* = (\mathfrak{C}^*)^{c=\text{id}}$, where c denotes the complex conjugation.

In terms of the intersection $\gamma^* = F^* \cap \bar{F}^*$ of the Hodge filtrations, we can write it as $\mathfrak{T}^* = \bigoplus_{*=m+2r} \mathfrak{T}^{m,2r}$, where

$$\mathfrak{T}^{m,2r} = \bigoplus_{p+q=m, k \geq \max\{0, 2r+m\}} \gamma^{m+r-k} \Omega^m(X) \otimes u^r \otimes h^k.$$

The \mathbb{Z} -graded complex vector space \mathfrak{C}^* is a subcomplex of C^* with respect to the differential d'_C . For p^\perp the orthogonal projection onto \mathfrak{C}^* under the inner product defined above, we define the second differential as $d'' = p^\perp d'_C$. Similarly, $d' = d'_C$ and $d'' = p^\perp d'_C$ define differentials on the \mathbb{Z} -graded real vector space \mathfrak{T}^* since the inner product is real on real forms and induces an inner product on \mathfrak{T}^* . Define the total differential as $\delta = d' + d''$.

The real vector spaces \mathfrak{T}^* can be described in terms of certain cutoffs on the indices of the complex C^* . Namely, define

$$\Lambda_{p,q} = \{(r, k) \in \mathbb{Z}^2 \mid k \geq l(p, q, r) \equiv \max\{0, 2r + m, \frac{|p - q| + 2r + m}{2}\}\},$$

and then identify \mathfrak{T}^* with the real vector space spanned by $\alpha \otimes u^r \otimes h^k$ for $(r, k) \in \Lambda_{p,q}$ and $\alpha = \xi + \bar{\xi}$ with $\xi \in \Omega^{p,q}(X)$. (No original figure).

◊ But may consider the lines in the (r, k) -plane defined as $k = 0$, $k = 2r + m$, and $k = r + 2^{-1}(|p - q| + m)$, all of which make a sort of the boundary of $\Lambda_{p,q}$ as well as the corresponding subspace $\mathfrak{T}_{p,q}^m$ of \mathfrak{T}^* , as cutoffs defining the complex at arithmetic infinity. \square

The logarithmic and Lefschetz operators. The complex (\mathfrak{T}^*, δ) has some interesting structures given by the action of certain linear operators, as follows. There are the (**logarithmic**) operators \mathfrak{n} and Φ that correspond to the logarithm of the monodromy and the logarithm of Frobenius, respectively. These are defined as the forms $\mathfrak{n} = uh$ and $\Phi = -u \frac{\partial}{\partial u}$, satisfying

$$[\mathfrak{n}, d'] = [\mathfrak{n}, d''] = 0 \quad \text{and} \quad [\Phi, d'] = [\Phi, d''] = 0,$$

and hence $[\mathfrak{n}, \delta] = 0$ and $[\Phi, \delta] = 0$, and thus they induce the corresponding operators in cohomology.

Moreover, there is another important (**Lefschetz**) operator \mathfrak{l} , which corresponds to the Lefschetz operator on forms, defined as

$$\mathfrak{l}(\eta \otimes u^r \otimes h^k) = (\eta \wedge \omega) \otimes u^{r-1} \otimes h^k,$$

where ω is the Kähler form on the complex manifold X . This satisfies that $[\mathfrak{l}, d'] = [\mathfrak{l}, d''] = 0$, so that it also descends to that in cohomology.

The pairs (\mathfrak{n}, Φ) and (\mathfrak{l}, Φ) of those operators satisfy the interesting commutation relations

$$[\Phi, \mathfrak{n}] = -\mathfrak{n} \quad \text{and} \quad [\Phi, \mathfrak{l}] = \mathfrak{l},$$

which can be seen as an action of the ring of differential operators

$$\mathbb{C}[\mathfrak{p}, \mathfrak{q}]/\mathfrak{q} \quad \text{with} \quad [\mathfrak{p}, \mathfrak{q}] = \mathfrak{p}\mathfrak{q} - \mathfrak{q}\mathfrak{p} = \mathfrak{q}.$$

As representations of $SL_2(\mathbb{R})$. As another important piece of the structure of the complex (\mathfrak{T}^*, δ) , there are two **involutions** defined as

$$\begin{aligned} \mathfrak{s}(\alpha \otimes u^r \otimes h^{2r+m+l}) &= \alpha \otimes u^{-r-m} \otimes h^l \quad \text{and} \\ \mathfrak{s}^{\sim}(\alpha \otimes u^r \otimes h^k) &= C(*\alpha) * \alpha \otimes u^{r-n+m} \otimes h^k \end{aligned}$$

(corrected). These maps, together with the nilpotent operators \mathfrak{n} and \mathfrak{l} , define two **representations** σ^L and σ^R , given by

$$\begin{aligned} \sigma^L(a(s)) &= s^{-n+m}, \quad \sigma^L(n(t)) = \exp(t\mathfrak{l}), \quad \sigma^L(r) = i^n C\mathfrak{s}^{\sim}, \\ \sigma^R(a(s)) &= s^{2r+m}, \quad \sigma^R(n(t)) = \exp(t\mathfrak{n}), \quad \sigma^R(r) = C\mathfrak{s}, \end{aligned}$$

where $SL_2(\mathbb{R})$ has the Iwasawa decomposition $KAN = SO(2) \cdot \mathbb{R}^* \cdot \mathbb{R}$ (Compact (or Kompakt in German) \cdot Abelian \cdot Nilpotent) with $SO(2) = \mathbb{P}^1(\mathbb{R}) = \mathbb{T} = S^1$, (partially) respectively generated by

$$r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad a(s) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \quad n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

for $s \in \mathbb{R}^*$ and $t \in \mathbb{R}$, where the matrix r is called the **Weyl** element.

The representation σ^L extends to an action by bounded operators on the Hilbert space completion of \mathfrak{T}^* with the inner product, while the action of the subgroup of $a(s)$, $s \in \mathbb{R}^*$ of $SL_2(\mathbb{R})$ under the representation σ^R on the same Hilbert space is given by unbounded densely defined operators.

Renormalization group and monodromy. That general structure for varieties in any dimension also has interesting connections to noncommutative geometry. For instance, we can see that the map \mathfrak{n} , in fact does play the role of the logarithm of the monodromy, using an analog in our context, of the theory of renormalization à la Connes-Kreimer [39].

In the classical case of a geometric degeneration on a disk, the **monodromy** around the special fiber is defined as the map $\mathfrak{t} = \exp(-2\pi i \text{res}_0(\nabla))$, where $\mathfrak{n} = \text{res}_0(\nabla)$ is the residue at zero of the connection ∇ , acting as an endomorphism of the cohomology.

Let $\mathfrak{T}_{\mathbb{C}}^*$ be the complexification of the real vector space \mathfrak{T}^* . Consider loops φ_{μ} with values in the group $G = \text{Aut}(\mathfrak{T}_{\mathbb{C}}^*, \delta)$, depending on a mass parameter $\mu \in \mathbb{C}^*$. The **Birkhoff** decomposition of a loop φ_{μ} is given as the multiplicative decomposition $\varphi_{\mu}(z) = \varphi_{\mu}^-(z)^{-1} \varphi_{\mu}^+(z)$, for $z \in \partial\Delta \subset \mathbb{P}^1(\mathbb{C}) \approx S^2$, where Δ

is a small disk with center zero. Of the two terms in the decomposition, φ_μ^+ extends to a holomorphic function on Δ and φ_μ^- does to a holomorphic function on $\mathbb{P}^1(\mathbb{C}) \setminus \Delta$ with values in G . The decomposition is normalized by requiring that $\varphi_\mu^-(\infty) = 1$.

By analogy with the Connes-Kremer theory of renormalization we require the following two properties:

- The time **evolution** defined by $\theta_t(a) = e^{-t\Phi} a e^{t\Phi}$ ($a \in G$) as a natural choice, given by the geodesic flow associated to the Dirac operator Φ , acts on loops by scaling as $\varphi_{\lambda\mu}(z) = \theta_{tz}\varphi_\mu(z)$, for $\lambda = e^t \in \mathbb{R}_+^*$ and $z \in \partial\Delta$.

- The term $\varphi_\mu^- = \varphi^-$ in the Birkhoff decomposition is independent of the energy (as mass) scale μ .

The **residue** of a loop $\varphi_\mu = \varphi$, as in [39], is given by

$$\text{res}(\varphi) = \frac{d}{dz} \varphi^-\left(\frac{1}{z}\right)^{-1} \Big|_{z=0} \quad (\text{in } G ?),$$

and the beta function of **renormalization** is defined as

$$\beta(= \beta(\varphi)) = \mathfrak{b} \text{res}(\varphi) \in G \quad \text{with } \mathfrak{b} = \frac{d}{dt} \theta_t \Big|_{t=0}.$$

It follows that

$$\mathfrak{b}(a) = \frac{d}{dt} \theta_t(a) \Big|_{t=0} = [a, \Phi].$$

There is the **scattering** formula (as in [39]), by which φ^- can be reconstructed from the residue. Namely, (in G)

$$\frac{1}{\varphi^-(z)} = 1 + \sum_{k \geq 1} \frac{d_k}{z^k}, \quad d_k = \int_{s_1 \geq \dots \geq s_k \geq 0} \theta_{-s_1}(\beta) \cdots \theta_{-s_k}(\beta) ds_1 \cdots ds_k.$$

The **renormalization** group (in G) is given by

$$\rho(\lambda) = \lim_{\varepsilon \rightarrow 0} \varphi^-(\varepsilon) \theta_{t\varepsilon}(\varphi^-(\varepsilon)^{-1}), \quad \lambda = e^t \in \mathbb{R}_+^*.$$

Thus, we only need to specify the residue in order to have the corresponding renormalization theory associated to $(\mathfrak{T}_\mathbb{C}^*, \delta)$. By analogy with the case of the geometric degeneration as $\mathfrak{n} = \text{res}_0(\nabla)$, it is natural to require that $\text{res}(\varphi) = \mathfrak{n}$. It then follows that ([63])

Proposition 4.6. *A loop φ_μ valued in $G = \text{Aut}(\mathfrak{T}_\mathbb{C}^*, \delta)$ with $\text{res}(\varphi_\mu) = \mathfrak{n}$, subject to the time evolution by scaling, and with $\varphi^- = \varphi_\mu^-$ independent of μ , satisfies $\varphi_\mu(z) = \exp(z^{-1}\mu^z \mathfrak{n})$ with Birkhoff decomposition (corrected)*

$$\varphi_\mu(z) = \frac{\varphi^+(z)}{\varphi^-(z)} = \frac{\exp(z^{-1}(\mu^z + 1)\mathfrak{n})}{\exp(z^{-1}\mathfrak{n})}.$$

Proof. (Edited). In fact, by $\mathfrak{b} = [\cdot, \Phi]$ and $\text{res}(\varphi_\mu) = \mathfrak{n}$, we have

$$\beta = \beta(\varphi) = \mathfrak{b} \text{res}(\varphi) = [\mathfrak{n}, \Phi] = \mathfrak{n}$$

and $\theta_t(\mathfrak{n}) = e^t \mathfrak{n}$. Hence, the scattering formula as above gives $\varphi^-(z) = \exp(z^{-1}\mathfrak{n})$ and the scaling property determines $\varphi_\mu(z) = \exp(z^{-1}\mu^z \mathfrak{n})$. \square

The (positive) part $\varphi_\mu^+(z)$ of the Birkhoff decomposition that is regular at $z = 0$ satisfies

$$\begin{aligned} \varphi_\mu^+(0) &= \mu^n = \exp(\mathfrak{n} \log \mu), \quad \text{with} \\ \lim_{z \rightarrow 0} \frac{\mu^z - 1}{z} &= \log \mu \lim_{z \rightarrow 0} \frac{e^{z \log \mu} - 1}{z \log \mu} = \log \mu. \end{aligned}$$

The renormalization group has the form $\rho(\lambda) = \lambda^n = \exp(t\mathfrak{n})$, with $\log \lambda = \log e^t = t$, which, through the representation σ^R of $SL_2(\mathbb{R})$, corresponds to the horocycle flow on $SL_2(\mathbb{R})$, as

$$\rho(\lambda) = n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The Birkhoff decomposition as above gives a trivialization of a principal G -bundle over $\mathbb{P}^1(\mathbb{C})$. Then the associated vector bundle \mathfrak{E}_μ^* with fiber $\mathfrak{T}_\mathbb{C}^*$ is considered. Moreover, a **Fuchsian** connection ∇_μ on this bundle is given by

$$\nabla_\mu : \mathfrak{E}_\mu^* \rightarrow \mathfrak{E}_\mu^* \otimes_{\mathcal{O}_\Delta} \Omega_\Delta^*(\log 0), \quad \nabla_\mu = \mathfrak{n} \left(\frac{1}{z} + \frac{d}{dz} \frac{\mu^z - 1}{z} \right) dz,$$

where $\Omega_\Delta^*(\log 0)$ denotes forms with logarithmic poles at 0, and local gauge potentials for ∇_μ are given as

$$-\varphi^+(z)^{-1} \frac{\log \pi(\gamma) dz}{z} \varphi^+(z) + \varphi^+(z)^{-1} d\varphi^+(z),$$

with respect to the monodromy representation $\pi : \pi_1(\Delta^*) \cong \mathbb{Z} \rightarrow G$, given by $\pi(\gamma) = \exp(-2\pi i \mathfrak{n})$, for γ the generator for $\pi_1(\Delta^*)$. This corresponds to the monodromy \mathfrak{t} in the classical geometric case. This Fuchsian connection has residue $\text{res}_{z=0} \nabla_\mu = \mathfrak{n}$ as in the geometric case.

Local factor and archimedean cohomology. It is shown by Consani [57] that the data (\mathcal{H}^m, Φ) can be identified with $(\mathfrak{H}^*(\mathfrak{T}^*, \delta)^{n=0}, \Phi)$, where $\mathfrak{H}^*(\mathfrak{T}^*, \delta)$ is the hyper-cohomology as the cohomology with respect to the total differential δ of the complex \mathfrak{T}^* , and $\mathfrak{H}^*(\mathfrak{T}^*, \delta)^{n=0}$ is the kernel of the map induced by \mathfrak{n} on the cohomology. The operator Φ is induced on the cohomology by $\Phi = -u \frac{\partial}{\partial u}$. The kernel is isomorphic to \mathcal{H}^m and called the **archimedean** cohomology. This also can be viewed as a piece of the cohomology of the **cone** of the monodromy \mathfrak{n} . This is the complex with differential as

$$\text{Con}(\mathfrak{n})^* = \mathfrak{T}^* \oplus \mathfrak{T}^*[+1], \quad D = \begin{pmatrix} \delta & -\mathfrak{n} \\ 0 & \delta \end{pmatrix}.$$

The complex $\text{Con}(\mathfrak{n})^*$ inherits a positive definite inner product from \mathfrak{T}^* , which descends on cohomology. The representation σ^L of $SL_2(\mathbb{R})$ on \mathfrak{T}^* induces a representation on $\text{Con}(\mathfrak{n})^*$. The corresponding representation

$$d\sigma^L : \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \rightarrow \text{End}(\mathfrak{T}^*)$$

of the Lie algebra \mathfrak{g} extends to a representation of the universal enveloping algebra $U(\mathfrak{g})$ on \mathfrak{T}^* as well as $\text{Con}(\mathfrak{n})^*$. This gives a representation in the algebra of bounded operators on the Hilbert space completion of $\text{Con}(\mathfrak{n})^*$ with the inner product.

Theorem 4.7. *The triple $(U(\mathfrak{g}), \mathfrak{S}^*(\text{Con}(\mathfrak{n})), \Phi)$ as a spectral triple (\mathcal{A}, H, D) has the following four properties:*

- (1) *Self-adjointness as $D = D^*$;*
- (2) *Unboundedness of D^2 as that $(1 + D^2)^{-\frac{1}{2}}$ is a compact operator;*
- (3) *Boundedness of $[D, \cdot]$ as that the commutators $[D, a]$ are bounded operators for any $a \in U(\mathfrak{g})$;*
- (4) *Summability as that the triple is $1+ = 1 + \varepsilon$ -summable for any $\varepsilon > 0$.*

Thus, the triple $(\mathcal{A}, H, D = \Phi)$ has most of the properties of spectral triples, confirming the fact that Φ as the logarithm of Frobenius should be thought of as a Dirac operator.

In any case, the structure is sufficient to consider zeta functions for that spectral triple. In particular, the alternating products of the local L -factors at infinity can be recovered from zeta functions of the spectral triple.

Theorem 4.8. *The zeta functions with respect to Φ defined as (corrected)*

$$\zeta_{a, \Phi}(s, z) = \sum_{\lambda \in \sigma_p(|\Phi|)} \text{tr}(a p(\lambda, |\Phi|)) \frac{1}{(s - \lambda)^z} \quad \text{with } a = \sigma^L(r)$$

give the formula

$$\det_{\infty, \sigma^L(r), \Phi}(s) \equiv \exp \left(-\frac{d}{dz} \zeta_{\sigma^L(r), \Phi}(s, z) \Big|_{z=0} \right) = \prod_{m=0}^{2n} L(H^m(X), s)^{(-1)^m}.$$

Arithmetic surfaces as homology and cohomology. In the particular case of arithmetic surfaces, there is an identification between $\mathfrak{S}^*(\text{Con}(\mathfrak{n}))$ and $\mathcal{H}^* \oplus (\mathcal{H}^*)^\vee$, where \mathcal{H}^* is the archimedean cohomology and $(\mathcal{H}^*)^\vee$ is its dual under the involution \mathfrak{s} defined above.

The identification

$$\mathfrak{u} : \mathcal{H}^1 \rightarrow \mathcal{V} \cong \bigoplus_{n \in \mathbb{Z}, n \geq 0} H^1(X(\mathbb{C})) \subset L^2$$

can be extended by considering a subspace \mathcal{W} of the homology $H_1(\mathfrak{S}_T)$ with $\mathcal{W} \cong (\mathcal{H}^*)^\vee$. The homology $H_1(\mathfrak{S}_T)$ can also be computed as a direct limit $\varinjlim_n \mathcal{K}_n$, where \mathcal{K}_n are free abelian groups of rank $(2g - 1)^n + 1$ for n even and of $(2g - 1)^n + (2g - 1)$ for n odd. The \mathbb{Z} -module \mathcal{K}_n is generated by the closed geodesics represented by periodic sequences in \mathfrak{S} of period $n + 1$. These need not be primitive closed geodesics. In terms of primitive closed geodesics, $H^1(\mathfrak{S}_T, \mathbb{Z})$ can be written equivalently as $\bigoplus_{n=0}^{\infty} \mathcal{R}_n$, where \mathcal{R}_n are free abelian groups of rank

$$\text{rk}(\mathcal{R}_n) = \frac{1}{n} \sum_{d|n} \mu(d) \text{rk}(\mathcal{K}_{\frac{n}{d}}),$$

with $\mu(d)$ the **Möbius** function satisfying $\sum_{d|n} \mu(d) = \delta_{n,1}$.

The **pairing** between the homology $H_1(\mathfrak{S}_T) = \varinjlim_{n'} \mathcal{K}_{n'}$ and the cohomology $H^1(\mathfrak{S}_T) = \varprojlim_n F_n$ is given by

$$\langle \cdot, \cdot \rangle : F_n \times \mathcal{K}_{n'} \rightarrow \mathbb{Z}, \quad \langle [f], x \rangle = n' f(\bar{x}).$$

This determines a graded subspace \mathcal{W} in $H_1(\mathfrak{S}_T, \mathbb{Z})$ dual to \mathcal{V} in $H^1(\mathfrak{S}_T)$. With the identification as

$$\begin{array}{ccc} \mathcal{H}^1 & \xrightarrow[\cong]{u} & \mathcal{V} \subset H^1(\mathfrak{S}_T) \\ \downarrow \cong & & \downarrow \langle \cdot, \cdot \rangle \\ (\mathcal{H}^1)^\vee & \xrightarrow[\cong]{u} & \mathcal{W} \subset H_1(\mathfrak{S}_T), \end{array}$$

yes we can identify the Dirac operator D on the Hilbert space $H = L^2 \oplus L^2$ with the logarithm of Frobenius, as $D|_{\mathcal{V} \oplus \mathcal{V}} = \Phi_{\mathcal{H}^1 \oplus (\mathcal{H}^1)^\vee}$ as restricted.

5 The vista around

May find more details in [64] and [46], or just invoking the following vista.

Noncommutative geometry for Shi-mura varieties. The results viewed as arising from a noncommutative version of the Shimura varieties $\text{Sm}(GL_1, \pm 1)$ and $\text{Sm}(GL_2, \mathbb{H}^\pm)$ are obtained by Bost-Connes [13], Connes-Marcolli [41], and CM-Ramachandran [48]. The results on the noncommutative boundary of the modular curves and the modular Hecke algebras of Connes-Moscovici [54], [55] are obtained by Manin-Marcolli [112], to contribute to illustrate the NC geometry associated to the modular tower, i.e., to the Shimura variety $\text{Sm}(GL_2, \mathbb{H}^\pm)$ (named as, East Mount Village, 2 Cho-me).

Then it is a natural idea to seek for quantum statistical mechanical systems QSMS associated to more general Shimura varieties, and to investigate the arithmetic properties of low temperature KMS states.

Shimura varieties have adelic groups of symmetries as a typical property, so that the problem of compatibility between adelic symmetries and the Galois action on the values of zero temperature KMS states on an arithmetic subalgebra may be formulated naturally in that context.

Generalizations of the Bost-Connes system and of the GL_2 system of Connes-Marcolli [41] to a class of Shimura varieties are constructed by Eugene Ha and Frédéric Paugam (HP) [76] as well as P [132], [133]. Their approach is to consider a more general Shimura datum (G, X) , in which the passing from \mathbb{A}_f^* to \mathbb{A}_f in the Bost-Connes, and from $GL_2(\mathbb{A}_f)$ to $M_2(\mathbb{A}_f)$ in the GL_2 is analogously achieved by considering a suitable enveloping semi-group M . To the resulting datum (G, X, M) , associated is a groupoid in the category of stacks, whose convolution algebra HP generalizes the algebra \mathcal{A}_2 of the GL_2 system, with variants depending on the choice of a level structure, which can be used to avoid the stacky singularities in the groupoid. There is a natural time evolution on the algebra HP , which is defined in terms of the data of the level structure,

and the resulting NC system has an adèlic group of symmetries. In particular, these results can be seen explicitly in the Hilbert module case. Also obtained through this general method are QSMS naturally associated to number fields. The construction agrees with that of CMR [48] for imaginary quadratic fields. The arithmetic properties of KMS states for these systems may be investigated or finished.

Classical Shimura varieties admit noncommutative boundaries, in the same sense as that a noncommutative boundary of modular curves is constructed by Manin-Marcolli [112]. Studied by Paugam [132], [133] are such noncommutative boundaries, described as the convolution algebras associated to the double cosets spaces $L \backslash G(\mathbb{R}) / P(K)$, where L is an arithmetic subgroup of a connected reductive algebraic group $G(\mathbb{Q})$ over \mathbb{Q} , and $P(K)$ is a real parabolic subgroup in $G(\mathbb{R})$. Described as moduli spaces are degenerations of complex structures on tori in multi-foliations. The point of view is from Hodge theoretic considerations.

In this general perspective, an interesting question seems to ask whether a connection between this type of noncommutative geometry of modular curves and the generalization of the theory of Heegner points by Darmon [67].

Sectral triples and trace formulae. Besides the spectral triples in the context of the archimedean factors at arithmetic infinity, mentioned in the last section, it would be interesting to seek other cases in the number theoretic context.

For example, there may be spectral triples naturally associated to the algebras of the quantum statistical mechanical systems for noncommutative Shimura varieties considered above.

In the case of the Bost-Connes system, this should be related to the noncommutative space of the Connes spectral realization of zeros of the Riemann zeta function, and for the Connes-Marcolli GL_2 system, also to the modular Hecke algebras of Connes-Moscovici, hence to modular forms. The more cases of general Shimura varieties should be interesting to consider in this perspective.

As an important topic related to the Connes spectral realization of zeros of the Riemann zeta function, considered is the trace formula in noncommutative geometry [29]. As an interesting direction of development of this idea, considered is a possible extension of the trace formula to the case of local L -factors of arithmetic varieties (cf. CCM such as [32], [33]). This approach may be suitable to merge the adèlic construction of the noncommutative space of commensurability classes of \mathbb{Q} -lattices, underlying the Connes trace formula, with the conjectural foliated space by Deninger [69], [70], underlying an arithmetic cohomology related to L -functions of arithmetic varieties.

Another class of spaces of arithmetic significance, that may carry more interesting spectral triples, contains spaces like the Mumford curves (cf. [60]), that admit p -adic uniformizations, for instance, in the case of the Cherednik-Drinfeld theory of p -adic uniformization of Shimura curves, as well as the generalizations of Bruhat-Tits trees with the action of p -adic Schottky groups, given by certain classes of higher rank buildings. Constructions of this type may be related to the Berkovich spaces, or to the Buium arithmetic differential invariants.

Noncommutative algebraic geometry. Noncommutative geometry initiated and developed by Connes is followed and considered so far, in the texts above. But to be mentioned, there are other variants of noncommutative geometry as the versions of noncommutative algebraic geometry, developed by Artin, Tate, and van den Bergh [5], by Manin [105], by Konsevich [90], by Rosenberg [143], and KR [91]. May also refer to Mahanta [101] as a guided tour to the literature limited in noncommutative algebraic geometry.

As noticed, more algebraic versions of noncommutative geometry may be more suitable for application to problems of algebraic number theory and arithmetic algebraic geometry. Certainly expected is that an interplay between the various versions of NG would help push the number theoretic applications further, besides being desirable in terms of the internal development of the field. Connes and Dubois-Violette [35], [36], [37] open up important new perspectives that combine these different forms of noncommutative geometry, to allow for the use of both analytic and algebro-geometric tools.

As an example of such an interplay, obtained is the result of Polishchuk [136], inspired by the Manin real multiplication program, where noncommutative projective varieties are naturally associated to noncommutative tori with real multiplication.

Hopf algebra actions. Symmetries of ordinary spaces are encoded by group actions, while symmetries of noncommutative spaces are also done by group actions, or given by Hopf algebra actions.

In the context of relations between noncommutative geometry and number theory, as being proved to be a powerful technique, the Holf algebraic structure of noncommutative spaces is exploited by Connes and Moscovici [54], [55] on the modular Hecke algebras. The product of modular forms is combined with the action of Hecke operators, to define the modular Hecke algebras, seen as the holomorphic part of the algebra of coordinates of the noncommutative space of commensurability classes of 2-dimensional \mathbb{Q} -lattices discussed above, endowed with symmetries given by the action of the Connes-Moscovici Hopf algebra \mathcal{H}_1^{CM} of transverse geometry in codimension 1. This Hopf algebra \mathcal{H}_1^{CM} arises in the context of foliations, acting on crossed product C^* -algebras, and is in a family of Hopf algebra \mathcal{H}_n^{CM} of transverse geometry in codimension n , obtained from the dual of the enveloping algebra of the Lie algebra of vector fields [52].

The action of that Hopf algebra \mathcal{H}_1^{CM} on the modular Hecke algebras is determined by (1) a grading operator, corresponding to the weight of modular forms, (2) a derivation introduced by Ramanujan, which corrects the ordinary differentiation by a logarithmic derivative of the Dedekind η function, and (3) an operator that acts as multiplication by a form-valued cocycle on $GL_2(\mathbb{Q})_+$ that measures the lack of modular invariance of $\eta^4 dz$. Allowed by this is to transfer notions and results from the transverse geometry of foliations to the context of modular forms and Hecke operators.

A tool useful in dealing with Hopf algebra symmetries of noncommutative spaces, not mentioned above, is the Hopf cyclic cohomology of Hopf algebras, introduced by Connes and Moscovici (cf. [51], [53], [56]) and as well refined

through a systematic foundational treatment by Khalkhali and Rangipour (cf. [89] as an overview). It can be thought of as the analog of group and Lie algebra cohomology in noncommutative geometry.

In the case of the modular Hecke algebras, the use of Hopf cyclic cohomology yields interesting results on modular forms. In the case of the Hopf algebra \mathcal{H}_1^{CM} , there are three basic cyclic cocycles, which respectively correspond to (1) the Schwarz derivative, (2) the Godbillon-Vey class, and (3) the transverse fundamental class, in the context of transverse geometry. May refer to [150].

(1) The cocycle associated to the Schwarz derivative is realized by an inner derivation, given in terms of the Eisenstein series $E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}$. This cyclic cocycle has an arithmetic significance, related to the data used in defining canonical Rankin-Cohen algebras in Zagier [159].

(2) The cocycle associated to the Godbillon-Vey class is expressed in terms of an 1-cocycle on $GL_2(\mathbb{Q})_+$ with values in Eisenstein series of weight two.

(3) The cocycle associated to the transverse fundamental class gives rise to a natural extension of the first Rankin-Cohen brackets of modular forms (cf. [159]).

Renormalization and motivic Galois symmetry. Seen in the last section is an application of the formalism of the Connes-Kreimer theory of perturbative renormalization in the context of arithmetic geometry. Revealed by a closer inspection of the CK formalism are the connections to arithmetic, in the context of perturbative renormalization of quantum field theories. This is investigated by Connes and Marcolli [40], [42], [43].

Their work is started from the Connes-Kreimer formulation of perturbative renormalization in terms of the Birkhoff decomposition of loops, as in the form $\varphi_{\mu}(z) = \frac{\varphi_{\mu}^{+}(z)}{\varphi_{\mu}^{-}(z)}$ for $\mu \in \mathbb{C}^*$ and $z \in S^1$, with values $\varphi_{\mu}(z)$ in the pro-unipotent Lie group $G(\mathbb{C})$ of complex points of the affine group scheme, associated to the Connes-Kreimer Hopf algebra of Feynman graphs of CK [38]. It is shown by CM that the scattering formula of Connes-Kreimer, as given above as the form

$$\frac{1}{\varphi^{-}(z)} = 1 + \sum_{k \geq 1} \frac{d_k}{z^k}, \quad d_k = \int_{s_1 \geq \dots \geq s_k \geq 0} \theta_{-s_1}(\beta) \cdots \theta_{-s_k}(\beta) ds_1 \cdots ds_k,$$

can be conveniently formulated in terms of iterated integrals and the time ordered exponential, or expansional. This follows from a calculation generalizing the Birkhoff decomposition given above as

$$\varphi_{\mu}(z) = \frac{\varphi^{+}(z)}{\varphi^{-}(z)} = \frac{\exp(z^{-1}(\mu^z + 1)\mathbf{n})}{\exp(z^{-1}\mathbf{n})},$$

which gives the time ordered exponential expression as

$$\varphi^{-}(z) = T \exp(-z^{-1} \int_0^{\infty} \theta_{-t}(\beta) dt).$$

Once expressed in the expansional form as above, we can understand the terms in the Birkhoff decomposition as solutions of certain differential equations. This leads to a reformulation of perturbative renormalization in terms of

the equivalence classes of certain differential systems with singularities, within the context of the Riemann-Hilbert correspondence and the differential Galois theory.

The nature of the singularities is determined by the two conditions on the behavior of the terms in the Birkhoff decomposition with respect to the scaling of the mass parameter, namely the condition $\partial_\mu \varphi_\mu^- = 0$ as the independence of the energy scale μ in $\varphi^- = \varphi_\mu^-$ and the time evolution scaling condition $\varphi_{e^t \mu}(z) = \theta_{tz} \varphi_\mu(z)$, as mentioned above.

Those two conditions are expressed geometrically through the notion of G -valued equi-singular connections on a principal \mathbb{C}^* -bundle B over a disk Δ , where G is the pro-unipotent Lie group of characters of the Connes-Kreimer Hopf algebra of Feynman graphs. The equi-singularity condition is the property that such a connection is \mathbb{C}^* -invariant and that its restrictions to sections of the principal bundle, that agree at $0 \in \Delta$, are mutually equivalent, in the sense that those are related by a gauge transformation by a G -valued \mathbb{C}^* -invariant map regular in B . It then follows that the geometric formulation equivalent to the data of perturbative renormalization is given by a class, up to the equivalence given by such gauge transformations, of flat equi-singular G -valued connections.

The class of differential systems defined by the flat equi-singular connections can be studied by techniques of the differential Galois theory. In particular, by this way, an underlying group of symmetries can be identified with the differential Galois group. In fact, it is shown that the category of equivalence classes of flat equi-singular bundles is a neutral Tannaka category, which is equivalent to the category of finite dimensional linear representations of an affine group scheme $U^* = U \rtimes G_m$, where U a pro-unipotent affine group scheme and G_m the multiplicative group. The renormalization group lifts canonically to a one-parameter subgroup of U , and this gives it an interpretation as a group of Galois symmetries.

The affine group scheme U corresponds to the free graded Lie algebra with one generator in each degree $n \in \mathbb{N}$. These corresponds to the splitting up the renormalization group flow in homogeneous components by loop number. This affine group scheme admits an arithmetic interpretation as the motivic Galois group of the category of mixed Tate motives on the scheme of N -cyclotomic integers, after localization at N , for $N = 3$ or $N = 4$.

The generator of the renormalization group lifted to U defines a universal singular frame, whose explicit expression in the generators of the Lie algebra has the same rational coefficients that appear in the local index formula of Connes-Moscovici [50]. This appears as an intriguing connection between perturbative renormalization (PR) and NG, possibly through an interpretation of the local index formula in terms of chiral anomalies [47]. Noncommutative geometry also appears to provide a geometric interpretation for the deformation to complex dime of dimensional regularization in perturbative quantum field theory, through the notions of spectral triples and dimension spectram (cf. [47]).

Remark. May recall from [116] the following. Let A be a unital commutative ring. The **spectrum** $\text{Sp}(A)$ of A is defined to be the set all non-trivial prime

ideals of A . The **Zariski** topology \mathcal{Z} on $\mathrm{Sp}(A)$ is defined as that for any subset $B \subset A$, the set $V(B)$ of all prime ideals of A containing B is a closed set. A closed point in $\mathrm{Sp}(A)$ corresponds to a maximal ideal of A . For any $a \in A$, let $O(a) = V(a)^c$ the complement in $\mathrm{Sp}(A)$. The set of all $O(a)$ for $a \in A$ is the open basis for \mathcal{Z} .

Denote by A_a the quotient ring of A by the multiplicative set $\{a^n \mid n \geq 0\}$. By associating $O(a)$ to A_a , obtained is the sheaf A_* over $\mathrm{Sp}(A)$. Then A_a is isomorphic to $\Gamma(O(a), A_*)$ of sections over $O(a)$. In particular, $\Gamma(\mathrm{Sp}(A), A_*) \cong A$.

◊ For instance, $V(0) = \mathrm{Sp}(A)$. Hence $O(0) = \emptyset$. Define $A_0 = \{0\}$. For $a, b \in A$, we have $V(\{a, b\}) \subset V(a)$, with $V(\{a, b\}) = V(a) \cap V(b)$. Thus, $O(a) \subset O(\{a, b\})$, with $O(\{a, b\}) = O(a) \cup O(b)$. There is a homomorphism from A_a to A_0 . \square

The **affine** scheme for A is defined to be $\mathrm{Sp}(A)$ as a space with A_* as a local ring.

A **scheme** is a space X with a local ring \mathcal{O} , which is locally an affine scheme. Namely, there is an open covering $\{U_j\}$ of X such that each U_j with the local ring $\mathcal{O}|_{U_j}$ restricted to U_j is isomorphic to an affine scheme as a space with a local ring. \blacktriangleleft

As closing the garden, more details may be included in the next time, if continued. There may be still some (possibly minor) mistakes found in the texts, because of the time and effort for editing, limited to the last minute.

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